Minimal Bootstrap in Meson-Baryon Systems

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We consider two-particle scattering systems for which the crossing matrices give a good estimate of the nature of forces. We define a bootstrap state to be an eigenvector of the total crossing matrix. It is shown that bootstrap schemes are generally not unique. We conjecture that the physical particles manifest themselves in a state of minimal bootstrap, that is, a scheme involving the participation of the least number of particles. By studying meson-baryon scattering in the $SU(2)$ and $SU(3)$ symmetries, we find substantial evidence in support of this conjecture. The results are not only that the minimal boostrap vectors accommodate the observed particles in the usual way, but also that the ratio of the coupling strengths of particles in different representations and the mixing parameter (D/F ratio) of the Yukawa-type octet coupling both turn out to agree with experiments.

I. INTRODUCTION

IN exploring the possibility that the existence of particles and resonances in strong interaction may be understood as a self-consistent bootstrap in the analytic S-matrix theory, a great number of investigations have been made[1–10] either to explain the existing particles or to predict new ones. These studies can broadly be classified into two categories: One involves detailed dynamical calculations using all the machinery of the S-matrix theory, viz., analyticity, unitarity, and crossing symmetry; the other is a qualitative study involving the examination of the crossing matrices from which is inferred the nature of the forces in the various states of a scattering process. The former aims at calculating the masses and coupling strengths of the particles, but because of the complications involved the considerations must per force be limited to simple systems only, or be content with partial self-consistency. The latter category deals with more complicated systems but abandons any attempt to answer questions concerning dynamical details such as mass values, thus permitting an understanding or prediction of the quantum numbers of the particle spectrum without first requiring the solutions of the complete problem. The purpose of this paper is to give a systematic and precise description of bootstrap within the confines of the considerations in the second category, and to consider in detail some aspects related to this description.

If we do not question the masses of the particles, but ask only in which angular-momentum state and irreducible representation of the internal symmetry group the particles are likely to occur, it is only necessary to examine the sign and strength of interaction in the various channels as dictated by the appropriate crossing-matrix elements. This has been borne out by the success of the Chew-Low static model[11] as well as by the qualitative results of many subsequent dynamical calculations. Thus the spin and the internal symmetry of the particles may perhaps be interpreted as a direct consequence of crossing symmetry and bootstrap in much the same sense that in atomic physics the multiplicity of spectral lines may be determined from the rotational symmetry of the interaction potential without the requirement that the Schrödinger equation must first be solved.

Let us then take seriously the notion of crossing symmetry and bootstrap, the latter being as yet not precisely defined. Assuming that the particles participating in the scattering process possess certain symmetry properties themselves, the content of crossing symmetry can partially be framed in the form of a crossing matrix relating the amplitudes in the direct process to those in the crossed process. The nature of the force corresponding to the exchange of certain particles in the crossed process is then inferred from the sign and magnitude of the corresponding crossing matrix element. We expect a state to bootstrap itself if the diagonal element for that state is positive and large; in the case of reciprocal bootstrap, two states support each other, so that the two corresponding off-diagonal elements should be positive and large. These are rough criteria that ignore the contribution of forces from other states which are often not negligible.

More specifically, Chew[1] has considered the meson-baryon system, and has shown that within the framework of the static model the exchange of the nucleon produces forces to form the (3,3) resonance and the exchange of the (3,3) resonance produces forces to form the nucleon bound state. In Chew's treatment and in subsequent considerations, however, only the submatrix of the crossing matrix is taken into account

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[1] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).
[2] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[3] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).
[4] P. Carruthers, Phys. Rev. Letters **10**, 538, 540 (1963); Phys. Rev. **133**, B497 (1964).
[5] R. E. Cutkosky, Ann. Phys. (N. Y.) **23**, 415 (1963).
[6] R. H. Capps, Phys. Rev. **134**, B460 (1964).
[7] I. P. Gyuk and S. F. Tuan, Phys. Rev. **134**, B1320 (1964).
[8] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963); Nuovo Cimento **31**, 1324 (1964).
[9] I. S. Gerstein and K. T. Mahanthappa, Nuovo Cimento **32**, 239 (1964).
[10] V. Singh, Nuovo Cimento **33**, 763 (1964).
[11] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

without regard to the $(I,J)=(\frac{1}{2},\frac{3}{2})$, $(\frac{3}{2},\frac{1}{2})$ states. Also it has been pointed out[9] that this treatment within the static model is equivalent to requiring the crossing submatrix to have an eigenvalue of 1. But in general the crossing submatrix need not satisfy this requirement; moreover, it is not consistent to ignore the $(\frac{1}{2},\frac{3}{2})$, $(\frac{3}{2},\frac{1}{2})$ states, since the nucleon and (3,3) exchanges produce certain interactions in the $(\frac{1}{2},\frac{3}{2})$ and $(\frac{3}{2},\frac{1}{2})$ states which then must be included in the self-consistent bootstrap for all the four states.

In order to treat the problem more completely, we adopt a more general notion of a bootstrap state vector, which we define to be an eigenvector of the total crossing matrix with eigenvalue $+1$. (We shall later show that the total crossing matrix can be diagonalized and that the eigenvalues can only be ± 1.) Clearly, if a state bootstraps itself, the eigenvector must lie mainly in the direction of that state. If two states bootstrap reciprocally, the eigenvector must have large components in these two states, and within the static model the ratio of the two components is equal to the ratio of the corresponding coupling constants squared. A generalization of this is that all the large components of a bootstrap state vector must be positive.

In our analysis we find that, in general, bootstrap schemes are not unique. In the language that we have adopted, this corresponds to the fact that there are generally more than one independent eigenvectors of the total crossing matrix having eigenvalue $+1$. It may be argued that this nonuniqueness is due to the approximations made in the analysis, such as the use of the static model and the pole approximation, which give the eigenvectors undue significance. It is reasonable to expect that a complete dynamical calculation can eliminate some of the arbitrariness. More specifically, one expects that among all possible linear combinations of the independent eigenvectors, a set of discrete eigenstates may emerge as consistent bootstrap solutions of the dynamical equations. Now, if the static model has any approximate validity at all, this set must contain more than one element when the crossing matrix has more than one eigenvector with positive eigenvalue. It is therefore very probable that bootstrap schemes are not unique even if relativistic effects are taken into account.

At this point we conjecture that the particles observed in nature correspond to a state of minimal bootstrap—i.e., a bootstrap scheme which involves a minimum number of particles. In other words, the eigenvector associated with a minimal bootstrap has large components in the least number of physically realizable states. We shall test this principle by applying it to the problem of pseudoscalar meson-baryon scattering in $SU(2)$ and $SU(3)$ symmetries, and show that the state of minimal bootstrap, in fact, has major components only in the usual irreducible representations which can accommodate the observed particles. It is assumed that the state of minimal bootstrap obtained by applying the principle directly to the eigenvectors of the crossing matrix is the same as the minimal scheme which the principle selects from the set of discrete bootstrap solutions of the complete dynamical problem where relativistic effects are fully taken into account. The eigenvector thus selected by the principle of minimal bootstrap also predicts the ratio of coupling strengths of particles in different representations and the mixing parameter of the Yukawa-type octet coupling (the D/F ratio), which agree well with the experimental numbers, thus lending greater credibility to the principle.

II. CROSSING MATRIX

We consider the scattering of two sets of particles, each corresponding to an irreducible representation of a simple Lie group. The direct product of these two irreducible representations can be reduced to a direct sum of irreducible representations, which are m in number, say. There are then at least m independent scattering amplitudes. If a representation of a certain dimension occurs r times, $r>1$, in the reduction of the direct product, then there are, in addition, $r(r-1)/2$ "reaction" amplitudes. Let the total number of independent amplitudes be n, so that we have T_α, $\alpha=1, \cdots, n$.

By crossing symmetry the amplitudes T_α in the direct process are related to the amplitudes in the crossed process, $T_\beta{}^{\text{cr}}$, $\beta=1, \cdots, n$, by

$$T_\alpha=\sum_\beta C_{\alpha\beta}T_\beta{}^{\text{cr}}, \tag{2.1}$$

where C is the crossing matrix, which can be determined in a variety of ways.[12,13] For certain processes there are crossed channels that are not spanned by the same number of independent amplitudes as are the direct channels; in such cases the crossing matrices are not square. We consider here only those crossed channels for which C is a square matrix.

The general properties of C are that its elements are real and that

$$C^2=1. \tag{2.2}$$

As we have set forth in the previous section, we define a bootstrap state to be an eigenvector of C corresponding to a positive eigenvalue. In general, a real matrix that is not symmetric cannot always be diagonalized. We show here that the condition (2.2) guarantees that C can always be diagonalized.

It is a theorem in matrix theory[14] that any real, square matrix A of dimension n can, by a similarity transformation, be brought to a block diagonal form

[12] A. O. Barut and B. C. Unal, Nuovo Cimento **28**, 112 (1963); A. O. Barut, Phys. Rev. **130**, 436 (1963).

[13] D. E. Neville, Phys. Rev. **132**, 844 (1963).

[14] See, for example, Turnbull and Aitken, *The Theory of Canonical Matrices* (Blackie and Son, London, 1932), p. 49.

$[A_1,A_2,\cdots,A_p]$, where A_i, $i=1,\cdots,p$, are the square matrices of dimensions d_i along the diagonal having the form

$$A_i=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_{d_i} & a_{d_i-1} & \cdot & \cdot & \cdot & a_1 \end{bmatrix}, \quad d_i \leq n. \qquad (2.3)$$

The dimension d_i is the degree of the reduced characteristic equation of the matrix A:

$$A^{d_i}-a_1A^{d_i-1}-a_2A^{d_i-2}-\cdots-a_{d_i}I=0. \qquad (2.4)$$

That is, for a given vector u_i, we can form a chain of linearly independent vectors: u_i, u_iA, u_iA^2, $\cdots$, $u_iA^{d_i-1}$. If $d_1<n$, other chain or chains may be formed. Since the crossing matrix C satisfies (2.2), no submatrix C_i in the block diagonal form can have a dimension greater than two; moreover, $a_1=0$, and $a_2=1$. Thus, if C_i is not a number, it is at most

$$C_i=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which can obviously be diagonalized. This completes the proof that C can always be diagonalized.

Again, on account of (2.2), the eigenvalues, λ_α, of C must be either $+1$ or -1. If $\mathrm{Tr}C=l$, which is necessarily an integer, then there are $(n+l)/2$ positive eigenvalues and $(n-l)/2$ negative eigenvalues.

III. MESON-BARYON SCATTERING IN $SU(2)$ AND $SU(3)$

As a simple example we consider first the p-wave pion-nucleon scattering in $SU(2)$. Bootstrap in the static model is then a simple eigenvalue problem which involves solving the equation

$$\gamma_\alpha=\sum_\beta C_{\alpha\beta}\gamma_\beta, \qquad (3.1)$$

where $C_{\alpha\beta}$ is the total crossing matrix given by the direct product of the isotopic spin and J-spin crossing matrices. Since pole approximation is consistent with the static-model approximation, the amplitudes T_α and T_β^{cr} in (2.1) are here replaced[9] by the residues γ_α and γ_β, which must therefore be positive. If the states characterizing the rows and columns are labeled by $(I,J)=(\frac{1}{2},\frac{1}{2})(\frac{1}{2},\frac{3}{2})$, $(\frac{3}{2},\frac{1}{2})$, and $(\frac{3}{2},\frac{3}{2})$, respectively, then the well-known crossing matrix is

$$C_{\alpha\beta}=\begin{bmatrix} 1/9 & -4/9 & -4/9 & 16/9 \\ -2/9 & -1/9 & 8/9 & 4/9 \\ -2/9 & 8/9 & -1/9 & 4/9 \\ 4/9 & 2/9 & 2/9 & 1/9 \end{bmatrix}. \qquad (3.2)$$

Since $\mathrm{Tr}C=0$, and since the eigenvalues can only be ±1 as is shown in the preceding section, there are two independent eigenvectors which have eigenvalues $+1$. The general eigenvector of arbitrary normalization may be represented by

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}+\lambda\begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \\ \frac{5}{2} \\ 1 \end{bmatrix}. \qquad (3.3)$$

We see from (3.3) that, if $\lambda=0$, the bootstrap vector has two components, namely, in the states $(\frac{1}{2},\frac{1}{2})$ and $(\frac{3}{2},\frac{3}{2})$. Although the general vector does not preclude possible interaction in the $(\frac{1}{2},\frac{3}{2})$ and $(\frac{3}{2},\frac{1}{2})$ states for nonzero values of λ, there is no value of λ for which any pair of states, other than $(\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$, is the only pair of states that has large components. That is, nonvanishing λ leads to at least three large components. Thus, by the principle of minimal bootstrap, the vector corresponding to $\lambda=0$ is preferred. We therefore obtain the result that the $(\frac{1}{2},\frac{1}{2})$ and $(\frac{3}{2},\frac{3}{2})$ states form the physical bootstrap vector in agreement with the original reciprocal bootstrap scheme of Chew[1] which involves only N and N^*. Note that for $\lambda\approx0$ the ratio of the residues γ_1 to γ_4 is about 2, which checks with the experimental result on the ratio of $NN\pi$ to $N^*N\pi$ couplings.

Consider now the scattering of pseudoscalar mesons and baryons in the octet model of $SU(3)$. In this case the isotopic-spin crossing matrix is replaced by the unitary-spin crossing matrix. Since we see from the decomposition of the direct product of two octets

$$8\times8=1+27+10^*+10+8_a+8_s \qquad (3.4)$$

that the eight-dimensional irreducible representations occur twice in the reduction, we have, in addition to the six "elastic" scattering amplitudes, a "reaction" amplitude T_Q: $8_a\rightleftharpoons8_s$. In an obvious notation we denote them, respectively, by T_1, T_{27}, T_{10^*}, T_{10}, T_A, T_S, and T_Q. The corresponding crossing matrix C_U has been given by Cutkosky[5]; we reproduce it in Table I.

Since the trace of C_U is $+1$, there are four eigenvectors with eigenvalue $+1$ and three with eigenvalue -1. The J-spin crossing matrix for scattering in the p wave is

$$C_J=\begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix},$$

which has eigenvalues $+1$ and -1. Thus, the total

TABLE I. Crossing matrix $C_{\alpha\beta}$ for octet-octet scattering in $SU(3)$.

$\alpha \backslash \beta$	1	27	10*	10	A	S	Q
1	1/8	27/8	−5/4	−5/4	−1	1	0
27	1/8	7/40	1/12	1/12	1/3	1/5	0
10*	−1/8	9/40	1/4	1/4	0	2/5	$-2/\sqrt{5}$
10	−1/8	9/40	1/4	1/4	0	2/5	$2/\sqrt{5}$
A	−1/8	9/8	0	0	1/2	−1/2	0
S	1/8	27/40	1/2	1/2	−1/2	−3/10	0
Q	0	0	$-\frac{1}{4}\sqrt{5}$	$\frac{1}{4}\sqrt{5}$	0	0	0

crossing matrix $C=C_U\times C_J$ has altogether seven independent eigenvectors with eigenvalue $+1$; they may be represented, apart from normalization, by the six-parameter vector:

$$\begin{bmatrix} x+5y+(2\sqrt{5})z-5 \\ [5x+9y+(2\sqrt{5})z-5]/9 \\ -4z/\sqrt{5}+1 \\ 1 \\ x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} (9r-5s)/2 \\ (-5r+s)/6 \\ r-s-t \\ t \\ r \\ s \\ (\sqrt{5})(r-s-2t)/4 \end{bmatrix} \times \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad (3.5)$$

where x, y, z, r, s, and t are independent real parameters. Let the states, in terms of which this vector is expressed, be denoted by the pair assignments (U,J), where $U=1$, 27, 10*, 10, A, S, Q, and $J=\frac{1}{2}, \frac{3}{2}$. Thus, for example, the component of this eigenvector in the state $(10,\frac{3}{2})$ is $1-t/2$.

To reduce the number of free parameters we observe that the amplitudes T_A, T_S, and T_Q, which correspond to the processes $8_a\rightleftharpoons 8_a$, $8_s\rightleftharpoons 8_s$, and $8_a\rightleftharpoons 8_s$, respectively, must satisfy the property of factorization of residues under the approximation of a single degenerate pole in all the three amplitudes. This implies the following constraints:

$$\begin{aligned} [z+(\sqrt{5})(r-s-2t)/4]^2 &= (x+r)(y+s)\,, \\ [z-(\sqrt{5})(r-s-2t)/8]^2 &= (x-r/2)(y-s/2)\,, \end{aligned} \quad (3.6)$$

for the $J=\frac{1}{2}$ and $\frac{3}{2}$ states, respectively. The 2×2 octet scattering matrix can be diagonalized, and because the determinant now vanishes, the matrix can be represented by one octet amplitude[5,9] T_8—i.e.,

$$\begin{pmatrix} T_S & T_Q \\ T_Q & T_A \end{pmatrix} \rightarrow \begin{pmatrix} T_8 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.7)$$

On account of this we need consider the vector (3.5) as having only five U components: $U=1$, 27, 10*, 10, and 8, where $T_8=T_S+T_A$. The constraints (3.6) reduce the number of independent parameters from six to four. We now apply the principle of minimal bootstrap to fix these remaining parameters.

Before proceeding further we remark that the S-matrix theory requires that the amplitude for scattering in a definite state in which a particle is known to exist must possess a pole at the mass of that particle. Thus for pseudoscalar meson-baryon scattering in the octet model, the $J=\frac{1}{2}$ octet state must contain the baryons. We assume that the system of particles under consideration is broad enough so that the baryons can be bootstrapped within the dynamical framework which involve only the baryons and pseudoscalar mesons. This assumption amounts to saying that the physical bootstrap vector must have a large (and positive) component in the baryon state.

A search for the minimal bootstrap state can, in principle, proceed as follows. Requiring the component in the $(8, \frac{1}{2})$ state to be large, one can first try to minimize the sum of the squares of all the other components. However, we see from (3.5) that the component in the $(10,\frac{1}{2})$ state is $1+t$, while the component in the $(10,\frac{3}{2})$ state is $1-t/2$. Clearly, there is no value of t for which both components can be small. It can be shown that the root-least-squares of $1+t$ and $1-t/2$ is comparable to the component in the $(8,\frac{1}{2})$ state. Thus, we expect that the minimal bootstrap state has at least two large components. One of these large components is, of course, required to be in the $(8,\frac{1}{2})$ state. For the other component to be in any given state, one can calculate the least square of the remaining eight components. Change the second large component to a different state and repeat the calculation. The optimum choice of the two large components clearly corresponds to the minimum least square of the eight small components.

For the case in which two of the components are large [one of them being in the $(8,\frac{1}{2})$ state] we find that the pair $(8,\frac{1}{2})$, $(10,\frac{3}{2})$ satisfies our requirements of minimal bootstrap far better than other allowed pairs. The search for the optimal pair is a straightforward but tedious calculation, which we simplify by requiring that the $(8,\frac{3}{2})$ component be exactly zero, thus eliminating two parameters. [The pair with $(8,\frac{1}{2})$, $(8,\frac{3}{2})$ components being large is eliminated by the same arguments which eliminated the case of component $(8,\frac{1}{2})$ alone being large.] The minimization of the sum of the squares of the small components yields, for the case in which $(8,\frac{1}{2})$ and $(10,\frac{3}{2})$ components are large, the following vectors for the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ states:

$$V_{1/2}=\begin{bmatrix} -0.1 \\ 0.0 \\ 0.7 \\ 0.0 \\ 2.6 \end{bmatrix}, \quad V_{3/2}=\begin{bmatrix} 0.0 \\ 0.5 \\ 0.0 \\ 1.5 \\ 0.0 \end{bmatrix}. \quad (3.8)$$

The sum of the squares of the small components for this solution is about 0.8. Compared to this, the least squares for the other allowed pairs of states are found to be all greater than 2.

Since we have not done a thorough least-square analysis, we do not claim the solution obtained to be the best one. However, it certainly contains the essential properties of the minimal bootstrap vector. Evidently, the two large components are in the $(8,\frac{1}{2})$ and $(10,\frac{3}{2})$ states, which are just the states that accommodate the observed baryons and baryon resonances. This is the generalization to $SU(3)$ of the nucleon and (3,3) resonance states forming the minimal bootstrap vector in $SU(2)$.

We do not suggest that the components in the $(10^*,\frac{1}{2})$ and $(27,\frac{3}{2})$ states, being only three or four times smaller than the largest components in the respective J-spin states, should be entirely ignored. Perhaps, they may indicate that some moderately significant interaction in these states is not ruled out. Their importance can be assessed only after a more complete dynamical calculation is made.

From the bootstrap vectors (3.8) we can also deduce the following consequences which are of significance. We note that the ratio of the components in the $(8,\frac{1}{2})$ to $(10,\frac{3}{2})$ states is 2.6:1.5. With the proper Clebsch-Gordan coefficients taken into account, this implies the following relationship between the coupling constants:

$$g_{NN\pi}^2 \approx 2.4 g_{N^*N\pi}^2, \tag{3.9}$$

which is in good agreement with the experimental situation.

We can also obtain from the octet components of $V_{1/2}$ the mixing parameter of the Yukawa-type octet coupling. If $(1-f)/f$ is the D/F ratio, then the three octet components are related to one another through the coupling coefficients as[8,15,16]

$$\begin{pmatrix} T_S & T_Q \\ T_Q & T_A \end{pmatrix} \propto \begin{pmatrix} (20/3)(1-f)^2 & (4\sqrt{5})f(1-f) \\ (4\sqrt{5})f(1-f) & 12f^2 \end{pmatrix}. \tag{3.10}$$

Corresponding to the minimal bootstrap vector $V_{1/2}$ in (3.8), the values of the components in the $U=A$, S, and Q states are 0.92, 1.69, and 1.24, respectively. Thus we find that

$$D/F = 1.83$$

or

$$f = 0.35\,,$$

which is again in agreement with the results of other analyses.[5,8]

In conclusion we make the following remarks. By defining a bootstrap state to be the eigenvector of the total crossing matrix, we have found that bootstrap schemes are not unique. In problems where static models are reasonable, this nonuniqueness is likely to persist even when relativistic effects are fully considered. We have proposed a principle of minimal bootstrap to select the physical solution. By studying meson-baryon scattering in $SU(2)$ and $SU(3)$ symmetries, we have found ample support for the validity of this principle.

Without some elaborate calculations, it is difficult to assess the credibility of this principle in problems where relativistic effects are important. This is because the crossing of angular-momentum states makes the use of only the isotopic- or unitary-spin crossing matrices unreliable for the estimation of forces in the various channels. However, if we nonetheless apply the considerations of this paper to, say, the $\pi\pi$ problem, we find that there are two possible minimal eigenvectors: $(1,\frac{2}{3},0)$ and $(1,0,\frac{2}{5})$, where the states are labeled by $I=0$, 1, 2, respectively. An extra constraint, namely, the existence of ρ in the $I=1$ state, is needed to select the first eigenvector. This constraint may be provided by a consideration of the $\pi\rho$ scattering problem where π is to appear as a bound state. The implication is then that a unique minimal bootstrap state may be obtained if one regards the mesons as having a higher symmetry where the existence of *both* π and ρ are to be explained by self-consistency. We can learn qualitatively from the eigenvector $(1,\frac{2}{3},0)$ for $\pi\pi$ scattering in the $SU(2)$ subspace that no particle is to be expected in the $I=2$ state and that even J-spin particles, such as f_0, must exist in the $I=0$ state. These properties seem to be borne out by experiments.

Note added in proof. It has been brought to our attention that Babu[17] has also diagonalized the crossing matrix for meson-baryon scattering in G_2 symmetry. An attractive bootstrap vector, i.e., eigenvalue $+1$, of course, exists also in that case as in any other symmetry. In this sense internal symmetry cannot be uniquely determined by merely requiring the existence of an attractive bootstrap state. However, minimal bootstrap may offer a unique selection. Babu finds that in G_2 a bootstrap state exists with baryons in the singlet and the 7-dimensional representation and with baryon resonances in the singlet and 15-dimensional representation. Clearly, minimal bootstrap prefers $SU(3)$ with only two multiplets to G_2 which requires four multiplets.

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[15] P. Tarjanne, Ann. Acad. Sci. Fennicae, Ser. A, VI, No. 105 (1962).

[16] M. Hosoda and Y. Miyamoto, Progr. Theoret. Phys. (Kyoto) 32, 263 (1964).

[17] P. Babu, Nuovo Cimento 34, 770 (1964).