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Practical Approximation to the Faddeev Equations*

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The form of the equations given by Faddeev for the problem of three-particle scattering is analyzed in the case in which the amplitudes of the two-body subsystems are dominated by a finite number of pole terms. It is shown that an important simplification can be made, reducing the Faddeev equations to a system of coupled integral equations in one variable only.

I. INTRODUCTION

THE numerous achievements of the ideas of Faddeev,[1] on the proper mathematical formulation of the scattering of three particles in terms of two-body interactions, have been pointed out by several authors.[2–5]

They are mainly due to the fact that all the two-body subsystems are taken into account exactly, so that the integral equations given by Faddeev involve no two-body potential at all, but only the actual exact solution of each two-body subsystem. Thus, the three-body problem appears to be formulated in such a way that, as long as one knows the exact two-body scattering amplitude off the energy shell, one should be able to derive all the properties of three-particle states.

In the domain of strong interactions, where the Faddeev equations will presumably receive much attention, one is faced with a quite hopeful situation. In fact, it is well known that in that domain one has much greater information about the properties of the scattering amplitude itself (on the energy shell) than about the original potentials which give rise to it; moreover, it has often proved quite satisfactory to assume that a two-body scattering amplitude is dominated by a certain number of poles that correspond to bound states and resonances.

Furthermore, the properties of the off-shell two-body amplitude have been studied in great detail by Lovelace.[2,3] He has shown, in particular, that in the neighborhood of a pole, the scattering amplitude factorizes in the initial and final off-shell momenta, and thus can be written in the form

$$\langle p'|T_l(s)|p\rangle \equiv T_l(p,p';s) \simeq g(p)t_l(s)g(p'), \qquad (1)$$

where p and p' are the off-shell initial and final momenta, s is the total energy, and l indicates the partial wave in which we find the pole; $g(p)$ and $g(p')$ are called the resonance or bound-state "form factors," and $t_l(s)$ the "propagator." In the case of a bound state, Eq. (1) is well known; the function $g(p)$ is related to the bound-state wave function $\psi_B(p)$ by

$$g(p) = -(p^2+E_B)\psi_B(p),$$

and $t_l(s)$ can, for instance, be given the simple form

$$t_l(s) = (s+E_B)^{-1},$$

where E_B is the energy of the bound state. The "form factor" $g(p)$ can still be defined for a resonance,[3] and the various forms one can give to $t_l(s)$ are discussed in great detail by Lovelace in Ref. 3.

It is on these grounds that Lovelace[3] was able to show an important simplification of the Faddeev equation. Assuming that the influence of regions far from the poles is not too great, so that one can give the amplitude any arbitrary form as long as it reproduces the known one near the pole, Lovelace noticed that a separable two-body potential gives satisfactory behavior of the amplitude in the vicinity of the pole, provided that it is chosen to give the two-particle bound-state wave function correctly. From this standpoint, Lovelace calculated the two-body scattering amplitude, shows that one can define some kind of "potentials" corresponding to the scattering of a bound state or a resonance by an elementary particle, and then derived from the Faddeev equations two-body Lippmann-Schwinger equations involving these so-called potentials.

In this paper, we want to show that the step of the separable potential is perhaps unnecessary, and that Eq. (1) can be directly inserted in the Faddeev equations without making any assumption regarding the propagator $t_l(s)$, and that this leads to considerable simplifications. We will give a particular example in Sec. II, while the general case will be dealt with in Sec. III.

Our assumptions are quite simple: We suppose that each two-body amplitude can be approximated by a finite number of pole terms, and that the contribution of a pole to the off-shell two-body amplitude is factorizable in the initial and final momenta, for all energy values. We will consider that Eq. (1) is the exact expression of $T(p,p';s)$, valid for all energies.

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; and Dokl. Akad. Nauk. SSSR **138**, 565 (1961); **145**, 301 (1962) [English transls.: Soviet Phys.—Doklady **6**, 384 (1961); **7**, 600 (1963)].

[2] C. Lovelace, Lectures at Edinburgh Summer School, 1963 (unpublished).

[3] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[4] R. Omnes, Phys. Rev. **134**, B1358 (1964).

[5] S. Weinberg, Phys. Rev. **133**, B232 (1964).

II. A SIMPLE CASE: $J=0$

For the sake of clarity, we will first show our result in a very simple example. Following the notations of Ref. 4, where the total angular momentum J and its projection on a body-fixed axis M are chosen as quantum numbers, we will suppose that the total angular momentum is $J=0$, so that we can suppress all indices but one in Eq. (44) of Ref. 4. The Faddeev equations thus are written, in the kernel notation

$$T^i(\omega',\omega)=\mathcal{T}^i(\omega',\omega)-\int K^i(\omega',\omega'')[T^j(\omega'',\omega)+T^k(\omega'',\omega)]d\omega'', \quad (2)$$

where ω represents the whole set $(\omega_1,\omega_2,\omega_3)$ and $d\omega \equiv d\omega_1 d\omega_2 d\omega_3$, ω_i being the energy of particle i, in the total center-of-mass system. Furthermore, we approximate each two-body amplitude by a single pole term, so that[4,6]

$$K^i(\omega',\omega'')=(m_1m_2m_3)(m_ip_i')^{-1}\delta(\omega_i'-\omega_i'')(\sum \omega_j''-z)^{-1} \times f_i^l(\omega',\omega'',z)Y_{l,0}^*(\gamma_i',0)Y_{l,0}(\gamma_i'',0), \quad (3)$$

where m_i is the mass of particle i and p_i' its momentum, z is the total energy of the three-particle system, the functions $Y_{l,m}$ are the spherical harmonics, l is the spin of the composite system of particles (j) and (k), each of which is assumed to be spinless, γ_i', defined in Ref. 4, is a function of ω' and γ_i'' is a function of ω''. The two-body amplitude is written, according to formula (1),

$$f_i^l(\omega',\omega'',z)=g(p_{jk}'^2)g(p_{jk}''^2)t(z-\omega_i'), \quad (4)$$

where p_{jk} is the relative momentum of particles j and k in their relative c.m. system, and is related to the momenta of these particles in the total c.m. system by

$$p_{jk}=(m_kp_j-m_jp_k)(m_j+m_k)^{-1},$$

for, as the angular momentum has been separated, the form factors depend only on the absolute values of the momenta, and we have replaced s by its value in terms of the total energy z and ω_i'. We can thus write

$$f_i^l(\omega',\omega'',z)=a_i(\omega',z)b_i(\omega''). \quad (5)$$

The kernel defined by Eq. (3), then, can be written in a simple form, omitting the variable z, which has no importance in this matter,

$$K^i(\omega',\omega'')=\delta(\omega_i'-\omega_i'')\phi_i(\omega'')A_i(\omega'); \quad (6)$$

and the system of integral equations in three variables then seems to be quite simple, as the kernel is separable except for a part which, in fact, will give rise to a convolution product in one variable. This convolution, furthermore, is very simple, as it involves a Dirac distribution. We are now able to write the Faddeev equations as

$$T^i(\omega',\omega)=\mathcal{T}^i(\omega',\omega)-A_i(\omega')\int \phi_i(\omega'')\delta(\omega_i'-\omega_i'') \times[T^j(\omega'',\omega)+T^k(\omega'',\omega)]d\omega'', \quad (7)$$

and the solution can be written quite naturally:

$$T^i(\omega',\omega)=\mathcal{T}^i(\omega',\omega)-A_i(\omega')B_i(\omega_i';\omega). \quad (8)$$

Now, inserting (8) into (7), and changing the names of the variables in a very obvious way, we obtain a new set of integral equations, involving the functions $B_i(x,\omega)$:

$$B_i(x,\omega)=\beta_i(x,\omega)-\int K_i{}^j(x,x')B_j(x',\omega)dx', \quad (9)$$

where we have introduced the functions

$$\beta_i(x,\omega)=\int \phi_i(\omega'')\delta(x-\omega_i'') \times[\mathcal{T}^j(\omega'',\omega)+\mathcal{T}^k(\omega'',\omega)]d\omega'' \quad (10)$$

[since the inhomogeneous terms $\mathcal{T}^i$ are known, $\beta_i(x,\omega)$ is a perfectly well-known function], and

$$K_i{}^j(x,x')=(1-\delta_{ij}) \times\int \phi_i(\omega)A_j(\omega)\delta(x-\omega_i)\delta(x'-\omega_j)d\omega. \quad (11)$$

In Eq. (9) ω has the importance of an index, and we can write that equation in the symbolic form

$$\begin{bmatrix} B_1(x,\omega) \\ B_2(x,\omega) \\ B_3(x,\omega) \end{bmatrix} = \begin{bmatrix} \beta_1(x,\omega) \\ \beta_1(x,\omega) \\ \beta_3(x,\omega) \end{bmatrix} - \begin{bmatrix} 0 & K_{12} & K_{13} \\ K_{21} & 0 & K_{23} \\ K_{31} & K_{32} & 0 \end{bmatrix} \begin{bmatrix} B_1(x,\omega) \\ B_2(x,\omega) \\ B_3(x,\omega) \end{bmatrix}. \quad (12)$$

It will be shown in Sec. III that the results of the very simple case considered here ($J=0$ and only one pole term in each two-body system) can perfectly well be extended to the general case in which the two-body amplitudes are approximated by the sum of a finite number of pole terms, whatever the angular momentum may be. The only change is, in fact, an increase of the dimensionality of the K_{ij} matrix considered above, as one increases the number of input pole terms and the angular momentum. On the other hand, it is obvious that all the reductions coming from the separation of parity and the identity of particles are applicable to these equations as well as to the original Faddeev equations.

The form of the equations we have obtained is quite analogous to that of the original Faddeev equations; what must be pointed out as extremely important from a practical point of view is that we have now a problem involving a system of coupled integral equations in one variable only. This means, in particular, that the use of

[6] Akbar Ahmadzadeh and Roland Omnes, Phys. Rev. (to be published).

a computer is now much easier and will lead to reliable numerical results. One can easily imagine the enormous difference between solving an integral equation in three variables and solving one in one variable only.

Our result is more general than that of Lovelace, who also obtains equations in one variable, for two reasons:

(i) We have made no assumption on the form of the propagator $t_l(s)$, while he has taken that given by a separable potential.

(ii) Our result (see Sec. III) is valid even when there is more than one pole in a given partial wave, whereas this cannot be taken into account by Lovelace's method.[3]

III. GENERAL CASE

We will derive our equations directly from the equations of Ref. 4, where angular momentum has been separated.

Let us suppress the index J, and make some slight modifications in the notations; the equations then appear to be

$$T_{M'M}{}^{i}(\omega',\omega)=\mathcal{T}_{M'M}{}^{i}(\omega',\omega)-\int K_{M'M''}{}^{i}(\omega',\omega'')\times[T_{M''M}{}^{j}(\omega'',\omega)+T_{M''M}{}^{k}(\omega'',\omega)]d\omega'', \quad (13)$$

where

$$K_{M'M''}{}^{i}(\omega',\omega'')=(m_1m_2m_3)(m_ip_i')^{-1}\delta(\omega_i'-\omega_i'')[\sum\omega_j''-z]^{-1}\times\sum_M\int F_{jk}(\omega',\omega'',z-\omega_i',u)\times d_{M'M}{}^{J}(-\alpha_i')e^{iMu}d_{MM''}{}^{J}(\alpha_i'')du. \quad (14)$$

Following Ref. 6, we can make a partial-wave expansion, and write

$$F_{jk}(\omega',\omega'',z-\omega_i',u)=\sum_l f_{jk}{}^{(l)}(\omega',\omega'',z-\omega_i')(2l+1)\times P_l(\cos\gamma'\cos\gamma''+\sin\gamma'\sin\gamma''\cos u). \quad (15)$$

Choosing, as in Ref. 6, the z axis to be perpendicular to the plane of the momenta, and integrating over u, we obtain

$$K_{M'M''}{}^{i}(\omega',\omega'')=(m_1m_2m_3)(m_ip_i)^{-1}\delta(\omega_i'-\omega_i'')(\sum\omega_j''-z)^{-1}\times\sum_l f_{jk}{}^{(l)}(\omega',\omega'',z-\omega_i')X_{M'M''}{}^{(l)}, \quad (16)$$

where (Ref. 6)

$$X_{M'M''}{}^{(l)}=\sum_\mu(-1)^\mu Y_{l\mu}{}^{*}(\gamma_i',0)Y_{l\mu}(\gamma_i'',0)\Delta_{M''\mu}{}^{*J}\Delta_{M'\mu}{}^{J}. \quad (17)$$

(The $\Delta_{M''M'}{}^{J}$ are defined in Ref. 7.) Now we assume that each partial-wave amplitude is dominated by a certain number of pole terms, characterized by an index of degeneracy s, so that, following Eq. (5), we can write

$$f_{jk}{}^{(l)}=\sum_s a_{jk}{}^{l,s}(\omega',z)b_{jk}{}^{l,s}(\omega''). \quad (18)$$

If we now make the assumption that only a finite number of pole terms will actually contribute significantly to the two-body amplitude in the energy range we are considering, the kernel of the Faddeev equations becomes, upon inserting (18) into (16),

$$K_{M'M''}{}^{i}(\omega',\omega'')=\delta(\omega_i'-\omega_i'')\sum_{l=0}^{n}\sum_{s=s_0}^{s=s_l}\phi_i{}^{l,s}(\omega'')\times A_i{}^{l,s}(\omega',z)X_{M'M''}(\omega',\omega''), \quad (19)$$

where we have transformed the pair index (j,k) into the single (i). From (17) and (19), we see that Eq. (13) now becomes

$$T_{M'M}{}^{i}(\omega',\omega)=\mathcal{T}_{M'M}{}^{i}(\omega',\omega)-\sum_{l,s}A_i{}^{l,s}(\omega',z)\sum_\mu Y_{l\mu}{}^{*}(\gamma_i',0)\Delta_{M'\mu}{}^{J}\times\int\phi_i{}^{ls}(\omega'')Y_{l\mu}(\gamma_i'',0)\Delta_{M''\mu}{}^{J*}\delta(\omega_i'-\omega_i'')\times[T_{M''M}{}^{j}(\omega'',\omega)+T_{M''M}{}^{k}(\omega'',\omega)]d\omega'', \quad (20)$$

and, in exactly the same way as in Eq. (8), the solution is

$$T_{M'M}{}^{i}(\omega',\omega)=\mathcal{T}_{M'M}{}^{i}(\omega',\omega)-\sum_{l,s}A_i{}^{l,s}(\omega',z)\times\sum_\mu Y_{l\mu}{}^{*}(\gamma_i',0)\Delta_{M'\mu}{}^{J}B_{(l,s)\mu M}{}^{i}(\omega_i',\omega), \quad (21)$$

where we insist on the fact that $B_{(l,s)\mu M}(\omega_i',\omega)$ depends only on one variable: ω_i', besides ω, which can here be considered as a simple index without any practical influence. We now insert (21) into (20) and, indentifying to zero the coefficients of the functions $A_i{}^{l,s}(\omega',z)Y_{l\mu}{}^{*}(\gamma_i',0)$, which are independent functions of three variables, we obtain the equations

$$B_{(l,s)\mu M}{}^{i}(x,\omega)=\beta_{(l,s)\mu M}{}^{i}(x,\omega)-\int\Gamma_{j(l,s)\mu}{}^{i(\lambda,\sigma)\nu}(x,y)B_{(\lambda,\sigma)\nu M}{}^{j}(y,\omega)dy, \quad (22)$$

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[7] For the definition of the rotation matrices $\Delta_{MM'}{}^{J}=d_{MM'}{}^{J}(\pi/2)$, we follow A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), Chap. 4.

where the definition of $\beta_{(l,s)\mu M}{}^{i}(x,\omega)$ is quite obvious and analogous to Eq. (10), and where the matrix kernel

$$\Gamma_{j(l,s)\mu}{}^{i(\lambda,\sigma)\nu}(x,y)$$

is defined by

$$\Gamma_{j(l,s)}{}^{i(\lambda,\sigma)\nu}(x,y) = (1-\delta_{ij}) \sum_{M''} \int \phi_i{}^{l,s}(\omega'')\Delta_{M''\mu}{}^{J*}Y_{l\mu}(\gamma_i'',0) \times\delta(x-\omega_i'')A_j{}^{\lambda,\sigma}(\omega'',z) \times Y_{\lambda\nu}{}^*(\gamma_j'',0)\Delta_{M''\nu}{}^{J}\delta(y-\omega_j'')d\omega''. \quad (23)$$

Equation (22) is closely analogous to (12), except that the dimensionality of the matrix is greater. In practical cases, one must say that these equations are much simpler than what they seem to be here, for the number of pole terms in each two-body channel will not be very large.

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New Measurement of the $\bar{K}^0$-K^- Mass Difference*

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The $\bar{K}^0$-K^- mass difference has been measured to be 3.90±0.25 MeV. The method employed involves the observation of nine examples of the reaction $K^-+p\rightarrow\bar{K}^0+n$, $\bar{K}^0\rightarrow\pi^++\pi^-$ in association with the proton recoil from the scattered neutron. The kinematic fitting of this event in the bubble chamber is very sensitive to the $\bar{K}^0$-K^- mass difference. Sources of systematic errors are also discussed.

I. INTRODUCTION AND EXPERIMENTAL METHOD

We have measured the $\bar{K}^0$-K^- mass difference using $\bar{K}^0$ produced by low-energy K^- interactions in the Saclay 81-cm hydrogen bubble chamber[1] at CERN.[2]

The reactions which we observed were

$$K^-+p\rightarrow\bar{K}^0+n, \quad \bar{K}^0\searrow\pi^++\pi^-, \quad (1)$$

$$n+p\rightarrow n+p. \quad (2)$$

We rescanned a sample of about 100 events which had been measured and fit the hypothesis of reaction (1). We searched for those events where there was, in addition, an example of reaction (2), an (n,p) scatter with a proton recoil, in the same photograph. We only considered events as candidates if the proton recoil was less than 7 cm away and if the proton recoil appeared to conserve momentum in reaction (1). Thirteen candidates were found. These events were remeasured to include the neutron direction and the recoil proton. The kinematic fitting program,[3] using the $\bar{K}^0$ momentum[4] as deduced from its decay and the neutron momentum as deduced from its recoil, tested the hypothesis of reaction (1). In this attempted kinematic fit all the vector momenta are known and therefore the results are very sensitive to the $\bar{K}^0$-K^- mass difference, provided, of course, that the recoil is truly associated. Our kinematic fitting program did not allow the $\bar{K}^0$ mass to be varied as an undetermined parameter so the following procedure was adopted. For each event, the kinematic fits were attempted in steps of 0.1 MeV over the region of $\bar{K}^0$ mass 491.0 to 504.0 MeV. The best value of the $\bar{K}^0$-K^- mass difference for each event was taken as that value of the mass difference for which the χ^2 of the fit was a minimum.[5] Accidental recoils and poor measurements were rejected by requiring that the goodness of this fit at the minimum correspond to a confidence level of greater than 1%. An error was assigned to each determination of the mass difference. This error was obtained by noting that the mass was being used as an independent degree of freedom and $(\chi^2)_{\min}+1$ corresponds to a change of a standard deviation in this one degree of freedom.

II. RESULTS

Table I lists data of the nine events which satisfy our criteria. The data listed are the average values obtained

* Supported in part by the U. S. Atomic Energy Commission.
[1] P. Baillon, thesis, University of Paris, 1963 (unpublished).
[2] A description of the beam is given by B. Aubert, H. Courant, H. Filthuth, A. Segar, and W. Willis, in *Proceedings of the International Conference on Instrumentation for High Energy Physics at CERN* (North-Holland Publishing Company, Amsterdam, 1963).
[3] For a description of the program, KICK, see Reference Manual for Kick IBM Program, edited by A. H. Rosenfeld, University of California Radiation Laboratory Report No. UCRL 9099 (unpublished); and A. H. Rosenfeld and J. N. Snyder, Rev. Sci. Instr. 33, 181 (1962).
[4] To the accuracy needed we can ignore the variation of the fitted $\bar{K}^0$ momentum with the assumed $K_1{}^0$ mass.
[5] The kinematic fitting program, KICK, uses linear constraints. We assume that this approximation does not systematically shift the position of $(\chi^2)_{\min}$.