# Three-Body Correlations in Nuclear Matter*

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The energy of nuclear matter which is due to the simultaneous interaction of three nucleons is calculated to all orders of Goldstone perturbation theory. The various orders form a divergent series with alternating signs. This series is summed by an integral equation due to Faddeev, and an approximate solution of this equation is found. According to this solution, the effect of the repulsive core on the potential energy of "particle" states is reduced to about one-third as compared with the usual third-order calculation, while the long-range attraction is essentially unchanged. The total potential energy of the important excited particle states is thereby reduced essentially to zero. Similarly, the total effect of three-body correlations on the energy of nuclear matter is very small, of the order of 3 to 6% of that of the two-body correlations. Thus the Goldstone series, if rearranged into a series in the number of interacting particles (which also corresponds roughly to powers of the density), converges very rapidly. The energy of nuclear matter is reduced substantially by our theory. When the theory is combined with Wong's idea of a "soft" repulsive core, the binding energy becomes roughly 13 MeV per particle, in much better agreement with observation than other recent estimates.

## 1. INTRODUCTION

TWO independent calculations of the binding energy of nuclear matter have been made with realistic nuclear forces, by Brueckner and Masterson[1] and by Razavy.[2] Both of these give too little binding energy, about 8 MeV per particle instead of the experimental value[3] of about 16 MeV. It is true that this discrepancy should in fairness be compared with the average potential energy of about 40 MeV and is then only about 20%. But most of the mathematics in the treatment of the many-body problem should be much more accurate, probably better than 1 MeV. Reasons for the discrepancy must therefore be found.

Three types of suggestions have been made. One possibility of great interest is that the two-body forces between nucleons should be changed in such a way as to preserve the agreement with nucleon-nucleon scattering but increase the binding energy of nuclear matter. In particular, Wong[4] has suggested that the short-range interaction should not be a completely hard core but a somewhat "softer" repulsion like $e^{-\mu r}/r$. This is very reasonable on theoretical grounds, because the repulsion may be due to the exchange of a heavy meson,[5] particularly the $\omega$, and $1/\mu$ is then the Compton wavelength of this meson, i.e., about 0.2 F. This modification leads to an increase in binding energy by 4–7 MeV (see Sec. 8 of this paper). However, Brown, Schappert, and Wong[6] have shown that a correction in the opposite direction, by 2–8 MeV, should be applied to the calculations of Brueckner and Masterson[1] and of Razavy,[2] so that a discrepancy of 5–10 MeV remains. Other modifications of the two-body potential are, of course, possible.

The second possibility is that the use of a potential is not permissible. Nucleon forces are derived from experiments on the energy shell, while in nuclear matter we are far off the energy shell. This extrapolation is commonly made by assuming that the forces are given by a potential. This may be incorrect, and this indeed might be the most interesting conclusion from the above-mentioned discrepancy, if it is real. A related possibility is true three-body forces.[7]

Before drawing these far-reaching conclusions, however, the third possibility should be investigated, viz., whether the mathematical treatment of nuclear matter is reliable. Some improved calculations have, in fact, been done, while others are under way: Dahlblom[8] has calculated the "ring diagram" (cf. Fig. 7) of three successive tensor interactions, Irwin[9] is investigating the Pauli and spectral corrections for tensor forces, while Sprung[10] has calculated the energy gap at the Fermi momentum. In addition, however, Rajaraman[11] has pointed out that there are many higher order diagrams (fourth order and higher) which do not necessarily give small results because they involve only three hole lines and are therefore proportional to the same power of the density as the third-order diagrams, which are known to be very important.

It is the purpose of this paper to give a numerical estimate of these diagrams. It will turn out that they are attractive and indeed very important. Our estimate is that they approximately cancel the correction of Brown, Schappert, and Wong[6] and bring the energy back to the value of Brueckner and Masterson.[1] To-

* Supported in part by the U. S. Office of Naval Research.
[1] K. A. Brueckner and K. S. Masterson, Phys. Rev. **128**, 2267 (1962). Quoted as BM.
[2] M. Razavy, Phys. Rev. **130**, 1091 (1963).
[3] P. A. Seeger, Nucl. Phys. **25**, 1 (1961).
[4] C. W. Wong, Nucl. Phys. (to be published).
[5] A. Scotti and D. Y. Wong, Phys. Rev. Letters **10**, 142 (1963); R. A. Bryan, C. R. Dismukes, and W. Ramsay, Nucl. Phys. **45**, 353 (1963).
[6] G. E. Brown, G. T. Schappert, and C. W. Wong, Nucl. Phys. (to be published).
[7] Bjorken has investigated some of the more important meson-theoretic three- and four-body forces. When this paper was written, the net effect of these forces was zero.
[8] T. Dahlblom, Nucl. Phys. (to be published).
[9] J. Irwin (private communication).
[10] D. Sprung, Ann. Phys. (N. Y.) **31**, 342 (1965).
[11] R. Rajaraman, Phys. Rev. **131**, 1244 (1963).

gether with Wong's[4] "soft" core, the binding energy comes out to be about 12 MeV, in not too bad agreement with experiment (Sec. 8).

The importance of the higher order diagrams arises from the fact that the nuclear binding energy, as is well known, is not simply a power series in the interaction potential, because such a series would not converge for the strong, short-range forces. Instead of this, as was first pointed out by Hugenholtz,[12] the effect of the short-range forces may be considered as a power series in the density. If $c$ is the radius of the repulsive core, and $r_0A^{1/3}$ is the nuclear radius, an appropriate expansion parameter is

$$\kappa=(c/r_0)^3. \tag{1.1}$$

For instance, according to Bethe *et al.*,[13] the effective mass in the limit of high momenta is[14] $m^*=1-2\kappa$. For $r_0$, the best available value[15] is 1.12 F; for $c$, a reasonable value is 0.4 F, although the modern calculations of Breit *et al.*[16] and of Hamada and Johnston[17] favor a somewhat larger value, $c=0.51$ and 0.49 F, respectively. Using $c=0.49$, we get $\kappa=0.08$ which is satisfactorily small.

The leading terms in the Goldstone expansion (first-order diagrams) contain two hole lines (actually two bubbles) and therefore give a total energy of the nucleus proportional to $\rho^2$, or an energy per particle proportional to $\rho$. The next order, viz., the third, involves three hole lines and thus gives an energy contribution per particle proportional to $\rho^2$. These diagrams have been amply discussed.[13,14] In fourth order, there are two classes of diagrams, some involving three hole lines, as in Fig. 1, and others involving four[18,19] as in Fig. 2. Every hole line implies integration over the volume of the Fermi sea, hence a factor $k_F^3$ which is proportional to $\rho$. We are only concerned with the diagrams involving three hole lines, Fig. 1; those involving four will be of relative order $\kappa$. There is also a diagram in third order involving four hole lines, Fig. 3. There are, however, diagrams with three hole lines in all higher orders as well, since the three particles created from the original three holes can interact any number of times, as, for example, in Fig. 4, the only condition being that the same pair must not interact twice in immediate succession. We should therefore calculate not only the fourth-order diagram, Fig. 1, but also the sum of all higher orders.

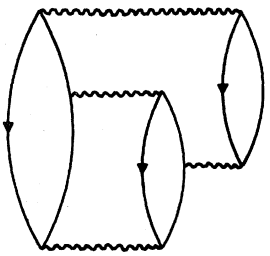

FIG. 1. A fourth-order diagram with three hole lines.

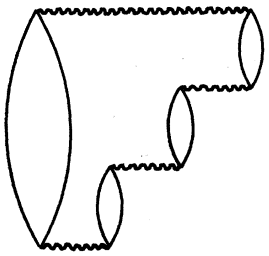

FIG. 2. A fourth-order diagram with four hole lines.

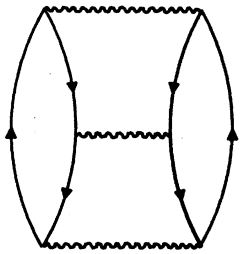

FIG. 3. A third-order diagram with four hole lines.

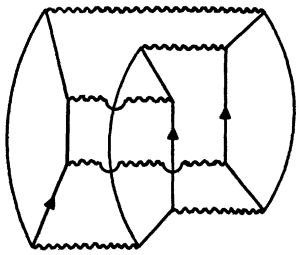

FIG. 4. A higher (sixth-order) diagram with three hole lines.

## 2. FOURTH-ORDER CALCULATION

We shall generally assume that all hole states have zero momentum. This is a good approximation, because only the short-range forces are strong enough to give an appreciable contribution in these high orders; a typical length for these forces is the core radius $c$; therefore our assumption amounts to $k_{Fc}\ll 1$. In fact, $k_{Fc}=0.54$, if $c=0.4$ F, but the corrections in third-order diagrams for finite momentum of the hole states are typically[14] of the order $0.2(k_{Fc})^2$ and thus only a few percent.

[12] N. M. Hugenholtz, Physica **23**, 533 (1957).
[13] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963); quoted hereinafter as BBP.
[14] R. Rajaraman, Phys. Rev. **129**, 265 (1963) shows that exchange effects change this result to $1-\kappa$. We shall find in Sec. 7 a value still closer to 1, but $1-m^*$ remains proportional to $\kappa$.
[15] B. H. Brandow, Ph.D. thesis, Cornell University, 1963 (to be published).
[16] K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).
[17] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 383 (1962). Quoted as HJ.
[18] Note that whenever a hole interacts, a new hole line is counted as starting. See J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).
[19] There is also a third-order diagram involving four hole lines, viz., the so-called hole-hole interaction, Fig. 3. This diagram gives an energy per particle proportional to $\rho^2$, because of the correlation of the momenta involved. However, the coefficient is very small, cf. Ref. 14.

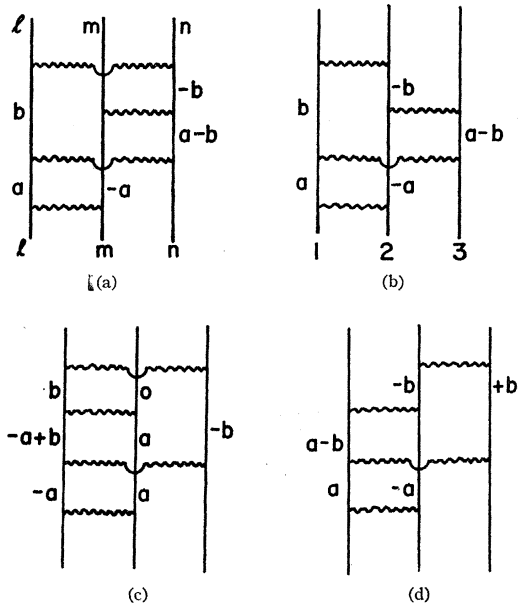


FIG. 5. Fourth-order diagrams with three interacting nucleons. The hole lines have been broken open. The four diagrams (a), (b), (c), (d), represent all possibilities apart from relabeling of lines. Each propagator labeled with its momentum.

A typical fourth-order diagram has the form of Fig. 5(a). Here the hole lines $l$, $m$, $n$ have purposely not been closed for the sake of clarity of the figure; this representation was first used by Rajaraman.[11] Momentum conservation has been taken into account in labeling the lines. Many different orders of interactions are conceivable, but many of these diagrams are identical. The sum over identical diagrams is carried out if we *define* particle $l$ as that particle which is involved in both the first and the second interaction, and particle $m$ as the other particle involved in the first interaction, so that particle $n$ participates in the second. Then we must obviously integrate over $l$, $m$, and $n$, independently; i.e., the diagram of Fig. 5(a) is multiplied by $\rho^3$ to give the energy per unit volume, and by $\rho^2$ to give the energy per particle. Since the same pair cannot interact twice in succession, there are then only 4 different diagrams, Figs. 5(a)–(d). In Fig. 5(b), we have labeled the particles 1, 2, 3 rather than $l$, $m$, $n$, and we shall keep the notation 1, 2, 3. The intermediate momenta $\mathbf{a}$, $\mathbf{b}$, $\pm(\mathbf{a}-\mathbf{b})$ are, of course, vectors.

All four diagrams have in common that the first interaction leads to a state of momenta $\mathbf{a}, -\mathbf{a}$, while before the last interaction we have the momenta $\mathbf{b}, -\mathbf{b}$. Since particles 1, 2, 3 are well defined by the fact that 1 is involved in both the first and the second interaction, we must integrate over the *entire* space of the vectors $\mathbf{a}$ and $\mathbf{b}$, not just over half the space. The diagrams can be considered as the various ways of going from a state $\mathbf{a}, -\mathbf{a}$ to $\mathbf{b}, -\mathbf{b}$. This cannot be done in third order; the only relevant third-order diagram is that of Fig. 6, the well-known three-body cluster, also

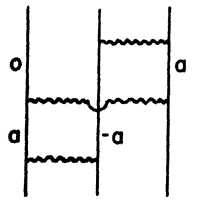


FIG. 6. Third-order ring diagram with hole lines broken open.

shown in the more conventional diagram of Fig. 7: This can only lead from $-\mathbf{a}, \mathbf{a}$ to another state $-\mathbf{a}, \mathbf{a}$. In the middle interaction, particles 1 and 3 simply exchange momenta; the transition goes from $(\mathbf{a},0)$ to $(0,\mathbf{a})$. The third order is therefore a special case, which will become even more important when we compare the third and fourth orders below. All orders beyond the fourth have the same structure as the fourth, namely, of leading from $\mathbf{a}, -\mathbf{a}$ to $\mathbf{b}, -\mathbf{b}$.

We have calculated the fourth-order diagrams of Fig. 5 approximately, by integrating over the momenta $\mathbf{a}$ and $\mathbf{b}$ explicitly. We have assumed the same potential as Moszkowski and Scott,[20] the "standard hard-core potential" (SHC), viz.,

$$\begin{aligned} &V=+\infty, \quad r<c, \\ &V=-Ae^{-\mu(r-c)}\tfrac{1}{2}(1+P), \quad r>c, \\ &A=260 \text{ MeV}=6.27 \text{ F}^{-2}, \\ &c=0.4 \text{ F}, \quad \mu=2.083 \text{ F}^{-1}, \\ &P=\text{Majorana exchange operator} \end{aligned} \tag{2.1}$$

(translation from MeV to $\text{F}^{-2}$ by factor $\hbar^2/M=41.5$ MeV $\text{F}^2$). We expected, and verified in the calculation, that the main contribution is due to the repulsive core, but that the attraction has a noticeable effect in keeping the result down. For the core, it can be shown on dimensional grounds that the result must be proportional to $c^4$; assuming this for the entire interaction, we find for the fourth-order energy

$$\Delta E_4=-30(c/r_0)^4(\hbar^2/Mr_0^2). \tag{2.2}$$

With $r_0=1.12$ F, and $c=0.49$ F, the Hamada-Johnston value, this gives

$$\Delta E_4\approx -35 \text{ MeV/particle}, \tag{2.3}$$

which is enormous.

Before discussing the consequences of this result, we want to make certain features of (2.2) plausible. The negative sign follows from the fact that three propagators (energy denominators) are involved in Fig. 5, each contributing a negative sign; the $G$ matrices at the vertices (wavy lines) are all positive because the repulsive core dominates. The dependence on $r_0$ follows from the fact that the energy per nucleon from our term is proportional to the square of the density, i.e., proportional to $r_0^{-6}$. The quantity (2.2) is an energy,

[20] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960).

hence of dimension (length)$^{-2}$, and since $c$ is the only other length available, it must be proportional to $c^4$.

Concerning statistical weights, we note that the first and last interaction in Figs. 5 involve a transition of two particles from states inside the Fermi sea to excited states. Since the repulsive core gives the main contribution, and since $k_F c \ll 1$, only $S$ states will be strongly affected. This means that the two interacting nucleons must have different spin or isospin, so that only three-fourths of all possible nucleon pairs will contribute. In all cases, except Fig. 5(b), the first and last interacting pairs are different, and the numerical calculation shows that Fig. 5(b) gives the smallest contribution of the four diagrams, about one-sixth of the total. We therefore assume that the first and last pairs are different. They will have one common nucleon, No. 1 in Figs. 5(a) and 5(c). Both the other nucleons must then differ in spin or isospin from nucleon 1, and we therefore have inserted a factor $(\frac{3}{4})^2$ for this fact in obtaining (2.2). In the second and third interaction, the initial and final relative momenta of the interacting particles are both large, so that any angular momentum is possible. Then there is no restriction on the spins and isospins of the interacting particles.

The very large result (2.3) would be far more than sufficient to explain the discrepancy between the previous calculations of the binding energy (8 MeV) and the observation (16 MeV). However, it indicates that the whole procedure of calculating successive orders in the Goldstone series is unreliable. We must be prepared that the fifth order is again very big, and the series appears to diverge (see Sec. 4). The only remedy is then to sum all the graphs containing three hole lines, over all orders. This will be done in the next section, and the very large result for the fourth order will then become understandable.

The large result for the fourth order may seem particularly surprising, in view of the fact that the ring diagrams of third order, Fig. 7, are known to give a small result, less than 1 MeV per particle.[21,22] This result, in fact, was the chief reason for the erroneous belief that higher order diagrams do not contribute much (and also that ring diagrams are less important than bubble diagrams). However, Köhler[22] pointed out that the small result for the ring diagram is due to the fact that the contributions to Fig. 7 from the short- and the long-range forces in the middle interaction almost cancel. This is best understood using Fig. 6, which is our new representation for Fig. 7. As we pointed out before, in the middle interaction here the two nucleons exchange the large momentum **a**. This substantially decreases the effect of the repulsive core which is an "ordinary" force, while the long-range, central force has a large effect, because it is essentially a Serber force, proportional to $1+P$, Eq. (2.1). The

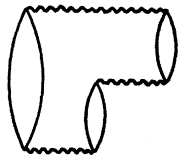

FIG. 7. Same as Fig. 6, in usual representation.

operator $P$, operating on the exchange of the two momenta 0 and **a**, gives zero momentum change, and hence a large matrix element. Therefore the attractive force, though fundamentally weaker than the repulsive core for high momentum, is, in this particular case, equally strong, leading to the cancellation discussed above. In fourth order, as we pointed out in connection with Fig. 5, the momentum changes in the two middle interactions are arbitrary, hence applying the exchange operator $P$ does not lead to zero momentum change. Then the attractive force contribution is only a small correction to the repulsive core, and therefore the net result of the fourth order is large; the same will hold in higher orders. So the third-order ring diagram is exceptional[23] because of the special character of the nuclear forces—repulsion plus Serber-type attraction—while the higher orders are regular.

### 3. SUMMATION OF THREE-BODY INTERACTIONS

We wish to sum all three-body interactions. In Goldstone language, this means all diagrams containing exactly three hole lines. There must be not more than three loops, because each loop must contain at least one hole line, and there must be at least three interactions because otherwise it is impossible for all nucleons to return to their original momentum. We break up the hole lines as in Fig. 5 and thus consider diagrams involving three nucleon lines. As in Sec. 2, we neglect the momenta of particles in the Fermi sea; so we start from and return to three nucleons of zero momentum.

A method to treat three-body interactions has been developed by Faddeev.[24–26] The total scattering matrix is written as the sum of three parts

$$T = T^{(1)} + T^{(2)} + T^{(3)}, \tag{3.1}$$

where $T^{(1)}$ means that particle 1 does not take part in the last interaction. If $g_{23}$ denotes the two-body reaction matrix, i.e., the quantity denoted by $G$ in BBP,[13] then

$$T^{(1)} = g_{23} - g_{23}(1/e)(T^{(2)} + T^{(3)}). \tag{3.2}$$

[21] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).
[22] H. S. Köhler, Ann. Phys. (N. Y.) **12**, 444 (1961).

[23] This is also in agreement with the result of G. A. Baker, J. L. Gammel, and B. J. Hill, Phys. Rev. **132**, 1373 (1963), who find that the ring diagrams are large for their case of a simple, weak, square-top repulsive potential.

[24] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; Dokl. Akad. Nauk SSSR **138**, 565 (1961) [English transl.: Soviet Phys.—Doklady **6**, 384 (1962)].

[25] I am indebted to R. Omnès for drawing my attention to Faddeev's work, in a lecture about a different application of this work.

[26] I am particularly grateful to A. M. Green who suggested that Faddeev's theory be applied to the entire three-body interaction, beginning with the third order. In my original version I had only applied it from the fourth order on.

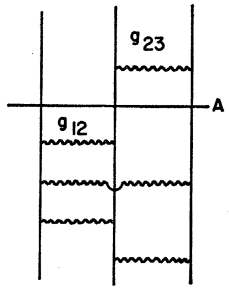


FIG. 8. Interaction of three nucleons representing $T^{(1)}$. The part below level $A$ represents $T^{(3)}$.

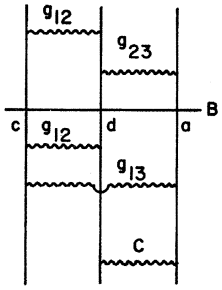


FIG. 9. Interaction of three nucleons. As described in the text, it is convenient to separate the two last interactions, $g_{12}$ and $g_{23}$, from the rest. At level B, the wave function is $\Psi^{(1)}$.

It is important that all matrices be calculated for the same starting energy,[13] which may be complex in Faddeev's theory (we shall not need this).

The result (3.2) is very easily understood (Fig. 8). In $T^{(1)}$, the last interaction is by definition $g_{23}$. This may stand by itself, or it may be preceded by other interactions. Because the same pair cannot interact twice in succession, the interaction just before $g_{23}$ must involve particle 1. (In Fig. 8, it is $g_{12}$.) This may again be preceded by an arbitrary ladder of $g$ interactions. In other words, the part of Fig. 8 below the line A has again the form of a complete three-body $T^{(i)}$ matrix, which in Fig. 8 must be $T^{(3)}$. Obviously, $T^{(2)}$ is equally possible.

Clearly, the notation (3.2) is highly abbreviated, and, in fact, we must pay close attention to the energy denominators, not only the $e$ occurring explicitly in (3.2) but also those occurring in the calculation of $g$ itself. The $e$ occurring in (3.2) involves the excitation of the two nucleons, 2 and 3. Any $e$ occurring in $T^{(2)}$ or $T^{(3)}$ involves the excitation of all three nucleons[11] and is therefore, in general, much larger; an exception is the last $e$ on the right, i.e., the $e$ after the first (bottom) interaction in Fig. 8 which again involves only two excited nucleons. Faddeev explicitly writes down an energy argument for each $T$ and $g$; we shall not do this but shall discuss the appropriate energies when indicated.

We shall find it convenient to introduce a three-body wave function

$$\Psi^{(1)}=[1-(1/e)(T^{(2)}+T^{(3)})]\Phi, \tag{3.3}$$

so that

$$T^{(1)}\Phi=g_{23}\Psi^{(1)}. \tag{3.4}$$

Then (3.2) may be rewritten

$$\Psi^{(1)}=\Phi-(1/e)g_{13}\Psi^{(2)}-(1/e)g_{12}\Psi^{(3)}. \tag{3.5}$$

Here we may write, in the notation of BBP,

$$(1/e)g_{12}=1-\Omega_{12}. \tag{3.6}$$

The quantities $g$, $e$, and $\Omega$ are operators. We shall show, however, that in good approximation $1-\Omega_{12}$ can be replaced by a suitable function $\zeta$ of the distance $r_{12}$.

To evaluate the operator $\Omega_{12}$ operating on $\Psi^{(3)}$, we imagine the latter function to be expanded in a Fourier integral, and consider the component

$$\Psi_{cda}=\exp i(\mathbf{k}_c\cdot\mathbf{r}_1+\mathbf{k}_d\cdot\mathbf{r}_2+\mathbf{k}_a\cdot\mathbf{r}_3), \tag{3.7}$$

where the notation is chosen to conform with BBP (7.10) (see also Fig. 9 in this paper). We use the reference approximation of BBP, then

$$(\gamma^2-\nabla_{12}^2)((1-\Omega_{12})\Psi_{cda})=m^*v\Omega_{12}\Psi_{cda}, \tag{3.8}$$

where $\nabla_{12}^2$ is the Laplacian with respect to the relative coordinate $\mathbf{r}_{12}$, $v$ is the potential acting between nucleons 1 and 2, and $m^*$ is the effective mass. $\Psi_{cda}$ now plays the role of the unperturbed function, $\Phi$ in BBP, $\Omega_{12}\Psi_{cda}$ of the perturbed wave function, $\psi$ in BBP, and $(1-\Omega_{12})\Psi_{cda}$ that of the $\zeta$ in BBP. The parameter $\gamma$ is given by BBP (7.10), viz.,

$$\gamma^2=m^*[E^R(a)+E^R(c)+E^R(d)-E(l)-E(m)-E(n)]-\tfrac{1}{4}(\mathbf{k}_c-\mathbf{k}_d)^2. \tag{3.9}$$

This can be evaluated, in a manner slightly different from BBP (7.14), giving

$$\gamma^2=\tfrac{3}{4}k_a^2+(3\Delta-0.45)k_F^2. \tag{3.10}$$

In (3.9), $E^R(a)$ is the reference spectrum energy of state $a$ above the Fermi sea, $E(n)$ is the actual (nuclear matter) energy of state $n$ in the Fermi sea, $l$, $m$, $n$ are the initial momenta of the three nucleons before any interaction, and $k_F^2\Delta$ is the effective gap between states inside the Fermi sea and the reference spectrum, as defined in BBP (7.3) and (more precisely) in this paper, Eq. (7.22). We have used the fact that the total momentum is conserved:

$$\mathbf{k}_a+\mathbf{k}_c+\mathbf{k}_d=\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n.$$

The function $(1-\Omega_{12})\Psi_{cda}$ can be obtained from (3.8) by expanding in spherical harmonics in the relative coordinate $r_{12}$. If $v$ is a pure hard-core potential, the radial functions for $r_{12}>c$ are given by spherical Hankel functions $h_L(i\gamma r)$, BBP Eq. (5.6). However, it is shown in BBP (5.25) that a good approximation for all of these $h_L$ is $h_0(i\gamma r)=e^{-\gamma r}/r$. [This could be improved by using an effective $\gamma$, somewhat larger than the true $\gamma$, and defined from BBP (5.28).] If we make this approximation, the operator $1-\Omega_{12}$ becomes simply a factor $\zeta$;

thus

$$(1-\Omega_{12})\Psi_{cda}/\Psi_{cda}=\zeta(r_{12})=1\,, \quad r<c\,, \qquad =e^{-\gamma(r-c)}(c/r)\,, \quad r>c. \tag{3.11}$$

Here $\gamma$ still depends, by Eq. (3.10), on the momentum $k_a$ of the nucleon not participating in the interaction. We may distinguish two cases, cf. Fig. 9: In the first and last interaction, the "spectator" nucleon is not excited, so that $k_a<k_F$ in (3.10). [Actually, (3.10) must then be somewhat modified, see (3.16) below.] Then $\gamma$ is rather small, and $\zeta$ decreases fairly slowly with the distance $r_{12}$ for $r_{12}>c$. We shall denote this $\zeta$ function by $\eta$. In all other interactions, the spectator nucleon is outside the Fermi sea, $k_a>k_F$. Now we know that the most important two-body interaction is the repulsive core because the weak, long-range interaction can be treated by perturbation theory. The repulsive core raises nucleons from the Fermi sea chiefly to states of momentum

$$k_a\approx\pi/2c\,, \tag{3.12}$$

because it favors states whose $S$-state wave function $\sin k_a r$ has a maximum near $r=c$. For $c=0.4$ F (as in the Gammel-Thaler potential), this means

$$k_a\approx 4\ \mathrm{F}^{-1}. \tag{3.13}$$

Figure 14 of BBP gives a quantity $F_0(k)$ whose square is the probability of transition to an excited state $k$; this has a maximum between 3 and 5 $\mathrm{F}^{-1}$, a fact which was used in BBP for the determination of the parameters of the reference spectrum. We now assume that even after repeated interactions, as in Fig. 9, momenta of the order (3.12) still dominate: One argument for this is that the momenta must be first generated by interactions and then be brough back to zero by the further interactions, so that very high momenta in intermediate states are not likely. On the other hand, very low momenta will not contribute much because of the volume in momentum space. Further, the wave function (3.11) is not terribly sensitive to $\gamma$, because its inner part ($r\lesssim c$) will turn out to be most important, and is independent of $\gamma$, and further, $\gamma$ by (3.10) is only moderately sensitive to $k_a$. We therefore assume that $k_a$ is generally given by (3.12), for all but the first and last interaction.

We may now compute $\gamma$. According to (7.17), the best value for $\Delta$ is about 0.55, and Brandow[15] has shown that $k_F\approx 1.36\ \mathrm{F}^{-1}$. Then using (3.10) and (3.12),

$$\gamma^2=3(\pi/4c)^2+1.2k_F^2\,, \tag{3.14}$$

$$\gamma c=1.49 \text{ to } 1.56 \quad \text{for} \quad c=0.4 \text{ to } 0.5\ \mathrm{F}\,, \qquad \gamma=3.7 \text{ to } 3.1\ \mathrm{F}^{-1} \quad \text{for} \quad c=0.4 \text{ to } 0.5\ \mathrm{F}. \tag{3.15}$$

This value of $\gamma$ is to be used for the function $\zeta$ describing all interactions except at the two ends. For the exceptional cases at the two ends of the ladder, we must calculate *on* the energy shell; then BBP (7.7) is valid, and

$$\gamma_2^2=2k_F^2\Delta-k_0^2\,, \tag{3.16}$$

where the subscript 2 means that only two nucleons are excited, not three as in (3.15). Taking the average value in the Fermi sea, $k_0^2=0.3k_F^2$, and again $\Delta=0.55$, we get

$$\gamma_2^2=0.8k_F^2, \quad \gamma_2=1.22\ \mathrm{F}^{-1}. \tag{3.17}$$

This is then the appropriate value of $\gamma$ for the function $\eta$; it is much smaller than (3.15), which holds for $\zeta$.

In the cases of both $\zeta$ and $\eta$, we can include some of the attractive force outside the repulsive core in calculating the wave function. In the case of $\zeta$, this makes little difference, because of the large value of $\gamma$. In the case of $\eta$, it makes a lot of difference. The most convenient procedure is the Moszkowski-Scott method,[20] modified as proposed by BBP Sec. 10, in which the wave function defect $\eta$, and its derivative, go to zero at the separation distance $d$. Since the function $\eta$ is an ordinary defect wave function for two interacting nucleons in the Fermi sea, the separation distance, for most of the acceptable potentials, is

$$d=1.0 \text{ to } 1.1\ \mathrm{F}. \tag{3.18}$$

A rough approximation to $\eta$ is

$$\eta=[(d-r)/(d-c)]^2 \text{ for } c<r<d\,, \qquad \eta=1 \text{ for } r<c, \quad \eta=0 \text{ for } r>d. \tag{3.19}$$

This function satisfies the boundary conditions $\eta=1$ at $r=c$, $\eta=\eta'=0$ at $r=d$. For $\zeta$, we may use a similar approximation, but with a different separation distance, $d_3<d$. The crude approximation (3.19), is, of course, not necessary but will be useful in Sec. 5.

We can now write the operator equation (3.5) in terms of the functions $\eta$ and $\zeta$. Let us assume for the moment that these are equal, then we can simply replace $(1/e)g$ in (3.5) by $\zeta$. Then (3.5) becomes

$$\Psi^{(1)}=\Phi-\zeta(r_{13})\Psi^{(2)}-\zeta(r_{12})\Psi^{(3)}\,, \tag{3.20}$$

where the $\zeta$'s are now known *functions*, rather than operators. We may write down the two corresponding equations for $\Psi^{(2)}$ and $\Psi^{(3)}$ and thus obtain three linear, algebraic equations for the three independent unknowns,[27] $\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}$. These can easily be solved, with the result

$$\frac{\Psi^{(1)}}{\Phi}=\frac{(1-\zeta_{12})(1-\zeta_{13})}{1-\zeta_{12}\zeta_{13}-\zeta_{13}\zeta_{23}-\zeta_{23}\zeta_{12}+2\zeta_{12}\zeta_{13}\zeta_{23}}, \tag{3.21}$$

where $\zeta_{12}\equiv\zeta(r_{12})$. This will be discussed in Sec. 4.

Equation (3.20), however, is oversimplified. We must take into account that the first and last interactions are described by $\eta$ rather than $\zeta$. This can be

[27] I am greatly indebted to D. Thouless for pointing this simple fact out to me, after I had tried for some time, and with some success, to solve (3.20) by approximate methods.

done most conveniently by calculating $\Psi^{(1)}$ at the level B of Fig. 9. Then $\Psi^{(1)}$ is *followed* (going upwards) by the interaction $g_{23}$, and this in turn by $g_{12}(1/e)=\eta(r_{12})$, by definition of $\eta$. The contribution of Fig. 9 to the energy is then

$$W=\int \eta(r_{12})g_3(r_{23})(\Phi-\Psi^{(1)})d\tau_1 d\tau_2 d\tau_3. \quad (3.22)$$

The subscript 3 on $g_3$ means that three nucleons are excited, so that (3.10) must be used for $\gamma$. We use $\Phi-\Psi^{(1)}$ in (3.22) rather than $\Psi^{(1)}$ itself, for the following reason: $W$ is to be the diagonal element of the energy, i.e., the nucleons should start, as well as end, in states **l, m, n**. The two topmost interactions of Fig. 9, however, involve nucleons 1 and 3 each only once; therefore, at level B, their momenta must be different from **l** and **n**. Hence, there must be at least one interaction below level B to return all nucleons to their "normal" states **l, m, n**; therefore, the unperturbed wave function $\Phi$ will not give a contribution to the diagonal element of the energy, but only the difference $\Phi-\Psi^{(1)}$ will.

The equation for $\Psi^{(1)}$ corresponding to (3.5) is now

$$\Psi^{(1)}=\Phi-\eta(r_{12})-\zeta(r_{12})(\Psi^{(3)}-\Phi) -\eta(r_{13})-\zeta(r_{13})(\Psi^{(2)}-\Phi). \quad (3.23)$$

The first term corresponds to the possibility that there is no interaction at all below the level B. The second term corresponds to a single interaction between nucleons 1 and 2; this must then be the first and is therefore described by $\eta(r_{12})$. The next term gives the contribution from more than one interaction: Then the interaction preceding level B is $\zeta$ because it is not the first one, but to the right of $\zeta$ we cannot have the unperturbed function but only the difference $\Psi^{(3)}-\Phi$, because at least one interaction must precede $\zeta$. The last two terms in (3.23) are the same as the preceding two, but with the nucleons 1 and 3 interacting.

Equation (3.23) for $\Psi^{(1)}$, and the two corresponding ones for $\Psi^{(2)}$ and $\Psi^{(3)}$, may again be considered as three linear equations for the three unknowns $\Psi^{(i)}$. The solution is

$$\Phi-\Psi^{(1)}=\frac{\eta_{12}u_{13}+\eta_{13}u_{12}-\eta_{23}(u_{12}+u_{13}-2u_{12}u_{13})}{u_{12}u_{13}+u_{13}u_{23}+u_{23}u_{12}-2u_{12}u_{13}u_{23}}, \quad (3.24)$$

where

$$u_{12}=1-\zeta_{12}=1-\zeta(r_{12}). \quad (3.25)$$

This, then, is our final solution for the three-body problem, with the approximations described above. It is linear in $\eta$, as it must be, but strongly nonlinear in $\zeta$ or $u$.

## 4. DISCUSSION

We shall first consider (3.24) for large distances $r_{12}$, $r_{13}$, and $r_{23}$. Then all the two-body functions $u_{12}$, $u_{13}$, and $u_{23}$ are close to one. The denominator in (3.24) is then unity, and the coefficient of $\eta_{23}$ in the numerator is zero. Then

$$\Phi-\Psi^{(1)}=\eta(r_{12})+\eta(r_{13}). \quad (4.1)$$

This also follows immediately from the fundamental equation (3.23), since the last term in each row of this equation is of order $\zeta\eta$ and hence negligible.

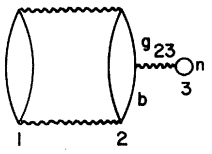


FIG. 10. Third-order bubble interaction.

We may insert (4.1) into (3.22) and obtain

$$W_1=\int \eta(r_{12})g_3(r_{23})[\eta(r_{12})+\eta(r_{13})]d\tau_1 d\tau_2, \quad (4.2)$$

where particle 3 may be considered as fixed (instead of the center of mass). This is exactly the contribution from the familiar third-order diagrams shown in Figs. 7 and 10. The particular form is similar to that first given by Köhler.[22] It is convenient to consider (4.2) as an integral over the two-body interaction $g_3(r_{23})$ with a weight factor; thus,

$$W_1=\int g_3(r_{23})F_1(r_{23})d\tau_{23}, \quad (4.3)$$

$$F_1(r_{23})=\int d\tau_1\eta(r_{12})[\eta(r_{12})+\eta(r_{13})], \quad (4.4)$$

where in (4.4) the distance $r_{23}$ is held fixed. When the correct $\Psi^{(1)}$ is inserted, $F_1$ must be replaced by

$$F(r_{23})=\int d\tau_1\eta(r_{12})[\Phi-\Psi^{(1)}(r_{12},r_{13},r_{23})]. \quad (4.5)$$

Equation (4.3) represents the third-order three-body energy in terms of the two-body interaction $g(r_{23})$ acting at the middle level of Fig. 10, i.e., the interaction of the particle $b$ with the hole $n$. Due to the higher order diagrams, the *weight* of each $r_{23}$ in this interaction is reduced by the factor

$$f(r_{23})=F(r_{23})/F_1(r_{23}), \quad (4.6)$$

which, as we shall see, is less than one.

Expanding (3.24) to second order in $\eta$ and $\zeta$, we get

$$\Phi-\Psi^{(1)}=\eta(r_{12})+\eta_{13}-\zeta_{12}(\eta_{13}+\eta_{23})-\zeta_{13}(\eta_{12}+\eta_{23}). \quad (4.7)$$

The same solution again follows more easily by iteration from (3.23). It will be noted that (4.7) contains two terms of first degree in $\eta$, and four terms of second degree in $\eta$ and $\zeta$. The former correspond to the third-order Goldstone diagrams, Figs. 7 and 10, the latter to the fourth-order diagrams, Figs. 5(a)–(d). It may be seen, most easily by continued iteration of (3.23), that

there are $2^n$ terms of $n$th degree in $\eta$ and $\zeta$; these correspond to the $n+2$nd Goldstone order. As long as $\eta$ and $\zeta$ are small, this series will converge; but $\eta$ and $\zeta$ each become equal to one when their argument $r<c$; and, in this case, the series of iterations, and hence the *Goldstone* perturbation *series*, obviously *diverges*.

We must then use the exact solution (3.24) without expanding it. The situation is simplest, if we assume

$$r_{12}=r_{13}=r_{23};$$

hence

$$u_{12}=u_{13}=u_{23}\equiv u=1-\zeta$$

and

$$\eta_{12}=\eta_{13}=\eta_{23}\equiv\eta. \tag{4.8}$$

Then (3.24) reduces to

$$\Phi-\Psi^{(1)}=2\eta/(3-2u)=2\eta/(1+2\zeta). \tag{4.9}$$

For small $\zeta$, we easily recover the results (4.1), (4.7). But when the three nucleons are very close together, then $u=0$, and therefore

$$\Phi-\Psi^{(1)}=2\eta/3, \tag{4.10}$$

which is only one-third of the first-order result (4.1). In this limit, also $\eta=\Phi$ and, therefore,

$$\Psi^{(1)}=\Phi/3, \tag{4.11}$$

again one-third of the "unperturbed" result.

We may get further insight by expanding (4.9) in a power series,

$$\Phi-\Psi^{(1)}=2\eta-4\eta\zeta+8\eta\zeta^2-16\eta\zeta^3+\cdots. \tag{4.12}$$

Each term corresponds to one order of Goldstone perturbation theory, the first to third order. The series is exactly that which is obtained by continued iteration of (3.23), in the case when all $\zeta$'s and all $\eta$'s are equal; it is just the series which we described below (4.7). For small $\zeta$, this series converges, but as $\zeta>\frac{1}{2}$, it diverges. It is, however, always summable as a geometric series, giving (4.9). Thus also (3.24) may be considered as the sum of a quasigeometric series consisting of the various orders of perturbation theory.

It is now clear why the calculation of the fourth order in Sec. 2 gave such a large result: We calculated the second term in (4.12), which (in the limit $\zeta=1$) is twice as large as the first term (third order), while the correct result (4.10) is only one-third as large as the first term.

The result (4.10), that the relevant quantity $\Phi-\Psi^{(1)}$ is cut down to one-third of its elementary value, can be given the following, somewhat naive interpretation: The quantity $g$ represents the interaction of one pair of particles, including their repulsive cores. When three nucleons are close together, an elementary treatment would give us three repulsive pair interactions. In reality, we cannot do more than exclude the wave function from the repulsive core region, hence we get only *one* core interaction rather than three.

We now turn to a more general discussion. $W$ in (3.22) represents the interaction of particle 2, when it is outside the Fermi sea, with particle 3 (Fig. 10). To lift particle 2 out of the Fermi sea, it must first have interacted with particle 1, by the factor $\eta(r_{12})$ in (3.22). Therefore, particle 1 must be close by particle 2. Since the interaction is mostly repulsive, the presence of particle 1 reduces the wave function for particles 2 and 3. This is especially true if $r_{23}$ is small, because then a small value of $r_{12}$ means also small $r_{13}$. The correlated wave function of all three particles is less than it would be if only 2 and 3 interacted, and the interaction between 2 and 3 is reduced accordingly. This would follow automatically from Jastrow's correlated wave function,[28] but calculations with that function are somewhat cumbersome because the normalization integral is already complicated. These difficulties are avoided by using the Goldstone method as we have done.

We have shown, however, that the Goldstone method must not be used blindly by considering order by order, as everybody including myself has done until now. Instead, we needed to sum all diagrams involving three hole lines at once. This procedure is analogous to the familiar summation of all ladder graphs to give the $G$ matrix for the two-body interaction which is the starting point of the Brueckner-Goldstone theory. The reason why the summation is necessary, however, is different. For a potential with a hard core, the matrix elements of the potential are infinite and hence meaningless; so we *must* form the two-body reaction matrix $G$ to get any sensible quantity at all. For the three-body interaction, each Goldstone diagram is perfectly meaningful, hence nobody ever noticed the trouble, except perhaps for Rajaraman,[11] who also showed that the sum of all diagrams is finite. The trouble is that the number of diagrams in $n$th order goes up as $2^{n-2}$ [cf. paragraph following Eq. (4.7)], and the contribution of each diagram remains almost the same. The three-body trouble would be absent if the potential were sufficiently weak, because the $\zeta$ would remain small for *all* values of $r$, and the denominators in (3.24) and (4.9) would be essentially one. In this respect, the three-body and the two-body problems behave similarly.

On the other hand, nothing would be gained in the treatment of the three-body problem by replacing the hard core by a strong but finite repulsion. Such a repulsion would still make the two-body wave function go nearly to zero at small $r$, hence make $\zeta\approx1$ and cause the series (4.12) to diverge. Therefore, it would still be necessary to solve the three-body problem completely, rather than by perturbation theory. Since the solution of the two-body problem with a hard core presents no difficulty, there is no advantage of mathematical simplicity to be gained by avoiding the hard core. The question of agreement with experiment, to be treated in Sec. 8, is an entirely different matter.

[28] R. Jastrow, Phys. Rev. **98**, 1479 (1955).

The three-body correction is considerably milder than the two-body one. For two bodies, the wave function inside the repulsive region is zero, and the potential $v(r)$ which is infinite for $r<c$ is transformed into the function (in the reference approximation)

$$g(r)=(\gamma^2-\nabla^2)\zeta(r), \tag{4.13}$$

which merely has a $\delta$ function at $r=c$, and thus is completely different from $v(r)$. The three-body wave function $\Psi^{(1)}$, on the other hand, is everywhere finite and nonzero, just like the unperturbed $\Phi$, and multiplies the already well-behaved $g_{23}$, Eq. (3.22).

It is likely that the analogous procedure should be repeated for four bodies. However, as we shall show in Sec. 8, the contribution of the three-body correlation to the *energy* seems to be only about 3% of the two-body potential energy. Hence it is likely that the four-body correlation is only about 0.1%, and therefore hardly worth calculating. It is also likely that the four-body modification of the wave function will be even milder than the three-body one.

We have shown that the Goldstone series does not converge, just as a number of physicists have suspected. However, these authors have nearly always emphasized the superconductivity phenomenon which depends on the presence of very many ($N$) nucleons, and on the fact that $N$ times the interaction energy is large.[29] It is now generally agreed that the BCS gap $\Gamma$ in nuclear matter at normal density is very small,[30,31] almost certainly much less than 1 MeV, and that the observed, larger gap in finite nuclei (about 2 MeV) is a surface effect.[32,33] The effect on the binding energy per nucleon is of order $\Gamma^2/2E_F$, where $E_F$ is the Fermi energy, about 40 MeV; this effect is then less than 10 keV per particle and therefore negligible.

Instead, the divergence of practical importance is in the three-body correlation, i.e., in low orders and *few* bodies. To my knowledge, this has not been suspected, except by Rajaraman.[11] It seems to me that only paitient evaluation of the theory and study of the physical and numerical importance of various terms will give us the answer, rather than use of formal mathematics. I believe that the Goldstone series is the most convenient and most physical enumeration of the terms which occur. We may have to continue to extract from it suitable subseries, which should and can be summed to obtain convergence.

[29] A very good mathematical discussion of this problem has been given by G. A. Baker, Phys. Rev. **131**, 1869 (1963), who also shows elegant mathematical methods for obtaining, in many cases, the correct result in spite of the divergence of the series.
[30] V. J. Emery and A. M. Sessler, Phys. Rev. **119**, 248 (1960).
[31] E. M. Henley and L. Wilets, Phys. Rev. **133**, B1118 (1964); R. Kennedy, L. Wilets, and E. M. Henley, *ibid.* **133**, B1131 (1964).
[32] L. Wilets, E. M. Henley, and R. C. Kennedy, Congrès International de Physique Nucléaire, Paris, 1964, Vol. II, p. 302 (unpublished).
[33] S. Nagata and H. Bando, Ref. 32, p. 293.

## 5. NUMERICAL EVALUATION

Thomas and Dahlblom have independently evaluated the function $F(r_{23})$ of (4.5), using the Cornell and Copenhagen computers, respectively. Kirson and Reid have independently checked the result for $r_{23}=0$ and $r_{23}=\infty$ by analytical integration. The agreement between all calculations is satisfactory (see below).

The integral (4.5), for fixed $r_{23}$, may be written

$$F(r_{23})=\frac{1}{2r_{23}}\int_0^\infty r_{12}dr_{12}\eta(r_{12})\int_{r_{23}-r_{12}}^{r_{23}+r_{12}} r_{13}dr_{13}(\Phi-\Psi^{(1)}). \tag{5.1}$$

Here $\Phi-\Psi^{(1)}$ is given by (3.24). For $\eta$ and $\zeta$, we use the approximation (3.19), viz.,

$$\begin{aligned}\eta&=[(d_2-r)/(d_2-c)]^2 \quad \text{for} \quad c<r<d_2,\\ \zeta&=[(d_3-r)/(d_3-c)]^2 \quad \text{for} \quad c<r<d_3,\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\eta=\zeta=1-a \quad \text{for} \quad r<c,\\ &\eta=\zeta=0 \qquad \text{for} \quad r>d_2, d_3, \text{ respectively.}\end{aligned} \tag{5.3}$$

The parameter $a$ in (5.3) prevents $u=1-\zeta$ from vanishing: Otherwise, if two (or all three) of the distances $r_{12}$, $r_{13}$, and $r_{23}$ are less than $c$, (3.24) would take the indeterminate form 0/0, which would cause great trouble for the numerical calculation. In Dahlblom's and one of Thomas' calculations, $a$ was taken to be 0.1; in the final calculation of Thomas, $a=0.001$; in the analytical work, $a=0$ presents no difficulty.

The parameters $d_2$ and $d_3$ were determined as follows: For $d_2$, we assumed the Moszkowski-Scott separation distance. We assumed that for a core size $c=0.4$ F, the separation distance $d_2=1.0$ F, while for $c=0.5$ F, we took $d_2=1.1$ F. Accordingly, we calculated the two cases:

$$\begin{aligned}&\text{Case (A)} \quad d_2/c=2.5, \quad c=0.4 \text{ F};\\ &\text{Case (B)} \quad d_2/c=2.2, \quad c=0.5 \text{ F}.\end{aligned} \tag{5.4}$$

The function (5.2) represents $\eta$ quite well, as explained below (3.19).

For $\zeta$, a better approximation would be the exponen-

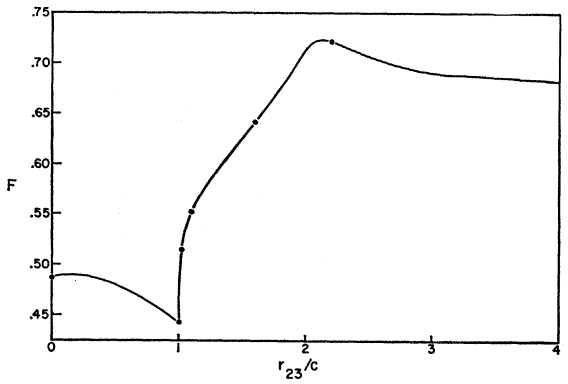


FIG. 11. The function $F(r_{23})$ including the correlation of three particles to all orders.

tial (3.11). Accordingly, we tried to fix $d_3$ in such a way that (5.2) represents (3.11) as well as possible. We chose $d_3$ such that the approximate expression (5.2) for $\zeta$ is equal to $\frac{1}{2}$ at the same value of $r$ as the "correct" expression (3.11), with $\gamma c$ given by (3.15). We found:

$$\begin{aligned}&\text{Case (A)},\quad \zeta=\tfrac{1}{2} \text{ for } r'=1.293c, \text{ then } d_3=2.00c;\\ &\text{Case (B)},\quad \zeta=\tfrac{1}{2} \text{ for } r'=1.284c, \text{ then } d_3=1.97c.\end{aligned} \tag{5.5}$$

To have round numbers, and appreciable difference between the two cases, we actually chose for the numerical work:

$$\begin{aligned}&\text{Case (A)},\quad d_3/c=2.0;\\ &\text{Case (B)},\quad d_3/c=1.9.\end{aligned} \tag{5.6}$$

With these parameters, the curve in Fig. 11 was obtained for $F(r_{23})$ in Case (B). Case (A) gives a similar curve, with the minimum at $r/c=1$ being $F(c)=0.773 \times F(\infty)$, instead of $0.647F(\infty)$ for Case (B). To interpret the result, we must also consider the "elementary" quantity $F_1$ [defined in (4.4)], which is given in Fig. 12 for Case (B). It decreases monotonically from $r_{23}=0$ to $\infty$. If we define also

$$F_0=\int d\tau_1 \eta^2(r_{12}), \tag{5.7}$$

which is obviously independent of $r_{23}$, then clearly

$$\begin{aligned}&F_1(0)=2F_0,\\ &F_1(\infty)=F(\infty)=F_0.\end{aligned} \tag{5.8}$$

Comparing Figs. 11 and 12, we find that $F(r)$ changes less with $r$ than $F_1(r)$, and in the opposite direction. Now $F_1-F_0$ represents the elementary ring diagram of Fig. 7, while $F_0$ represents the bubble, Fig. 10: Therefore it seems to be a better approximation to consider only the simple Fig. 10 than to include the ring diagram of Fig. 7, a surprising result.

However, this result is not of much practical use because of the many exchange diagrams which must be taken into account. Their examination (Sec. 6) shows that we should start from a complete third-order calculation, such as that of Razavy,[2] and then correct it by the ratio

$$f(r)=F(r)/F_1(r), \tag{5.9}$$

rather than calculate only the bubble diagram (Fig. 10) and then correct it by the factor

$$f_0(r)=F(r)/F_0(r). \tag{5.10}$$

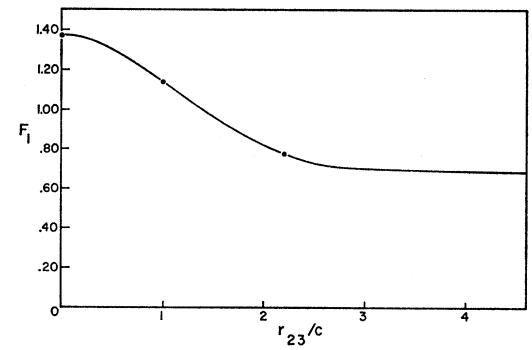


FIG. 12. The function $F_1(r_{23})$ giving the correlation in the third (i.e., the lowest nonvanishing) order.

TABLE I. Three-body correlation function $f$.

| $r/c$ | $f$ | $r/c$ | $f$ |
|---|---|---|---|
| 0 | 0.356 | 1.8 | 0.776 |
| 0.6 | 0.374 | 2.0 | 0.873 |
| 1.0 | 0.389 | 2.2 | 0.930 |
| 1.02 | 0.454 | 2.4 | 0.963 |
| 1.1 | 0.498 | 2.8 | 0.986 |
| 1.2 | 0.540 | 3.6 | 0.996 |
| 1.4 | 0.612 | 4.4 | 1.000 |
| 1.6 | 0.690 | | |

Representative values of the function $f$ for Case (B) are listed in Table I. It is seen that for $r=0$, $f(r)$ is actually about $\frac{1}{3}$, which was the estimate (4.10) of the ratio $[\Phi-\Psi^{(1)}]/[\eta(r_{12})+\eta(r_{13})]$ for small $r$. It is reasonable that $f(0)$ is somewhat larger than $\frac{1}{3}$, because larger $r_{12}$ contribute to the integral (5.1). The sharp rise of $f$ for $r/c>1$ may be noted. For $r=d=2.2c$, $f$ is within a few percent of its asymptotic value, 1. The increase of $f$ with $r$ is monotonic, while $f_0$ (see Fig. 11) has a minimum at $r=c$ and overshoots its asymptotic value for $r\approx d$.

## 6. SYMMETRY CONSIDERATIONS

We must now consider the exchange terms and the nucleon spin. In third order, this has been done by Rajaraman,[14] who found a surprisingly simple result: The interaction between the excited particle, $b$ in Fig. 10, and the sea nucleon $n$ is obtained by counting only the interaction in even angular momentum states, but with statistical weight 1 (rather than the customary $\frac{3}{4}$).

We shall generalize this result to the interaction of three particles in arbitrary order of perturbation theory, making the same assumptions as Rajaraman, viz., (1) the two-nucleon potential does not depend on spin, and (2) $k_F c \ll 1$. As in the rest of this paper, we shall use open diagrams, cf. Fig. 13.

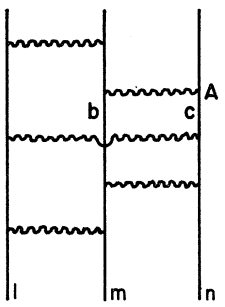


FIG. 13. Interaction of the three particles. A is the terminal interaction.

Since by assumption the potential does not depend on spin, we may assign a definite spin (up or down) and isospin (proton or neutron) to each of the three nucleon lines, and retain spin and isospin for the entire length of each line.

It is well known that the symmetry of the wave function does not need to be taken into account in intermediate states, because an integral is to be taken over *all* intermediate momenta at each level. The symmetry of the final state must be taken into account; since there are three particles, six permutations are involved. Once this symmetry has been properly considered, we may use an ordinary product wave function, unsymmetrized, for the initial state. The initial states of the three interacting particles are then definite and are denoted $l, m, n$ for particles 1, 2, 3, in Fig. 13. The final state may then be any of the permutations of $l, m, n$, and the respective diagrams enter with plus sign for even and with minus sign for odd permutations.

Let us assume, generally, that particles 1 and 2 are the first pair to interact. Let us further assume for the moment that at least one of these returns at the end to its initial state (this will be amended later). Now the final state of particle 3 can be $n$, $l$, or $m$. The first of these is always possible, but the final state $l$ (or $m$) is only permitted, if state $n$ has the same spin and isospin as $l$ (or $m$), because particle 3 retains its spin and isospin along its whole line. Assuming the spins and isospins of $l$ and $m$ to be given (they may be equal or different), and letting the spin-isospin of $n$ take all four possible values, we get four direct diagrams (final state of particle 3 is $n$) and two exchange diagrams (final state, $l$ or $m$).

Now consider the interaction which brings particle 3 into its final state; we call this the terminal interaction and denote it by A in Fig. 13. It may, in fact, be the last interaction in the ladder of Fig. 13, or it may be the next-to-last; this does not matter for the following. (The "terminal" interaction may be followed by one interaction of nucleons 1 and 2, but only by one, because a given pair cannot interact twice in succession.) Now assume that in the terminal interaction, particle 3 interacts with 2. The final state of particle 3 may still be $n$, $m$, or $l$; we shall call these "cases $n$, $m$, and $l$." All three possibilities are compatible with our assumption that at least one of the two particles 1 or 2 returns to its initial state: The final states of the three particles (1,2,3) are, respectively, $(l,m,n)$, $(l,n,m)$, and $(n,m,l)$, for the three cases.

Preceding the terminal interaction, particles 2 and 3 will be in some states of high momentum $k_b$, $k_c$ (Fig. 13). The matrix element of the terminal interaction will be nearly the same, in the three cases $n$, $m$, and $l$, because the momenta of the states $n$, $m$, and $l$ are rather small, in the sense

$$k_n c \ll 1, \quad k_m c \ll 1, \quad k_l c \ll 1. \tag{6.1}$$

If we then compare three diagrams, identical in the intermediate momenta, but leading to the three final states $l$, $m$, and $n$ for particle 3, they will give essentially the same result. But cases $l$ and $m$ carry a negative sign, because one exchange has taken place from initial to final state, while case $n$ carries a positive sign. Now we have shown that, if we sum over all spins and isospins of state $n$, there are four direct diagrams, leading to state $n$, and two exchange diagrams, leading to states $l$ and $m$. Since all these diagrams give nearly the same result except for sign, the result is equivalent to just two direct diagrams. Thus exchange simply has the effect of multiplying the final answer by one-half. Evidently, this result is a direct consequence of the existence of four states of spin and isospin. If there were $n$ states, exchange would reduce the direct diagram by a factor $1-2/n$.

There is an interesting corollary of our result, if we now fix the isospin (but not the spin) of state $n$. If states $l$ and $m$ are both proton states, then the interaction with a *neutron* state $n$ is not reduced by exchange; but if $n$ is a proton state, the entire interaction is canceled by exchange. Similarly, if $l$ and $m$ are both neutron states, the interaction with $n$ exists only if $n$ is a proton. If one of the states $l$ and $m$ is a neutron and the other a proton, then the interaction with state $n$ is reduced to one-half, whether $n$ is a neutron or a proton. All these results are compatible with the interpretation that the potential on an *excited* proton state ($k>k_F$) arises only from its interaction with the neutrons in the Fermi sea and vice versa. This result is particularly useful when the number of neutrons and protons is different, as in real, heavy nuclei.

The cancellation of direct and exchange terms is, of course, not exact. For a specific set of intermediate momenta, the difference will be of order $|\mathbf{k}_n-\mathbf{k}_m|c$, assuming that the important interactions are of range $c$. This assumption is justified because, at least for high intermediate momenta, the most important forces are the repulsive core and the strong attraction immediately outside it (see Secs. 3–5). If we average over all directions of the momentum $\mathbf{k}_c$, Fig. 13, the first-order term in $\mathbf{k}_n-\mathbf{k}_m$ will cancel, and the remaining effect will be of the order

$$|\mathbf{k}_m-\mathbf{k}_n|^2c^2. \tag{6.2}$$

That this is actually so was shown, in the particular case of third order, by Rajaraman[14] in the calculation leading to his Eq. (12). He also shows that the coefficient of (6.2) is quite small, $\frac{1}{10}$ to $\frac{1}{6}$, arising from power series expansions of $\sin x$ or $\cos x$, and it is easily seen that this result should be general. He finds in Eq. (12) that

$$\text{exchange/direct}\approx 1-0.15(k_Fc)^2. \tag{6.3}$$

Probably the exchange term is always less than the direct one. The difference, direct minus exchange, then has the same sign as the direct term, and is a few per-

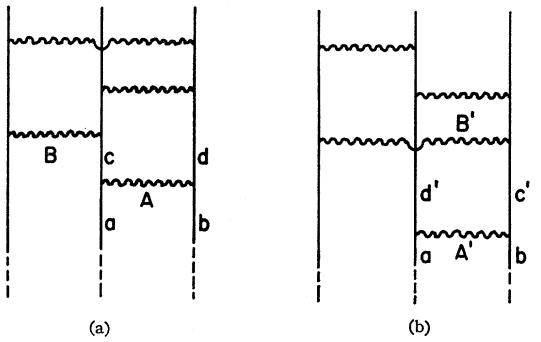


FIG. 14. Two diagrams, identical below the interaction A, but with the roles of particles 2 and 3 interchanged above this interaction. Both diagrams, (a) and (b), lead to the same final state.

cent of the latter, for reasonable values of $k_{FC}$ (0.5 to 0.7). A detailed investigation of this point will be necessary.

So far we have assumed that at least one of the first two nucleons has a final state equal to its initial one. To each such final state, there is another obtained by exchange of final nucleon states $l$ and $m$, provided their spins and isospins are equal. In this case, then, the final spatial wave function of the two nucleons must be taken antisymmetric, i.e., it will contain only odd $L$. Then also, in the initial state, by symmetry, only odd $L$ need be considered. If the spin or isospin of states $l$ and $m$ differ, both even and odd $L$ contribute. This argument leads to the well-known result (BBP Sec. 5) that, for the interaction of $l$ and $m$, we should take

$$(3/4)(\text{even } L)+(5/4)(\text{odd } L). \tag{6.4}$$

This is the same as for their first-order interaction. The odd-$L$ contribution is, of course, small, especially if we consider only central forces which are quite small in odd states.

We may now sum all diagrams belonging to a given order of perturbation theory. For this purpose, consider an arbitrary diagram $D$ [Fig. 14(a)], and within this diagram consider an interaction A between particles 2 and 3 corresponding to the transition [Fig. 14(a)]

$$k_a, k_b \rightarrow k_c, k_d. \tag{6.5}$$

Then there exists always another diagram $D'$, in which all the interactions of particles 2 and 3 subsequent to the interaction A are interchanged; i.e., if nucleon 2 interacts with 1 at a certain level B in diagram $D$, then 3 interacts with 1 at the same level in $D'$, and vice versa [Fig. 14(b)]. Diagram $D'$ is distinct from $D$, and if it ends up with the same final state, i.e., the same permutation of $l, m, n$, it will have the same sign as 14(a). In order to have in $D'$ the same sequence of interactions as in $D$ but interchanged, we should approximately interchange the momenta (6.5) after interaction A—approximately only, because in the *final* state, the momenta of particles 2 and 3 should be the *same* in Fig. 14(b) as in 14(a), they should not be interchanged. Assuming that the final states of particles 2 and 3 are $m, n$, we may take, e.g.,

$$\begin{aligned}\mathbf{k}_d' &= \mathbf{k}_d+\mathbf{m}-\mathbf{n},\\ \mathbf{k}_c' &= \mathbf{k}_c-\mathbf{m}+\mathbf{n}.\end{aligned} \tag{6.6}$$

Since $|\mathbf{k}_m-\mathbf{k}_n|c \ll 1$, the final state in A′ is obtained from that in A *almost* by spatial exchange of the two particles. Note that spins are not exchanged, and the spins and isospins of particles 2 and 3 need not be the same. The contributions of Figs. 14(a) and (b) must be added; this can be done approximately by adding the matrix elements for A and A′, which gives

$$\begin{aligned}&(\text{direct interaction A})+(\text{exchange interaction A}')\\ &\quad=\sum(\text{even } L+\text{odd } L)+\sum(\text{even } L-\text{odd } L)\\ &\quad=2\times(\text{interaction in even } L \text{ states only}).\end{aligned} \tag{6.7}$$

This is the contribution from the two diagrams, Figs. 14(a) and (b). It is most convenient to say that *each* diagram gives simply

$$1\times(\text{contribution from even } L \text{ states only}). \tag{6.8}$$

This may be used in conjunction with our theory of Secs. 3 to 5. It is clear that addition of interactions like A and A′ at *all* levels will give all diagrams of the given order.

If we have attractive Serber forces, these act anyway only in even $L$ states, and for these states they given the same result as an ordinary force. Of the repulsive core, only the even-$L$ part contributes. This is in agreement with the assumption of Brueckner and Gammel,[34] but it is in contrast to BBP who emphasized in their Sec. 8 that the odd-$L$ part of the repulsive core is important. The statistical weight of the even states is 1 in our theory, not $\frac{3}{4}$ as in BBP Fig. 10, so that our core contribution is $\frac{4}{3}$ times that of "even $L$ only" in that figure.

Our result was previously obtained, for the third order of perturbation theory by Rajaraman[14] and used by Razavy.[2] In Rajaraman's paper, the diagrams corresponding to Figs. 14(a) and (b) are the bubble diagram (our Fig. 10) and the ring diagram (our Fig. 7). In both of these, as in Figs. 14(a) and (b), there is no restriction on the spins of the interacting particles. In the bubble interaction, nucleons 2 and 3 retain their momenta, $b$ and $n$, respectively. In the middle interaction of the ring diagram, they approximately exchange it, viz., $b \rightarrow m$ and $n \rightarrow c$.

The bubble and the ring diagram are distinct diagrams, and if we neglect exchange (which has already been taken into account), they are the only such diagrams in third order. This is in agreement with the fact, stated in Sec. 4, that there are $2^{n-2}$ distinct diagrams in $n$th order. Each of these two diagrams may be considered to give a contribution (6.8). But we must still consider the factor $\frac{1}{2}$ from the exchange of particle

[34] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

3 in the final state (see above). Thus we find that, in third order, the two diagrams together give just the contribution (6.8) which is Rajaraman's result.

We must emphasize once more that our simple results depend on two important assumptions, viz.:

(1) The nuclear forces do not depend on spin or isospin, which means charge independence must be fulfilled, and the forces must be the same in triplet and singlet states. In particular, tensor and spin-orbit forces are neglected.

(2) The Fermi momentum and the range of the important interactions are both small, i.e., $k_F c \ll 1$.

## 7. ESTIMATE OF THE THREE-BODY CONTRIBUTION

Using (4.3), (4.5), and (4.6), the three-body contribution to the energy may be written

$$W=\int g(r_{23})F_1(r_{23})f(r_{23})d\tau_{23}. \tag{7.1}$$

In third order, i.e., when only the "bubble" and ring diagrams of Figs. 10 and 7 are included, $W$ is replaced by

$$W_1=\int g(r_{23})F_1(r_{23})d\tau_{23}. \tag{7.2}$$

Thus $gF_1(r)$ represents the interaction at distance $r$ in third order, and this is multiplied by $f(r)$ when the complete three-body interaction is taken into account.

We have shown in Sec. 6 that the proper way to treat the third order is that of Rajaraman,[14] i.e., to consider both bubble and ring diagram and the associated exchange diagrams Only in this way do the higher orders join smoothly to the third; this would not be the case if we used only the bubble diagram in third order. Now $g(r)F_1(r)$ is just the third-order interaction including both bubble and ring; therefore, $f(r)$ is the correct factor to go from third order to the complete interaction.

The only calculation to third order which follows the prescription of Rajaraman,[14] is that of Razavy.[2] He used the potential of Hamada and Johnston, with a repulsive core $c=0.49$ F, and found a separation parameter $d_2$ varying with relative momentum $k_0$ from 1.08 to 1.20 F (for the $^1S$ state, Razavy's Table I). This corresponds reasonably well to Case (B) of Sec. 5, i.e., $d_2/c=2.2$ (2.3 would have been better). From our Table I we see that $f$ varies from 0.355 to 0.389 in the core, and then rapidly increases to 1.

Razavy's calculations do not explicitly give $g(r)F_1(r)$ and thus cannot be used directly in (7.1). He, as well as other authors,[1,13] give instead the potential energy of particle states ($k>k_F$). BBP[13] separate this energy into the contribution from the core (volume and surface term) and from the outer potential. The same separation can be made for Razavy, since his core contribution is just that from the states of even $L$ which is given in BBP Eq. (8.4), except that (8.4) should be multiplied by $\frac{4}{3}$ [see the paragraph below our Eq. (6.8)].

Since the third-order contribution is only separated into a core and an outer part, we should use in (7.1) an average of $f(r)$ over the core, and then an average over the outside. The former is easy to take, since $f$ does not change very much. In fact, more than half of the core contribution comes from the surface and should thus be multiplied by $f(c)$; the rest is a straight average over the volume. The over-all average is about 0.384. For the outside $f$ changes from 0.39 to 1, so that the average is more difficult to take. We need to know the weight factor as a function of $r$. We use the first modified Born approximation of BBP, Sec. 5. According to their (5.36), the first approximation to the radial wave function is

$$u_L=\mathcal{J}_L-\mathcal{K}_L \tag{7.3}$$

(for notation see BBP). We insert this into BBP (5.35) and make the approximation

$$\mathcal{K}_L/\mathcal{J}_L=(c/r)e^{-\gamma(r-c)}. \tag{7.4}$$

This is correct for $L=0$ and $k_0=0$; we assume it for all $L$ and general $k_0$. We take the statistical weight

$$\begin{aligned}\nu(L,S)&=1, \text{ if } L \text{ is even}\\ &=0, \text{ if } L \text{ is odd},\end{aligned} \tag{7.5}$$

in accord with our Sec. 6. Then we can sum over $L$, using BBP (5.30). Then BBP (5.35) becomes

$$W_1=4\pi\int g(r)F_1(r)dr, \tag{7.6}$$

with

$$g(r)F_1(r)=v(r)[1+(\sin 2k_0r/2k_0r)]\times[1-(c/r)e^{-\gamma(r-c)}]^2. \tag{7.7}$$

This is then used to get the ratio $f_{\text{av}}$ of (7.1) to (7.2). It is interesting that $gF_1$ is not large for $r$ slightly greater than $c$, due to the last factor in (7.7); this is true, in spite of the fact that $v(r)$ is very large there. Therefore, the "weight" $gF_1$ is mostly at larger $r$, and the average of $f$ over the attractive force is nearly 1. Explicit calculation by T. Dahlblom gave

$$\begin{aligned}\langle f(r)\rangle_{\text{av, outer pot.}}&=0.86,\\ \langle f(r)\rangle_{\text{av, core}}&=0.384.\end{aligned} \tag{7.8}$$

The other difference between our treatment until now and that of Razavy is that we calculate the total energy due to three-body correlations, while he evaluates the potential energy of "particle" states, i.e., states $k_b$ outside the Fermi sea. Since the interaction of a particle $k_b$ with the states $n$ in the Fermi sea does not depend much on the momentum of the latter, we may write for the potential energy

$$U(k_b)\approx\rho G(k_b), \tag{7.9}$$

where $G(k_b)$ is the interaction of $k_b$ with an average state $n$ in the sea. The latter may then be written

$$G(k_b)=\int g(k_b,r_{23})d\tau_{23}, \tag{7.10}$$

where we have put in evidence the fact that $g$ depends on $k_b$, while in our Sec. 3 we assumed $k_b=4$ F$^{-1}$ and therefore simply wrote $g(r_{23})$. Otherwise, (7.10) differs from (7.2) only by the absence of the factor $F_1$ which, by (4.4), is related to $\eta(r_{12})$. Now the probability for a particle 2 to get into the intermediate state of momentum $k_b$ is proportional to the square of the Fourier transform of $\eta$,

$$(k_b|1-\Omega|m)\sim\int\eta(r_{12})e^{-i\mathbf{k}_b\cdot\mathbf{r}_{12}}d\tau_{12}. \tag{7.11}$$

Therefore

$$\int(k_b|1-\Omega|m)^2d^3k_b\sim\int\eta^2(r_{12})d\tau_{12}=F_0; \tag{7.12}$$

and we find, using (7.9) and (7.10),

$$\int(k_b|1-\Omega|m)^2U_b(k_b)d^3k_b\sim\int g(k_{b\,\mathrm{av}},r_{23})F_0d\tau_{23}. \tag{7.13}$$

This calculation corresponds to taking into account only the bubble diagram, hence the subscript $b$ (bubble) on $U$. The left-hand side of (7.13) corresponds to the procedure of Razavy and other earlier work, the right-hand side to our calculation. The left-hand side is the first-order contribution of the potential energy of particle states $k_b$ to the energy of the whole nucleus; it is sufficient to use the first order, because the total contribution from three-body correlations will turn out to be small, and hence its square can be neglected. [Of course, (7.13) is also the third-order Goldstone contribution to the nuclear energy.] If the ring diagram is included, (7.13) should be replaced by

$$\int(k_b|1-\Omega|m)^2U(k_b)d^3k_b\sim\int g(k_{b\,\mathrm{av}},r_{23})F_1d\tau_{23}, \tag{7.14}$$

where $U$ denotes the total potential energy of state $k_b$ (bubble plus ring diagram), which is the quantity calculated by Razavy.

We have thus established the equivalence of the Razavy procedure and ours. His gives more information because it also gives the potential energy of particle states. Although this quantity has no clear physical significance and is only a calculational device, it is still useful for a picture of the physical situation, and for comparison with earlier work. Moreover, if the energy $W$ in (7.1), and $U(k_b)$ in (7.9), turn out not to be small in some problem, it is probably somewhat more accurate to use $U(k_b)$ in the standard way by considering it as part of the energy $E(k_b)$ of the particle state $k_b$. This takes into account some of the interactions of more than three particles, cf. Sec. 9.

Therefore we take Razavy's particle energies and correct them, according to (7.8), as follows:

$$U=0.384U_{R\,\mathrm{core}}+0.86U_{R\,\mathrm{outer}}, \tag{7.15}$$

where the subscript $R$ stands for Razavy. As already mentioned, the core contribution is calculated as $\frac{4}{3}$ times BBP Eq. (8.4), with

$$k_F=1.3\text{ F}^{-1},\quad c=0.49\text{ F},\quad k_Fc=0.637. \tag{7.16}$$

($k_F=1.3$ is chosen, rather than 1.36, because Razavy gives explicit results for 1.3.) To find the outer contribution, we took Razavy's value for the total potential at $k=4$ F$^{-1}$, $k_F=1.3$, which is +49 MeV. Since $U_{R\,\mathrm{core}}(k=4\text{ F}^{-1})=140$ MeV (with Razavy's value $m^*=0.90$), it follows that $U_{R\,\mathrm{outer}}=-91$ MeV at this value of $k$. Now it is reasonable to assume that $U_{\mathrm{outer}}$ is nearly independent of $k$; this is true in BBP Table II to within 3 MeV for $k>3$ F$^{-1}$. Then the momentum dependence of Razavy's potential is entirely due to the core, and this can be shown to give $m^*=0.93$, slightly higher than Razavy's value of 0.90. With this, we obtain the results in Table II. (The outer contribution for $k=2$ F$^{-1}$ was estimated to be 10 MeV larger than for $k=4$ F$^{-1}$, in analogy with BBP Table II. For $k=0$, we have inserted the usual $U_{R\,\mathrm{outer}}=-91$, for future use.)

The "Razavy total" does not agree with Razavy's own figures because (1) we find a slightly different effective mass $m^*$, and (2) Razavy assumed a quadratic dependence of $U$ on $k^2$, while we take the outer contribution constant. In any case, the Razavy total is positive (repulsive). Our new calculation, using (7.15), gives a negative (attractive) potential ("new total") throughout. The last line in Table II gives a potential energy which is derived from that of Brueckner and Masterson (BM). These authors give $U(k)$ for $r_0=1.00$ F, which means $k_F=1.524$ F$^{-1}$. To find $U$ for our case, we assume that it is proportional to the density, i.e., that the average $G$ is independent of $\rho$, cf. (7.9). This is exactly true for the first modified Born approximation (MBA) (see Ref. 13, p. 238) for the outer (attractive) potential and approximately for the higher MBA's which anyway give a small contribution (BBP Table II). The core contribution to $G$, BBP Eq. (8.4), depends on $k_0=\frac{1}{2}k_b$ and on $\gamma$; the former is independent of $\rho$; the latter,

TABLE II. Potential energy of particle states, in MeV ($k_F=1.3$ F$^{-1}$).

| | | | | | | |
|---|---|---|---|---|---|---|
| 1. $k_b$ (F$^{-1}$) | 0 | 2 | 3 | 4 | 5 | 6 |
| 2. Razavy core | 91 | 101 | 119 | 135 | 145 | 154 |
| 3. Razavy outer | −91 | −101 | −91 | −91 | −91 | −91 |
| 4. Razavy total | 0 | 0 | 28 | 46 | 54 | 63 |
| 5. New total | | −48 | −32.5 | −26 | −22.5 | −19 |
| 6. Reference approx. | | −33.5 | −30.5 | −26.5 | −21 | −14.5 |
| 7. Wong | | −30 | −29 | −27.5 | −25.5 | −23 |
| 8. Brueckner and Masterson | | −43 | −22 | −7.5 | 0 | 0 |

according to (3.10), has two terms, of which the larger is again $3k_0^2$ and the smaller (10–20% of the total) is proportional to $k_F^2$; only this last term depends on density. Hence it should be a good approximation to assume that $U(k)$ for *particle* states is proportional to $\rho$; we shall assume this for BM, for our new total, and later (Sec. 8) for the Wong potential; for hole states this approximation is not good, because the MBA does not converge well.

Table II shows that our new total is *lower* than the BM potential energy, by amounts varying from 5 to over 20 MeV. This is in contrast to the old "Razavy total," which was *higher* than BM by about 50 MeV. The effect of $U$ on the nuclear binding energy is mainly due to the "dispersion effect" of the Moszkowski-Scott theory, which gives [see BBP Eq. (A14) and Eq. (8.4) of this paper]

$$E_U=\tfrac{3}{4}[U(k')_{\rm av}-\bar{U}]\rho\int\eta\eta_0 d\tau. \tag{7.17}$$

Here $\bar{U}$ is the average potential energy of a nucleon in the sea, $U(k')_{\rm av}$ is that of an average excited state, $\eta$ is the reference-spectrum "wave function defect," as in Sec. 3, and $\eta_0=\phi-\psi_0$ is the corresponding defect for two free nucleons interacting only by their short-range forces. We make the approximation $\eta_0=\eta$ and take for $U(k')$ the average over the region $k'=3$ to 5 F$^{-1}$, which is the most important intermediate momentum range for the short-range forces, cf. (3.12). Then, changing $U(k')$ from one theory to another gives a change of binding energy:

$$\Delta E=\tfrac{3}{4}\Delta U(k')\rho\int\eta^2 d\tau. \tag{7.18}$$

The change from BM to "new Razavy" is $\Delta U\approx-18$ MeV, on the average, for $k_F=1.3$ F$^{-1}$. The value of $\int\eta^2 d\tau$ for the Breit potential has been calculated by Wong and is 0.87 F$^3$, Eq. (8.2). For $k_F=1.3$ F$^{-1}$, we have $r_0=1.17$ F, $\rho=0.148$ F$^{-3}$ and (7.18) becomes

$$\Delta E=-1.74\text{ MeV}\quad(r_0=1.17\text{ F}). \tag{7.19}$$

According to our discussion, $U$ is about proportional to the density, hence $\Delta E$ to $\rho^2$. At $r_0=1.28$ F, the minimum of the BM energy curve for the Breit potential, we have then $\Delta E=-1.01$ MeV. The BM binding energy for this $r_0$ is thus increased from 8.3 to 9.3 MeV. The actual binding energy will be slightly greater, perhaps 9.5 MeV.

This is in contrast to the theory of Brown, Schappert, and Wong.[6] They used the potential $U$ of BBP, which is essentially that of Razavy, so that $\Delta U=+50$ MeV, according to Table II. Then $\Delta E\approx+5$ MeV for $r_0=1.17$ F and $+3$ MeV for $r_0=1.28$ F, which are the results of Brown *et al.* As they point out, the binding energy is reduced to about 5 MeV or less.

We return to a discussion of our new particle-state potential, Table II. We may fit this potential by a reference spectrum. The best fit to the region from $k=3$ to 5 F$^{-1}$ is

$$U_R=-36+0.6k^2. \tag{7.20}$$

Line 6 in Table II gives the reference potential energy calculated from (7.20); the agreement with the "new total" potential (line 5) is fair. The coefficient of $k^2$ is very small and corresponds to an effective mass

$$m^*=0.97. \tag{7.21}$$

This is remarkably close to unity, in spite of the large core size $c=0.49$ F. This is due (1) to the reduction of the core effect by the factor $f=0.384$, Eq. (7.15), (2) to the elimination of the interaction in states of odd $L$, cf. BBP Sec. 8 and Fig. 10, and (3) to the generally smaller third-order interaction due to Rajaraman's prescription, cf. Sec. 6 of this paper.

The constant $\Delta$ of BBP Eq. (7.3) may best be defined by

$$U_R(k_{\rm av})-\bar{U}=(\hbar^2k_F^2/Mm^*)\Delta, \tag{7.22}$$

where $k_{\rm av}=(0.6)^{1/2}k_F=1.05$ F$^{-1}$ is the average momentum of a nucleon in the Fermi sea. $\bar{U}$ is the actual average potential energy. For the equilibrium density, $k_F=1.36$ F$^{-1}$, this can be obtained from Weisskopf's argument[35] ($\bar{T}$=average kinetic energy, $B$=binding energy):

$$\begin{aligned}&\bar{T}+\tfrac{1}{2}\bar{U}=-B,\\ &B=16\text{ MeV},\\ &\bar{T}=0.3\hbar^2k_F^2M^{-1}=23.0\text{ MeV},\\ &\bar{U}=-78\text{ MeV}.\end{aligned} \tag{7.23}$$

We assume that $B$ is not very different for $k_F=1.30$ F$^{-1}$. The reference energy for $k_{\rm av}$ is $U_R=-35.3$ MeV. With $m^*=0.97$ this gives

$$\Delta=\left[\frac{-35.3+2B}{\hbar^2k_F^2/M}+0.60\right]m^*=0.535. \tag{7.24}$$

This is substantially lower than the value $\Delta=0.75$ obtained by BBP from their Eq. (8.24). This is to be expected, because the potential energy of particle states has been very much lowered. The value (7.24) of $\Delta$ is just sufficient to make $\gamma^2$ for hole states always positive; according to BBP (7.7),

$$\gamma_m^2=2k_F^2\Delta-k_0^2\geqslant k_F^2(2\Delta-1). \tag{7.25}$$

Since $m^*\approx1$, we may, approximately, replace the potential energy for particle states by a constant. Since $k=4$ F$^{-1}$ is the most important momentum, we choose the value of $U$ for this momentum: This is $-26$ MeV, according to Table II.

This approximation, $U(k)=$const for $k>k_F$, is about as simple as the assumption $U(k)=0$, which has often been used. Martin and de Dominicis[36] used it in their

[35] V. Weisskopf, Nucl. Phys. **3**, 423 (1957).
[36] P. C. Martin and C. de Dominicis, Phys. Rev. **105**, 1417 (1957).

early work chiefly for reasons of simplicity. Puff[37] and Mohling[38] made the same assumption because it fitted into their quite different formalism. The only added problem in our theory is that the value of $U$ has to be calculated for one representative $k$, let us say 4 F$^{-1}$, but this is very easy because modified Born approximation converges well.

The hole states, of course, have their usual potential energy. As Brueckner and Goldman[39] first made plausible, and BBP proved, the potential energy of the hole states is obtained as a sum over the appropriate $G$ matrices *on* the energy shell. This means that the interaction of two hole states is taken without regard to any possible third hole state which may interact with one of the first two; the potential energy of holes is a true two-body interaction, like the main part of the potential energy of the entire nucleus. No third particle needs to be close, therefore the interaction is not cut down by the factor $f(r_{23})$, Eq. (4.6), which comes from the proximity of the third particle to the first two. Thus the fact that the hole interaction must be calculated on the energy shell makes a difference in principle, and is not merely a matter of convenience.

The fact that the potential energy of hole states is not cut down by a large factor[40] is essential (1) in order to provide stability of the calculation by means of the self-consistency requirement which has often been emphasized by Brueckner and co-workers; (2) to obtain a gap between hole and particle single-particle energies, which prevents formal difficulties from spurious BCS phenomena[41]; and (3) to obtain saturation of nuclear forces.

## 8. SOFT CORE

We have shown, after Eq. (7.19), that the binding energy in the Breit potential is about 9.5 MeV, still short of the observed 16 MeV. To obtain a larger binding energy, it seems necessary to change the nucleon-nucleon interaction. The most promising modification is the reduction of the influence of the repulsive core, suggested by Wong.[4,42] He replaces Breit's hard core of $c=0.51$ F by a softer core, of Yukawa shape. In particular, Wong has considered a combination of three Yukawa potentials, that of longest range being simply the one-pion-exchange potential, another attractive one of intermediate range representing the exchange of two pions between the nucleons, and a short-range repulsive Yukawa potential assumed to represent the exchange of vector mesons,[5] chiefly $\omega$. The reciprocal range of the repulsive term was chosen to be 5 F$^{-1}$, in rough accord with the $\omega$ mass. The strength of this potential, and both strength and range of the intermediate attractive potential, were chosen arbitrarily to fit the experimental data. With these three arbitrary constants, Wong gets a good fit to the $^1S$ phase shifts of Moravcsik.[43] Now Wong's potential greatly reduces the wave function defect $\eta$ for an average pair of nucleons. Using $k_F=1.36$ F$^{-1}$ and a relative momentum of the two nucleons $k_0=0.55k_F=0.75$ F$^{-1}$, Wong finds[44]

$$1-z\equiv\rho\int_0^d\eta^2 d\tau=0.147 \text{ for Breit's potential} \\ =0.054 \text{ for Wong's potential}. \tag{8.1}$$

Here, $d$ is the Moszkowski-Scott separation distance, and the expression (8.1) is just the probability of finding a nucleon temporarily in a state $k>k_F$ rather than in the Fermi sea, due to short-range forces. This probability is thus now only about 6%, or the effective occupation number[45] $z=0.94$. The "effective volume" of the "wound" in the wave function[46] is

$$\int\eta^2 d\tau=0.87 \text{ F}^3 \quad \text{(Breit)} \\ =0.32 \text{ F}^3 \quad \text{(Wong)}. \tag{8.2}$$

(These values are appreciably less than those at $k_0=0$ because the relative momentum of the two nucleons was taken to be $0.55k_F=0.75$ F$^{-1}$.)

The reduction of the size of the "wound," according to Wong,[42] is only to a small extent due to the use of a Yukawa core instead of a hard core. The main effect is due to a reduction of the effective radius of the core. Breit's original[16] potential fits the observed $^1S$ phase shift quite well up to about 200 MeV but gives too much repulsion[42,47] at 300 and especially at 400 MeV. Since these energies are important for the determination o,

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[37] R. D. Puff, Ann. Phys. (N.Y.) **13**, 317 (1961).
[38] F. Mohling, Phys. Rev. **122**, 1043, 1062 (1961); **124**, 583 (1961); **128**, 1365 (1962).
[39] K. A. Brueckner and D. T. Goldman, Phys. Rev. **117**, 207 (1960).
[40] The potential may be cut down by a factor $z(k)$ representing the probability that the state $k<k_F$ is actually occupied which is about 0.85 [or, if Wong's soft core is used, 0.94, Eq. (8.1)]. This factor appears in the shell-model potential for finite nuclei, according to Brandow, Ref. 15. Whether it should also be used in nuclear matter is not clear to me, but Brandow has shown that it does not make any appreciable difference for the total binding energy of nuclear matter whether the factor is used or not because there are compensations. In any case, the factor is not very different from one, unlike the factor of 0.37 in the three-body interaction.
[41] J. S. Bell and E. J. Squires, Advan. Phys. **10**, 211 (1961).
[42] C. W. Wong, Nucl. Phys. (to be published).
[43] M. J. Moravcsik, *The Two-Nucleon Interaction* (Clarendon Press, Oxford, 1963).
[44] We use an average of Wong's potentials $V_{pp}^2$ and $V_{np}^2$. The relevant constants are given in his Table V. His $D$ is our $1-z$, our $\Delta E_l$ is twice his because he considers only the effect of the $^1S$ state, while we assume that of the $^3S$ state to be equally large.
[45] Actually, to calculate $z$ we should also take into account the tensor forces which also raise particles from the Fermi sea to states above it. The correct $z$ is probably $<0.9$.
[46] M. A. Preston and R. K. Bhaduri, Phys. Letters **6**, 193 (1963), have already suggested that the size of the "wound" determines the saturation properties of the potential.
[47] This is evident from detailed curves sent to me kindly by Professor Breit.

the core radius $c$, a smaller core radius, about $c=0.45$ F, will improve the fit. The Moravcsik[43] phase shifts are generally more attractive than those of Breit's group. The soft Yukawa core has the advantage of giving relatively less repulsion at higher energy than the hard core, in agreement with observation.

We can now calculate how BM's nuclear matter binding energy must be modified if we change the nucleon-nucleon interaction from Breit's hard core to Wong's soft one. This is most easily done by using the Moszkowski-Scott (MS) method. According to this, the two main contributions to the energy are (1) the effect of the long-range potential ($r>d$), which may be calculated in Born approximation, and (2) the "dispersion effect" due to the short-range forces, $r<d$ ($d$=separation distance). (The other MS terms are small enough so that we may assume that they are unchanged by the change of the interaction.) Wong has calculated the Born integral for the long-range forces, taking into account only the $S$ states, and finds[44]

$$\Delta E_l=-1.45 \text{ MeV/particle} \tag{8.3}$$

for $k_F=1.36$ F$^{-1}$, meaning that Wong's long-range forces are slightly *more* attractive than Breit's. To calculate the effect for other values of $k_F$, we assumed that $\Delta E_l$ is proportional to the nuclear density, which should be a good approximation. The dispersion effect of MS is given by

$$E_s=\tfrac{3}{8}\times 2[U(k_{\text{av}}')-\bar{U}]\rho\int\eta^2 d\tau, \tag{8.4}$$

where $U(k_{\text{av}}')$ is the potential energy for an average excited state (let us say $k'=4$ F$^{-1}$) and $\bar{U}$ the average potential energy in the Fermi sea. The factor 2 in (8.4) comes from the excitation of two nucleons, while $\frac{3}{8}$ is the sum of the statistical weights of the $^1S$ and $^3S$ states, it being assumed arbitrarily that their $\eta$'s are equal.

To compare the Breit and the Wong potential, we must calculate (8.4) for both of them. The integral has already been given in (8.2). Because the integral is larger for the BM calculation, we must know the factor $U(k_{\text{av}}')-\bar{U}$ more accurately in this case. In agreement with the discussion in Sec. 7, we take

$$U(k_{\text{av}}')_{\text{BM}}=-(8.5 \text{ MeV})(1.17/r_0)^3. \tag{8.5}$$

To calculate $\bar{U}$, we used the first Eq. (7.22). For consistency, $B$ should be that binding energy (B.E.) which results from the calculations, not the observed B.E. This binding energy was read from the curves and tables of BM; it is, of course, a function of $r_0$. The kinetic energy $T(r_0)$ was directly calculated.

$U(k_{\text{av}}')$ for the Wong potential was estimated crudely as follows: First, an effective core radius for the Wong potential was estimated. For this purpose, we calculated $\int\eta^2 d\tau$ with the simple expression (5.2) for $\eta$ and found

$$\int\eta_s^2 d\tau=\frac{4\pi}{15}c[d^2+dc+3c^2]+\frac{4\pi}{105}(d-c)^3 \tag{8.6}$$

($s$ for simplified). For Breit's potential, $c=0.51$, $d=1.10$, $\int\eta_s^2 d\tau=1.11$. Comparing with Wong's result (8.2),

$$\int\eta_s^2 d\tau\Big/\int\eta^2 d\tau=1.28. \tag{8.7}$$

Assuming the same *ratio* to hold for the Wong potential, we find $\int\eta_s^2 d\tau=0.41$ for this potential. Wong gives $d=1.06$ F for his potential, then (8.6) gives $c=0.23$ F. This seemed unreasonably small, so we chose arbitrarily $c=0.30$ F. With this value and BBP Eq. (8.4), we then calculated the full core contribution to $U(k)$, and found it to be 56 MeV at $k=0$. We then assumed that the total $U(k=0)$ has the same value as for the Razavy calculation, which happens to be zero (Table II); therefore we assumed $U_{\text{outer}}=-56$ MeV, for the Wong potential. Then, similarly to (7.15), we took the three-body interaction into account by writing

$$U_{\text{Wong}}(k)=0.384U_{\text{core}}+0.90U_{\text{outer}}. \tag{8.8}$$

The coefficient of $U_{\text{outer}}$ was chosen larger than in (7.15) because, due to the small size of the Wong core, the main contribution from the outer potential comes from larger values of $r/c$ than for the Hamada-Johnston (HJ) potential, hence $f_{\text{av}}$ is closer to one. With these assumptions we obtained, for $k_F=1.3$, the values of $U$ listed in Line 7 of Table II. For $k_F\gtrsim 4$ F$^{-1}$, these are still lower than for the HJ potential, Line 5 of Table II.

A reference spectrum may be fitted to the Wong potential energies:

$$U^R(\text{Wong})=-31+0.22k^2. \tag{8.9}$$

This corresponds to an effective mass

$$m^*=0.990. \tag{8.10}$$

This reference spectrum fits $U(k)$ of Line 7, Table II, within $\frac{1}{2}$ MeV for all $k$. The dependence of (8.9) on $k$ can now safely be neglected, and the potential energy of excited states replaced by its value for $k=4$ F$^{-1}$, viz., $-27.5$ MeV. As in Sec. 7, we assume that $U$ is proportional to density, so that

$$U(k_{\text{av}}')_{\text{Wong}}=-27.5(1.17/r_0)^3. \tag{8.11}$$

Table III gives the steps in the calculation of the binding energy. The values of $r_0$ were chosen to make it easy to read BM's binding energy from their table and figure (second line of Table III). The average potential energy in the Fermi sea, $\bar{U}$, changes slowly with $r_0$; it is given for the BM calculation in the third line, and for the Wong potential in the fourth (the latter can only be obtained after the binding energy is calculated, so an iteration is necessary but easy). The average

TABLE III. Binding energy with soft core (MeV).

| | | | | | |
|---|---|---|---|---|---|
| 1. $r_0$ (F) | 1.00 | 1.12 | 1.17 | 1.28 | 1.40 |
| 2. BM binding energy | 0.3 | | 6.8 | 8.3 | 7.1 |
| 3. $-\bar{U}$ (BM) | 59 | | 56 | 52 | 43 |
| 4. $-\bar{U}$ (Wong) | 79 | 72 | 69 | 62 | 52 |
| 5. $-U(k_{av}')$ (BM) | 13.6 | | 8.5 | 6.5 | 5.0 |
| 6. $-U(k_{av}')$ (Wong) | 44 | | 22.5 | 20.9 | 16.0 |
| 7. $E_s$ (BM) | 7.08 | | 4.64 | 3.37 | 2.16 |
| 8. $E_s$ (Wong) | 2.00 | | 1.66 | 1.09 | 0.73 |
| 9. $-\Delta E_l$ | 2.04 | | 1.27 | 0.97 | 0.74 |
| 10. Wong binding energy | 7.4 | | 11.0 | 11.5 | 9.2 |
| 11. $E_c$ (Wong) | 2.10 | 1.06 | 0.82 | 0.48 | 0.28 |
| 12. $2E_c/\bar{U}$ (%) | 5.3 | 2.9 | 2.4 | 1.5 | 1.05 |
| 13. $1-z$ (%) | 7.6 | 5.4 | 4.7 | 3.6 | 2.7 |

potential energy for intermediate states, $U(k_{av}')$, is given for the BM calculation [Line 5, Eq. (8.5)] and for Wong's potential [Line 6, Eq. (8.11)]. The effect of short-range forces, (8.4), is then listed for BM (Line 7) and for Wong (Line 8). The correction for long-range forces, derived from (8.3), is listed in Line 9. The Wong energy per nucleon is then the BM energy, minus $E_s$ for BM forces, plus $E_s$ for Wong forces, plus the correction for long-range forces. The resulting Wong binding energy is listed in Line 10.

Since both short- and long-range corrections are attractive, the binding energy is substantially increased. Since both corrections increase with density, the maximum binding energy is shifted to larger density; but the shift is small, the maximum lies now at about $r_0=1.25$ F instead of BM's value of 1.28 F, still far from the observed value of 1.12 F. The binding energy itself is about 11.6 MeV, a significant improvement over BM's value of 8.3, but still far short of the observed 16 MeV.

It is somewhat surprising that the improvement over the calculation of Sec. 7 is only 2.3 MeV at $r_0=1.28$ F. Moreover, of this amount, nearly 1.0 MeV are due to the long-range forces, Line 9. The softening of the core therefore contributes only 1.3 MeV, after the correction for the three-body correlations has been applied as in Sec. 7.

The compression modulus is

$$-r_0^2(d^2B/dr_0^2)=250 \text{ MeV}. \quad (8.12)$$

This is even larger than the 190 MeV of Brueckner and Gammel[34] and disagrees badly with the empirical value, <50 MeV, quoted by Ford and Hill.[48]

Obviously, our estimates are extremely rough. A completely new calculation should be made, starting from the best two-nucleon potential, including noncentral forces, and taking into account the three-body theory of this paper. Perhaps a recalculation of the tensor forces will improve the agreement: Since the energy of particle states $E(k)$ is now lower than in the BM and the Razavy calculation, the tensor force which acts only in second order should be more effective. Wong[42] shows that the $D$ states should give more binding, because the "quadratic spin-orbit" potential is reduced. Azziz and Signell[49] have pointed out that the $^1P$ state gives a large repulsive contribution (7.1 MeV) to the binding energy in Razavy's calculations, and that the potentials of HJ and of Breit may be too repulsive in this state. Wong suggests that also other odd states may be less repulsive than is now believed. It is certainly possible that an accumulation of all these corrections may bring about agreement with the observed binding energy, but only a complete calculation can show.

Finally, we want to estimate the effect of the three-particle diagrams on the binding energy. For this purpose, it is again useful to divide the nuclear force into a short- and a long-range part. The long-range part, being weak, can be treated by ordinary Goldstone perturbation theory, which is known to converge very rapidly; then the three-body effect will be given almost completely by the third order which should be quite small (1–2 MeV). Therefore we need only consider the contribution of the short-range forces. We exaggerate this contribution by taking only the repulsive core. The contribution of the core to the particle state potential can either be calculated from BBP (8.4), or more conveniently deduced from the total $U(k)$ for the Wong potential, Table II, Line 7, by means of (8.8): $0.90U_{outer}=-50.4$ MeV, so that the desired core contribution is

$$U_c=U(k)+50.4=+23 \text{ MeV} \quad \text{for} \quad r_0=1.17. \quad (8.13)$$

Using (8.4), we can then calculate the contribution of $U_c$ to the energy per particle, $E_c$, which is listed in Line 11 of Table III. This $E_c$ may be considered as the three-body contribution to the nuclear energy, arising from the repulsive core. It varies strongly with $r_0$, from 2.1 to 0.28 MeV. It is usually smaller than $E_s$: The two quantities are different because $E_s$ is proportional to $U(k_{av}')-\bar{U}=U_c+U_{outer}(k')-\bar{U}$, and $U_{outer}-\bar{U}$ is usually positive but may have either sign.

$E_c$ may be compared with the two-body contribution to the potential energy which is given by $\frac{1}{2}\bar{U}$. The ratio is given in Line 12 of Table III. The ratio is generally about one-half of $1-z$, cf. (8.3) and Line 13 of Table III; at the observed density, $r_0=1.12$, the ratio is 2.9%. We may then expect that the four-body interaction will again be about 2.9% of the three-body contribution, which would make it 0.03 MeV, extremely small. This will be further discussed in Sec. 9.

If the short-range, attractive force is included with the core, $U_c$ and $E_c$ will be reduced to about one-half, so that the three-body short-range contribution is then only 0.6 MeV, at the observed density. The four-body contribution, estimated as before, is then 0.008 MeV, totally negligible. Thus we have shown that the three-body (and even more the four-body) diagrams are

[48] K. W. Ford and D. L. Hill, Ann. Rev. Nucl. Sci. **5**, 46 (1955).

[49] N. Azziz and P. Signell, Nucl. Phys. **59**, 444 (1964) (quoted in Ref. 42).

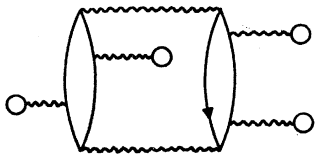

FIG. 15. A higher order bubble interaction.

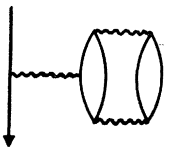

FIG. 16. Main contribution to Brandow's saturation potential.

small compared with the two-body ones. The Goldstone expansion, if reordered in terms of the number of interacting particles, thus appears to converge very rapidly. This is very satisfactory, and shows that nuclear matter is indeed a "low-density" system. It gives confidence in the use of the Goldstone method, as modified in this paper. It also encourages a new look at the problem of liquid $He^3$, where the ratio of core radius to $r_0$ is much larger, of order one.

## 9. NECESSARY IMPROVEMENTS

The theory developed here is obviously very rough. The integral (7.1) should actually be found by numerical calculation. Then the entire nuclear matter calculation should be repeated.

The calculation must also be done with $\zeta$ and $\eta$ appropriate for a soft core. Nonvanishing momenta must be taken for the three particles. The actual $\zeta$ depends on $k$, so that $1-\Omega$ in (3.8) is an operator rather than a function; the effect of this fact must be investigated.

We have treated spin sums very superficially. If the forces depend on spin, and even more in the presence of tensor forces, the three-body problem is likely to be much more complicated.

Another question is whether we should still express three-particle diagrams in the form of potential energies of "particle" states. This concept was originally introduced so that two or more successive interactions, as in Fig. 15, could be conveniently treated: If the bubbles are considered as inserts into particle lines, any number of such inserts are simultaneously treated. However, with our treatment of three-body interactions, particles 1, 2, and 3 get inextricably mixed up, so that we can no longer easily consider the three-body diagrams as inserts into one or the other particle line.

In exploring this question, we should distinguish long- and short-range interactions. The long-range interaction of particle 3 in Fig. 10 is not appreciably affected by the higher order diagrams; in the language of Eqs. (4.4) and (4.5), we have $F(r)\approx F_1(r)$ if $r>d$. Therefore, for the long-range interaction, we may use inserts as usual. For the short-range interaction, this is probably not legitimate, but it is better to calculate four-body, short-range diagrams on their own merit. However, as we pointed out at the end of Sec. 8, these diagrams are quite unimportant (probably less than 0.1 MeV per particle). It is therefore probably a fair approximation to treat interactions like those of Fig. 10 as inserts into particle lines, as has been customary.

Brandow[15] has emphasized the importance of defining the potential on a particle in a *finite* nucleus, for the purpose of obtaining the wave function. This is, however, important only for the filled orbitals (hole states), and thus presents no difficulty in our theory. Also, the main part of Brandow's saturation potential, Fig. 16, can still be defined and calculated, but it is greatly diminished for Yukawa cores, because the probability of finding a state $k<k_F$ empty is now only 6%, if we take just the short-range forces. Tensor forces will add to this, but the total will be considerably less than the 20% of Brandow's theory. Whether the resulting saturation potential will be strong enough is not clear at present.

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