Nuclear-Matrix Elements in First-Forbidden Beta Decay—Rb^{86} and Rb^{84} †

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Theoretical formulas that include finite-nuclear-size corrections and higher-order matrix elements are given for the analysis of first-forbidden beta transitions. The formulas are arranged so that the contribution of the higher-order matrix elements can be seen clearly. Examples are given that illustrate when the higher-order terms must be included and when they can be neglected. The procedure for determining nuclear-matrix elements with a computer is discussed. Particular attention is given to the problem of setting limits of error. Nuclear-matrix elements for Rb^{86} and Rb^{84} are discussed. The results show that the B_{ij} matrix element dominates in both transitions. The agreement with the predictions of conserved-vector-current theory is good. Further experiments are suggested which would set better limits on the nuclear-matrix elements.

I. INTRODUCTION

THIS is the first paper in a series that will report on a systematic analysis of the nuclear-matrix elements of first-forbidden beta transitions. When matrix elements are known for many nuclei, systematic trends should be evident which will suggest improved nuclear models. The matrix elements are also useful to test predictions based on the conserved-vector-current (CVC) theory. $_{37}Rb_{49}^{86}$ is an interesting nucleus to study because it has one neutron hole in the 50 shell. The shell model predicts that the beta transition will be dominated by the B_{ij} matrix element. The same prediction applies to $_{37}Rb_{47}^{84}$ except that in this case there are three neutron holes and it is less likely that the prediction will be correct. Since there are many nuclear-matrix elements to be determined, it is necessary to use as many experimental parameters in the analysis as possible. The energy dependence of the β-γ directional correlation, the angular dependence of the β-γ circular polarization, and the shape-correction factor were used for the analysis reported in this paper.

The analysis was performed on an IBM-7094 computer using formulas given by Bühring.[1] These formulas include the third-forbidden matrix elements and corrections for the finite size of the nucleus. In addition to the usual six matrix elements for a first-forbidden transition, there are a number of higher-order matrix elements to consider. The effect of the higher-order matrix elements is discussed so that one can see when these matrix elements must be included and when they can be neglected. In his original paper Bühring gave formulas so that the electron wave functions of Bhalla and Rose[2] could be used to calculate numerical parameters. The screening of the nuclear charge by the atomic electrons was not included in the tables of Bhalla and Rose. In a later paper Bühring[3] gives formulas for the parameters which include the effect of screening. He also gives some numerical examples. The effect is largest at low energy; it is greater for positrons than electrons, and of course it increases with the nuclear charge. For Rb^{84} and Rb^{86}, the correction is not large in the energy range of the experiments. However, it will be significant for many other isotopes. The computer program was designed so that limits of errors on the results could be given that were accurate and easy to interpret. The predictions of the conserved-vector-current (CVC) theory were investigated, and calculations were performed to consider the possibility of further experiments which could set better limits on the nuclear-matrix elements. An analysis of these isotopes has been reported previously,[4] but precise formulas were not used and the meaning of the limits of error was unclear. Furthermore the results reported here differ substantially from those previously reported.

II. THEORY

The first complete presentation of the theoretical formulas for first-forbidden transitions was given by Morita and Morita.[5] Later a valuable series of papers by Kotani and Ross[6,7] arranged the formulas so that the major characteristics of the experimental observables were easily seen. The application of these formulas has been discussed in the preceding paper. The notation of Kotani and Ross has been widely used, but their formulas are not in the most convenient form to include the effect of higher-order matrix elements. In a more recent paper, Bühring[1] gives formulas which are both precise and convenient. These formulas include many more matrix elements than the six which are usually considered for first-forbidden transitions. The additional matrix elements are of two types—those due to the finite nuclear-size corrections and the third-forbidden matrix elements from the usual multipole expansion. The most important matrix elements are listed with their selection rules in Table I. Matrix elements which contain (r/ρ) to a power of four and higher have been

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[3] W. Bühring, Nucl. Phys. (to be published).
[4] J. Eichler and S. Wahlborn, Phys. Letters **4**, 344 (1963).
[5] M. Morita and R. S. Morita, Phys. Rev. **109**, 2048 (1958).
[6] T. Kotani and M. Ross, Progr. Theoret. Phys. (Kyoto) **20**, 643 (1958).
[7] T. Kotani, Phys. Rev. **114**, 795 (1959).

dropped. All of the higher-order matrix elements are listed on the right side of the table and the parameters associated with them are identified by a prime mark.

The general notation of Kotani and Ross has been retained in defining parameters to represent the matrix elements. Two significant changes in notation have been made. The matrix elements that contain the radius vector are normalized and the parameter D is used in the same general way as the parameter ξ was used by Kotani and Ross:

$$D \equiv \tfrac{1}{2}\alpha Z + \tfrac{1}{3}W_0\rho, \tag{1}$$

Z is the charge of the daughter nucleus. It is taken as positive for β^- decay and negative for β^+ decay. α is the fine-structure constant, W_0 is the maximum energy of the beta particle, ρ is the nuclear radius. These changes in notation are intended to make it easier to estimate the order of magnitude of the higher-order terms and to see the energy dependence which they introduce. If there are no selection rules operating, the matrix elements that do not contain the relativistic operators will have an order of magnitude of unity. Of course the relativistic matrix elements will be considerably smaller. Thus the factor D' has been included in the definitions so that the relativistic matrix-element *parameters* will also have an order of magnitude of unity. A convenient and reasonable procedure for numerical estimates is to set $D' = |D|$. Nevertheless D' will be distinguished from D in the formulas as a reminder of the physical reason for the relative size of the various terms. All of the parameters have been defined as the ratio of the matrix element to a standard matrix element η which will be determined from the ft value. The notation of Bühring for the matrix elements is the same as that used by Morita but it differs slightly from that of Kotani and Ross. Kotani and Ross changed the phase of the matrix elements by a factor of "i," and they defined the operator α such that it has the opposite sign.

If all of the fifteen matrix-element parameters were treated as unknowns to be determined, the analysis would be unnecessarily complex. One must realize that combinations of matrix elements are responsible for the size, energy dependence, and angular dependence of the experimental observables. The individual matrix elements in a combination can be determined if they occur in a different arrangement in other combinations which can also be measured experimentally. It is also possible to extract the individual matrix elements if the relative order of magnitude of the matrix elements in a combination can be estimated theoretically. The CVC theory is one valuable aid in this type of calculation. The first step in arranging the theoretical formulas is to select combinations of matrix elements which one cannot expect to break up by practical experimental measurements. In this process, it is necessary to make reasonable estimates of the relative order of magnitude of the various terms. The basic assumption will be that only

TABLE I. Matrix elements and matrix-element parameters for first-forbidden beta decay.

For $\Delta J=0$	$\Delta\pi=-1$
$D'v=\frac{1}{\eta}(-C_A)\int\gamma_5$	$D'v'=\frac{1}{\eta}(-C_A)\int\gamma_5\left(\frac{r^2}{\rho}\right)$
$w=\frac{1}{\eta}(-C_A)\int i\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{\rho}$	$w'=\frac{1}{\eta}(-C_A)\int i\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{\rho}\left(\frac{r}{\rho}\right)^2$
For $\Delta J=1$	$\Delta\pi=-1$
$D'y=\frac{1}{\eta}(-C_V)\int\boldsymbol{\alpha}$	$D'y'=\frac{1}{\eta}(-C_V)\int\boldsymbol{\alpha}\left(\frac{r}{\rho}\right)^2$
$x=\frac{1}{\eta}(-C_V)\int i\frac{\mathbf{r}}{\rho}$	$x'=\frac{1}{\eta}(-C_V)\int i\frac{\mathbf{r}}{\rho}\left(\frac{r}{\rho}\right)^2$
$u=\frac{1}{\eta}(-C_A)\int\frac{\boldsymbol{\sigma}\times\mathbf{r}}{\rho}$	$u'=\frac{1}{\eta}(-C_A)\int\frac{\boldsymbol{\sigma}\times\mathbf{r}}{\rho}\left(\frac{r}{\rho}\right)^2$
	$D's'=\frac{1}{\eta}(-C_V)\int\frac{(\boldsymbol{\alpha}\cdot\mathbf{r})\mathbf{r}-\frac{1}{3}\boldsymbol{\alpha}r^2}{\rho^2}$
For $\Delta J=2$	$\Delta\pi=-1$
$z=\frac{1}{\eta}(-C_A)\int i\frac{B_{ij}}{\rho}$	$z'=\frac{1}{\eta}(-C_A)\int i\frac{B_{ij}}{\rho}\left(\frac{r}{\rho}\right)^2$
	$D'r'=\frac{1}{\eta}(-C_A)\int\gamma_5\frac{R_{ij}}{\rho^2}$
	$D't'=\frac{1}{\eta}(-C_V)\int i\gamma_5\frac{T_{ij}}{\rho^2}$

the selection rules on total angular momentum and parity apply. The consequences of other selection rules will be considered later. All of the numerical estimates will be for the case of Rb^{86}. The values for Rb^{84} are similar, and it should be evident how to apply these estimates to other isotopes. The parameters V (for $\Delta J=0$) and Y (for $\Delta J=1$) are defined to represent the two combinations which are frequently most important.

$$DV \equiv D'[v+v'(a-D\hat{q}+\hat{q}^2)]+D[w+dw'], \tag{2}$$

$$DY \equiv D'[y+y'(a+\tfrac{1}{3}D\hat{q}-\tfrac{1}{3}\hat{q}^2)] - D[(x+dx')+(u+du')], \tag{3}$$

where

$$a=-\tfrac{1}{6}[(W\rho+\tfrac{3}{2}\alpha Z)^2-\rho^2],$$
$$\hat{q}=\tfrac{1}{3}\rho q,$$
$$\hat{p}=\tfrac{1}{3}\rho p,$$
$$d=-\tfrac{1}{5}[\tfrac{1}{2}\alpha Z-a(3D+2\hat{q})]/[\tfrac{1}{2}\alpha Z+\tfrac{1}{3}W_0\rho],$$

W and p are the electron energy and momentum, respectively, and q is the neutrino momentum. If there is no selection rule operating and there is no internal cancellation, V and Y are defined such that their order of magnitude is unity. It is convenient to use natural

units ($m_0=c=\hbar=1$) for numerical estimates. Then

$$W=1+E(\text{MeV})/0.511, \quad p^2=W^2-1,$$
$$q=W_0-W, \quad \rho=(0.00312)A^{1/3},$$

where the value of the nuclear radius ρ is that used by Bhalla and Rose.[2] For Rb^{86},

$$W_0=2.39, \quad \rho=0.0135, \quad D=0.149,$$
$$a\approx-0.032, \quad d\approx-0.162.$$

The notation $\hat{q}$ and $\hat{p}$ is convenient because it shows the energy dependence and serves as a reminder that these terms are very small. The order of magnitude of $\hat{q}$ or $\hat{p}$ for 0.5 MeV is 0.0045.

One might ask why these matrix elements have been collected together when in principle the higher-order matrix elements could be separated from the first-order matrix elements by the energy dependence which they introduce. It can easily be seen that the energy dependence of V and Y is small in any case and most likely completely negligible. The most important contribution to the energy dependence of V comes from the term $(v'a)$. Thus the relative change in V as a function of energy can be estimated from the following formula:

$$R_V\equiv\frac{(\Delta V/\Delta W)}{V}\approx\frac{-\frac{1}{2}\alpha Z\rho v'}{v+av'+w+dw'}.$$

Whenever the higher-order matrix elements are not much larger than the first-order matrix elements,

$$R_V\approx\tfrac{1}{2}\alpha Z\rho<0.004/\text{MeV}.$$

Even if the higher-order matrix elements are much larger than the first-order matrix elements,

$$R_V\approx\alpha Z\rho/2d<0.025/\text{MeV}.$$

Only in the extreme case where v' is much larger than v, w, and w' will the energy dependence be significant. In that case

$$R_V\approx\alpha Z\rho/2a<0.12/\text{MeV}.$$

The same arguments apply to the energy dependence of Y. Unless measurements can be made that are an order of magnitude more accurate than those presently available, V and Y can be considered to be energy-independent.

All of the matrix-element combinations used by Bühring in his formulas for the transition probability are listed below. In Bühring's notation for the combinations $M_{k_e k_\nu}{}^{(K)}$, the tensor rank of the matrix elements is given by K while k_e and k_ν are the quantum numbers for the electron and neutrino partial waves.

$$(1+a)M_{11}{}^{(0)}=-DV, \tag{4}$$

$$(1+a)m_{11}{}^{(0)}=-\tfrac{1}{3}\rho(w+\tfrac{3}{5}aw'), \tag{5}$$

$$(1+a)M_{11}{}^{(1)}=DY+2\hat{q}[(u+au')-D's'(D-\hat{q})], \tag{6}$$

$$(1+a)m_{11}{}^{(1)}=-\tfrac{1}{3}\rho[(x+\tfrac{3}{5}ax')+(u+\tfrac{3}{5}au')], \tag{7}$$

$$(1+a)M_{12}{}^{(1)}=(1/\sqrt{2})\hat{q}[2(x+ax')+(u+au')+(D-\hat{q})D'(s'+\tfrac{4}{3}y')], \tag{8}$$

$$(1+\tfrac{3}{5}a)M_{21}{}^{(1)}=(1/\sqrt{2})\hat{p}[2(x+\tfrac{3}{5}ax')-(u+\tfrac{3}{5}au')+(9/5)s'D'(D-\hat{q})], \tag{9}$$

$$(1+a)M_{12}{}^{(2)}=\tfrac{1}{2}\sqrt{3}\hat{q}[(z+az')+2r'D'(D-\tfrac{2}{5}\hat{q})-t'D'(D-(8/5)\hat{q})], \tag{10}$$

$$(1+\tfrac{3}{5}a)M_{21}{}^{(2)}=\tfrac{1}{2}\sqrt{3}\hat{p}[z+\tfrac{3}{5}az')+2r'D'(\tfrac{3}{5}D+\tfrac{2}{5}\hat{q})-t'D'(\tfrac{3}{5}D-(8/5)\hat{q})], \tag{11}$$

$$(1+a)m_{12}{}^{(2)}=\tfrac{1}{2}\sqrt{3}\hat{q}(\tfrac{1}{3}\rho)D'(2r'-t'), \tag{12}$$

$$(1+\tfrac{3}{5}a)m_{21}{}^{(2)}=\tfrac{1}{2}\sqrt{3}\hat{p}(\tfrac{1}{5}\rho)D'(2r'-t'). \tag{13}$$

The difficulties in determining the first-order matrix elements will be considered first. Then the contribution due to the higher-order matrix elements will be discussed. The combination of matrix elements has been arranged so that one can see the energy dependence that they introduce. When there is no selection rule or cancellation effect, $M_{11}{}^{(0)}$ and $M_{11}{}^{(1)}$ are of order of magnitude D and there is only a minor energy dependence in $M_{11}{}^{(1)}$. Even though most of the other matrix elements depend directly on p or q, they are a factor ρ/D smaller than $M_{11}{}^{(0)}$ and $M_{11}{}^{(1)}$. Thus it is clear that, when V and Y dominate the transition, it is difficult to determine the other combinations by the additional energy dependence that they introduce. If V and Y are the only parameters which can be measured experimentally, the individual matrix elements cannot be determined unless the relative order of magnitude of the matrix elements in V and Y can be estimated theoretically.

Next, one needs to consider what will happen to the experimental observables if V and Y do not dominate the transition. If only one of these combinations is reduced, it is likely that it will still be difficult to determine the matrix elements. This is particularly true if V or Y is reduced by a selection rule. In that case, the corresponding energy-dependent combinations may also be reduced. However, one should keep in mind that evidence for selection rules is very important for nuclear-model considerations even though the matrix elements cannot be accurately determined. When one observes that V or Y is small and the energy-dependent terms are *not* present, this is a clear indication of a selection rule.

Certainly the best situation for determining the matrix elements occurs when V is small and there is a cancellation in Y. Then $M_{11}{}^{(1)}$ is energy-dependent, and the combinations $M_{12}{}^{(1)}$, $M_{21}{}^{(1)}$, $M_{12}{}^{(2)}$, and $M_{21}{}^{(2)}$ are important. This situation has been observed in most cases where it was possible to determine the nuclear-matrix elements. When one considers the prediction of the CVC theory, it is not surprising that a cancellation occurs in Y. Fujita[8] and Eichler[9] have calculated the

[8] J. I. Fujita, Phys. Rev. **126**, 202 (1962).
[9] J. Eichler, Z. Physik **171**, 463 (1963).

ratio of the matrix elements $\int \boldsymbol{\alpha}$ and $\int i\mathbf{r}/\rho$ on the basis of the CVC theory. With the parameters used here, the prediction is that

$$D'v/x=\pm 1.2\alpha Z+\rho(W_0\mp 2.5)\,, \tag{14}$$

where the upper sign is for β^- decay and the lower sign for β^+ decay. When terms of the order $(W_0\rho)$ are dropped, it is seen that

$$DY\approx(\pm 1.2\alpha Z)x-(\pm\tfrac{1}{2}\alpha Z)(x+u)\,.$$

Thus if $x\approx u$, a cancellation should occur.

The contribution of the higher-order matrix elements to V and Y has already been considered. It was shown that it is usually experimentally impossible to break up combinations of the type $(x+ax')$ unless x' is much larger than x. Fortunately this is not a serious problem. Even if the ratio (x'/x) can not be determined accurately, the uncertainty is quite small ($\sim 3\%$ for Rb^{86}). Thus, unless there is a theoretical reason to suspect that the first-order matrix elements are smaller than the corresponding higher-order matrix elements it is certainly safe to replace these combinations by parameters of the following type:

$$v_0=(v+av')\,;\quad w_0=(w+aw')\,;\quad y_0=(y+ay')\,;\quad \text{etc.} \tag{15}$$

The contributions of the third-forbidden matrix elements s', t', and r' are seen to be of the order

$$D'D\approx D^2=0.022\,.$$

The additional energy-dependence they introduce is even smaller.

$$\tfrac{1}{3}D'\rho\approx\tfrac{1}{3}D\rho=0.0007\,.$$

Unless the first-order matrix elements are greatly reduced, the third-forbidden matrix elements also represent only a small uncertainty.

It should now be clear that if the first-forbidden matrix elements are not abnormally small, the combinations of matrix elements can be simplified considerably.

$$DV=D'v_0+D[w+dw']\,, \tag{16}$$

$$DY=D'y_0-D[(x+dx')+(u+du')]\,, \tag{17}$$

$$M_{11}^{(0)}=-DV\,, \tag{18}$$

$$m_{11}^{(0)}=-\tfrac{1}{3}\rho w_0\,, \tag{19}$$

$$M_{11}^{(1)}=DY+2\hat{q}u_0\,, \tag{20}$$

$$m_{11}^{(1)}=-\tfrac{1}{3}\rho(x_0+u_0)\,, \tag{21}$$

$$M_{12}^{(1)}=(1/\sqrt{2})\hat{q}(2x_0+u_0)\,, \tag{22}$$

$$M_{21}^{(1)}=(1/\sqrt{2})\hat{p}(2x_0-u_0)\,, \tag{23}$$

$$M_{12}^{(2)}=\tfrac{1}{2}\sqrt{3}\hat{q}z_0\,, \tag{24}$$

$$M_{21}^{(2)}=\tfrac{1}{2}\sqrt{3}\hat{p}z_0\,. \tag{25}$$

These equations were obtained by making substitutions of the general form given in Eq. (15) and by neglecting the third-forbidden matrix elements s', r', and t'. One should notice that even though x_0 and u_0 occur in several combinations so that they can be determined, there is still a rather large uncertainty in determining y_0. Since d is approximately 20% for any nucleus, x_0 and u_0 cannot be substituted into Y in order to accurately determine y_0 unless x' and u' can be estimated theoretically. The same limitation applies to a determination of v_0 from V.

The following formulas for the particle parameters $b_{KK'}{}^{(n)}$ in terms of the matrix-element combination are those given by Bühring.[1] Note that it was necessary to change the sign of $b_{KK'}{}^{(1)}$ and $b_{KK'}{}^{(3)}$ to be consistent with the usual convention for circular polarization. A few other minor changes have been made to be consistent with the definition of the angular correlation function given in Eq. (39).

$$b_{00}{}^{(0)}=(M_{11}{}^{(0)})^2+(m_{11}{}^{(0)})^2-2\mu_1\gamma_1(1/W)M_{11}{}^{(0)}m_{11}{}^{(0)}\,, \tag{26}$$

$$b_{11}{}^{(0)}=(M_{11}{}^{(1)})^2+(m_{11}{}^{(1)})^2-2\mu_1\gamma_1(1/W)M_{11}{}^{(1)}m_{11}{}^{(1)}+(M_{12}{}^{(1)})^2+\lambda_2(M_{21}{}^{(1)})^2\,, \tag{27}$$

$$b_{22}{}^{(0)}=(M_{12}{}^{(2)})^2+\lambda_2(M_{21}{}^{(2)})^2\,, \tag{28}$$

$$b_{01}{}^{(1)}=-2(p/W)[M_{11}{}^{(0)}M_{11}{}^{(1)}-m_{11}{}^{(0)}m_{11}{}^{(1)}]+2\sqrt{2}[\eta_{12}M_{11}{}^{(0)}M_{21}{}^{(1)}-\hat{\eta}_{12}(1/W)M_{21}{}^{(1)}m_{11}{}^{(0)}]\,, \tag{29}$$

$$b_{11}{}^{(1)}=-\sqrt{2}(p/W)[(M_{11}{}^{(1)})^2-(m_{11}{}^{(1)})^2]-2[\eta_{12}M_{11}{}^{(1)}M_{21}{}^{(1)}-\hat{\eta}_{12}(1/W)M_{21}{}^{(1)}m_{11}{}^{(1)}]+(1/\sqrt{2})(p/W)[(M_{12}{}^{(1)})^2-\lambda_2(M_{21}{}^{(1)})^2]\,, \tag{30}$$

$$b_{12}{}^{(1)}=-2\sqrt{5}[\eta_{12}M_{11}{}^{(1)}M_{21}{}^{(2)}-\hat{\eta}_{12}(1/W)M_{21}{}^{(2)}m_{11}{}^{(1)}]+\sqrt{10}(p/W)[M_{12}{}^{(1)}M_{12}{}^{(2)}+\tfrac{1}{5}\lambda_2M_{21}{}^{(1)}M_{21}{}^{(2)}]\,, \tag{31}$$

$$b_{22}{}^{(1)}=(5/2)^{1/2}(p/W)[(M_{12}{}^{(2)})^2+\tfrac{3}{5}\lambda_2(M_{21}{}^{(2)})^2]\,, \tag{32}$$

$$b_{11}{}^{(2)}=-2\sqrt{3}[\nu_{12}(p/W)M_{11}{}^{(1)}M_{21}{}^{(1)}+\hat{\nu}_{12}\alpha^2Z^2(1/W)M_{21}{}^{(1)}m_{11}{}^{(1)}]+(\sqrt{3}/\sqrt{2})\lambda_2(M_{21}{}^{(1)})^2\,, \tag{33}$$

$$b_{02}{}^{(2)}=-2\sqrt{2}[\nu_{12}(p/W)M_{11}{}^{(0)}M_{21}{}^{(2)}+\hat{\nu}_{12}\alpha^2Z^2(1/W)M_{21}{}^{(2)}m_{11}{}^{(0)}]\,, \tag{34}$$

$$b_{12}{}^{(2)}=-2\sqrt{3}[\nu_{12}(p/W)M_{11}{}^{(1)}M_{21}{}^{(2)}+\hat{\nu}_{12}\alpha^2Z^2(1/W)\times M_{21}{}^{(2)}m_{11}{}^{(1)}]-(6)^{1/2}\lambda_2M_{21}{}^{(1)}M_{21}{}^{(2)}\,, \tag{35}$$

$$b_{22}{}^{(2)}=-(\sqrt{7}/\sqrt{2})\lambda_2(M_{21}{}^{(2)})^2\,, \tag{36}$$

$$b_{12}{}^{(3)}=-(6\sqrt{3}/\sqrt{5})\lambda_2(p/W)M_{21}{}^{(1)}M_{21}{}^{(2)}\,, \tag{37}$$

$$b_{22}{}^{(3)}=-(3\sqrt{2}/\sqrt{5})\lambda_2(p/W)(M_{21}{}^{(2)})^2\,. \tag{38}$$

The numerical parameters L_0, μ_1, γ_1, λ_2, ν_{12}, $\hat{\nu}_{12}$, η_{12}, and $\hat{\eta}_{12}$ are defined by Bühring.[1,3] They are all of order one and are only weakly dependent on Z and W. Results for the parameters without screening corrections are given in Tables II and III. The parameters γ_1, $\hat{\nu}_{12}$, and μ_1 were set equal to one since they only occur in terms which are very small.

TABLE II. Numberical values of L_0, λ_2, ν_{12}, η_{12}, η_{12} for Rb^{84}.

W	P	L_0	λ_2	ν_{12}	η_{12}	$\hat{\eta}_{12}$
1.077	0.4	0.983	1.247	1.023	0.919	0.886
1.166	0.6	0.984	1.043	0.967	0.922	0.889
1.280	0.8	0.984	0.977	0.950	0.925	0.893
1.414	1.0	0.985	0.948	0.943	0.928	0.896
1.562	1.2	0.986	0.934	0.940	0.930	0.898
1.720	1.4	0.987	0.927	0.939	0.932	0.900
1.886	1.6	0.988	0.923	0.939	0.933	0.902
2.059	1.8	0.989	0.921	0.939	0.935	0.903
2.236	2.0	0.990	0.920	0.940	0.936	0.904
2.416	2.2	0.992	0.920	0.940	0.937	0.907
2.600	2.4	0.994	0.921	0.942	0.939	0.906

The general beta-gamma angular correlation function in terms of the particle parameters is presented in the following formulas:

$$N(W,\theta,S)=A_0(W)+SA_1(W)P_1(\theta)+A_2(W)P_2(\theta)+SA_3(W)P_3(\theta), \quad (39)$$

$$A_0(W)=b_{00}{}^{(0)}+b_{11}{}^{(0)}+b_{22}{}^{(0)}, \quad (40)$$

$$A_1(W)=b_{01}{}^{(1)}G_{01}{}^{(1)}+b_{11}{}^{(1)}G_{11}{}^{(1)}+b_{12}{}^{(1)}G_{12}{}^{(1)}+b_{22}{}^{(1)}G_{22}{}^{(1)}, \quad (41)$$

$$A_2=b_{11}{}^{(2)}G_{11}{}^{(2)}+b_{02}{}^{(2)}G_{02}{}^{(2)}+b_{12}{}^{(2)}G_{12}{}^{(2)}+b_{22}{}^{(2)}G_{22}{}^{(2)}, \quad (42)$$

$$A_3=b_{12}{}^{(3)}G_{12}{}^{(3)}+b_{22}{}^{(3)}G_{22}{}^{(3)}. \quad (43)$$

$P_n(\theta)$ are Legendre polynomials, and $G_{KK'}{}^{(n)}(J_0,J_1,J_2)$ are combinations of Racah and Clebsch-Gordan coefficients which are defined and tabulated for some important spin sequences ($J_0\rightarrow J_1\rightarrow J_2$) by Kotani.[7] The helicity factor s is $+1(-1)$ for right- (left-) hand circular polarization. For beta-gamma directional-correlation measurements, the circular polarization of the gamma rays is not observed, and $s=0$. The directional-correlation function it usually written in the following form:

$$N(W,\theta)=1+\epsilon(W)P_2(\theta), \quad (44)$$

where

$$\epsilon(W)=A_2(W)/A_0(W).$$

The beta-gamma circular polarization correlation can be defined as

$$P_\gamma(W,\theta)=\frac{N(W,\theta,+1)-N(W,\theta,-1)}{N(W,\theta,+1)+N(W,\theta,-1)}. \quad (45)$$

Then,

$$P\gamma(W,\theta)=\frac{A_1(W)P_1(\theta)+A_3(W)P_3(\theta)}{A_0(W)+A_2(W)P_2(\theta)}. \quad (46)$$

If the direction of emission and the spin of the electron and neutrino are not observed, the transition probability λ is

$$\lambda=\frac{1}{\pi^3}\int_1^{W_0}F_0(Z,W)pWq^2S(W)dW, \quad (47)$$

$S(W)$ is the shape-correction factor defined in terms of the matrix elements, *not* the matrix-element parameters—that is,

$$S(W)=L_0\eta^2A_0(W). \quad (48)$$

For first-forbidden transitions, the following definition of the Fermi integral is most consistent with the definition for allowed transitions:

$$f\equiv(\bar{S})^{-1}\int_1^{W_0}F_0(Z,W)pWq^2S(W)dW, \quad (49)$$

where the average value of S is defined as

$$\bar{S}=(W_0-1)^{-1}\int_1^{W_0}S(W)dW. \quad (50)$$

The coupling constants can be removed from S by setting $C_V=1.0$ and $C_A=-1.2$ and using the experimental value[10] for the vector coupling constant. Then

$$ft=6150/\bar{S}. \quad (51)$$

The scale for the first-forbidden matrix elements is defined relative to the Fermi matrix element for super-allowed transitions [i.e., $(\int 1)=\sqrt{2}$]. Thus

$$\eta^2=6150\left[t\int_1^{W_0}F_0(Z,W)pWq^2C(W)dW\right]^{-1}, \quad (52)$$

where $C(W)$ is the shape-correction factor defined in terms of the matrix-element parameters.

$$C(W)=L_0A_0(W)=S(W)/\eta^2. \quad (53)$$

In these formulas p, W, q and the Fermi function $F_0(Z,W)$ are in natural units and the half-life t is in seconds.

There are several reasons why it seems desirable to use the formulas presented here rather than those of Kotani. Certainly it is easier to see the effect of the higher-order matrix elements. Even if the first-order matrix elements are large, the uncertainty in determining v_0 and y_0 is important. The particle parameters given by Bühring include small terms which were previously

TABLE III. Numerical values of L_0, λ_2, ν_{12}, η_{12}, $\hat{\eta}_{12}$ for Rb^{86}.

W	P	L_0	λ_2	ν_{12}	η_{12}	$\hat{\eta}_{12}$
1.077	0.4	0.963	1.272	1.005	0.980	0.855
1.166	0.6	0.963	1.066	0.958	0.905	0.869
1.280	0.8	0.962	0.997	0.944	0.913	0.877
1.414	1.0	0.961	0.970	0.939	0.920	0.883
1.562	2.5	0.960	0.956	0.938	0.924	0.887
1.720	1.4	0.960	0.949	0.938	0.927	0.890
1.886	1.6	0.958	0.947	0.939	0.931	0.892
2.059	1.8	0.957	0.946	0.940	0.933	0.895
2.236	2.0	0.956	0.947	0.941	0.936	0.897
2.416	2.2	0.955	0.948	0.942	0.938	0.899

[10] R. K. Bardin, C. A. Barnes, W. A. Fowler, and P. A. Seeger, Phys. Rev. **127**, 583 (1962).

neglected, and they contain better approximations for the electron wave function. Since the formulas are so complex, it is difficult to make a general statement about how much more accurate the results are with these improvements. The results reported in this paper differ substantially from the previous results[4] which were obtained using Kotani's formulas, so it would seem that the improvements are important. A final point of comparison is that the corrected Fermi integral f_C defined by Kotani is *not* equal to the f defined here. f_C will be equal to f only when $C(W)$ average is one.

When the particle parameters of Bühring are used, it is still necessary to decide whether or not the simple expressions [Eqs. (16)–(25)] for the combinations of matrix elements are adequate. Before the analysis is begun, the ft value is the only guide that is available. If the ft value is not much larger than $(6150/\rho^2)$, the simple formulas will probably be satisfactory. However, one should remember that the results of the analysis will provide the best check on the validity of the approximation. For example, if (ηx_0) turns out to be approximately one, the contribution of x' or s' to $M_{21}{}^{(1)}$ could not be significant. The matrix element $(\int i\mathbf{r}/\rho)$ has the expected physical size and the higher-order matrix elements are not important. There are many cases where the ft value is much larger than $(6150/\rho^2)$. The more accurate formulas given in Eqs. (4) through (13) must be used for these isotopes. The analysis for this type of transition will be considered in a subsequent paper.

III. PROCEDURE FOR ANALYSIS OF THE EXPERIMENTAL DATA

The 2^--2^+ transitions considered in this paper are considerably more difficult to analyze than the 3^--2^+ transitions which have received most of the attention in this field. Even with the simplification made in the preceding section, there are six unknowns to consider rather than four. It is usually not difficult to find sets of matrix elements which fit the experimental data satisfactorily. The difficulty comes first in ascertaining that no important sets of matrix elements have been missed and second in establishing meaningful limits of error on the results. Since acceptable sets of matrix elements are located by trial and error, the obvious procedure is to take fine steps over a wide range for all six unknowns. Even with a high-speed computer, this procedure consumes too much time. Fortunately the combination of matrix elements presented by Bühring suggest a procedure which reduces the time required by an order of magnitude.

An examination of the formulas shows that at $W=W_0$ the value of the directional correlation depends primarily on V, Y, z_0 and the combination $(2x_0-u_0)$. That is, when $q=0$,

$$M_{11}{}^{(1)}=Y \quad \text{and} \quad M_{12}{}^{(1)}=0.$$

It is reasonable to assume that the contribution from $m_{11}{}^{(0)}$ and $m_{11}{}^{(1)}$ is small. Thus z_0 is set equal to 1.0, 0.1 or 0.01, and V and Y are stepped over the range 1.0 to 0.01. The quantity $(2x_0-u_0)$ can be estimated from the experimental value of $\epsilon(W_0)$.

There are several reasons why this procedure saves time in the analysis. In the initial phase of the program, x_0 and u_0 do not need to be considered separately, and w_0 is neglected. This reduces the number of parameters to be stepped from six to four. Even the combination $(2x_0-u_0)$ does not need to be stepped at random, but rather can be stepped around the value estimated for each set (z_0, V, Y). Not only is the number of steps reduced, but also it is possible to test the validity of the value of V and Y chosen in the initial phase of the program. The expression for $(2x_0-u_0)$ is quadratic, so the sets of (z_0,V,Y) which give an imaginary solution can most likely be discarded. Before they are discarded, further precautions must be taken to insure that no valid sets are being overlooked. One source of error occurs because $\epsilon(W_0)$ can not be measured directly. It is obtained by extrapolating the experimental data for $W<W_0$. Also, the small terms containing $m_{11}{}^{(0)}$ and $m_{11}{}^{(1)}$ have been neglected. Before a set (z_0,V,Y) is discarded, the computer tries to find a real solution to the quadratic equation by making reasonable adjustments in the uncertain parameters.

For each set of z_0, V, Y, and $(2x_0-u_0)$, the energy dependence of $\epsilon(W)$ is used to estimate a starting point around which u_0 can be stepped. The remaining parameter w_0 is taken through very course steps since the formulas are very insensitive to $m_{11}{}^{(0)}$. Each set (z_0,V,w_0, Y,x_0,u_0) which is developed in this process is used to calculate the experimental observables, and a chi-squared test is performed on the experimental data. Any set of parameters which does not satisfy the desired statistical criterion is discarded and the remaining sets are retained for the second stage of the program.

An important point must be made concerning the calculation of the angular dependence of the circular polarization. These measurements are usually made with integral beta-energy selection. Serious errors can result if the circular polarization is calculated at the average energy quoted with the experiment. In this program the average value of $P_\gamma(\theta)$ is calculated by numerical integration over the energy range of the experiment.

A section has been included in the first stage of the program to assist in the design of future experiments. The energy dependence of $P_\gamma(\theta,W)$ is calculated for each set of matrix elements which fit the experimental data. In this way it can be seen whether or not measurements of the energy dependence of P_γ would be useful in setting better limits on the matrix elements. The program is designed to use this type of experimental data when it is available.

The second stage of the program is used to set limits of error on the results. The first set of matrix elements

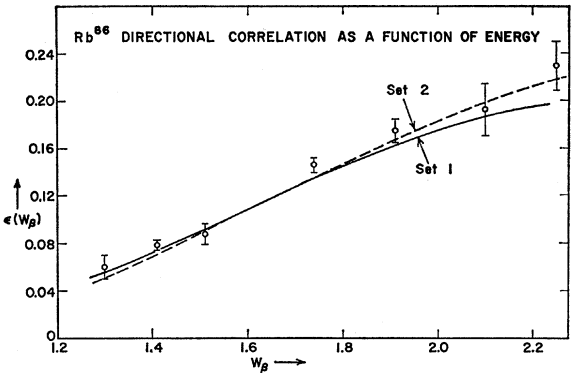

FIG. 1. Rb^{86} beta-gamma directional-correlation parameter as a function of beta energy.

which fit the experimental data is taken as a starting point. Z_0 is held constant, and the parameters V, y_0, x_0, and u_0 are stepped in a random manner through progressively larger ranges until no new set can be found that fits the experimental data. If necessary, the steps are made smaller until at least 20 values of each matrix element are found which fit the data. Since the steps are small, many thousands of sets are frequently produced which fit the experimental data. It is not necessary to record all of these sets. Rather, one would like to take these sets and determine the most probable value for each matrix element and the range into which

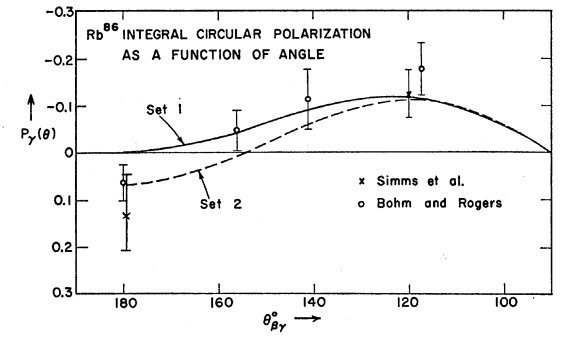

FIG. 2. Rb^{86} beta-gamma circular polarization as a function of angle. Integral pulse-height selection was used for the beta detector.

each matrix element can be restricted by the experimental data.

It is not difficult to determine the range of the matrix elements. The computer simply keeps track of the maximum and minimum values that occur for the matrix elements. There are certainly many ways to estimate the most probable value of the matrix elements. The following point of view was adopted here: If the value M_{jk} of the matrix element M_j occurs in only a few sets, then M_{jk} cannot be the correct value unless all of the other matrix elements are restricted to the values they have in these sets. However, the experimental data does not restrict the matrix elements to these values. Unless there is some theoretical reason to believe that they are so restricted, it is improbable that M_{jk} is the correct value. On the other hand, if M_{jk} occurs in many sets, it is likely that one of these is the real one, and the probability function for M_{jk} should be large. Thus the probability P_{jk} should be proportional to the number n of acceptable sets in which M_{jk} occurs, and it should be influenced by the quality of the fit. P_{jk} was defined as

$$P_{jk}=\sum_{j=1}^{n} P_{jki}. \tag{54}$$

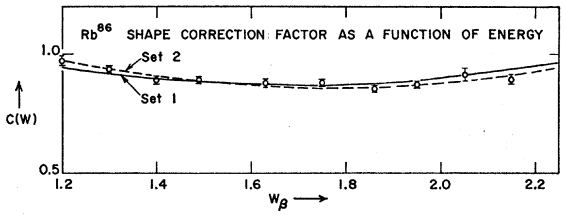

FIG. 3. Rb^{86} shape-correction factor as a function of beta energy. The verticle scale was chosen arbitrarily.

P_{jki} is inversely proportional to the sum of the residuals for the experimental data.

$$P_{jki}=[\sum_{m=1}^{m'} R_m]^{-1}, \tag{55}$$

where there are m' types of experimental data. The general definition of the residual R is,

$$R=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{c_i-e_i}{\Delta_i}\right]^2, \tag{56}$$

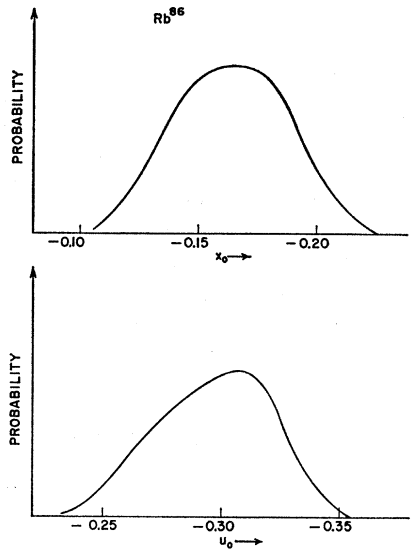

FIG. 4. The probability function [defined in Eq. (49)] for the matrix-element parameters x_0 and u_0 of Rb^{86}. The scale on the horizontal axis gives the ratio of the parameter to z_0.

TABLE IV. Matrix-element parameters for Rb^{86}.

Set	z_0	V	w_0	Y	y_0	x_0	u_0	$D'y_0/x_0$
1	1.0	−0.0524	+0.02	−0.0107	−0.426	−0.170	−0.245	0.366
2	1.0	−0.0524	+0.04	−0.0086	−0.445	−0.132	−0.304	0.490
E. & W.[a]	1.0	−0.06±0.012	−0.12±0.06	−0.07±0.015		−0.14±0.07	−0.30±0.10	

[a] See Ref. 4.

where there are n different measurements e_i of the experimental parameter [e.g., $\epsilon(W)$, $C(W)$, $P_\gamma(\theta)$]. c_i and Δ_i are the calculated value and the experimental error of the parameter, respectively.

Each set of matrix elements that was retained in the first section of the program is considered in turn. The set is ignored if it falls within the limits of a probability distribution which was calculated previously. Thus if the sets of matrix elements found in the first section are all similar, only one probability distribution will be calculated. Otherwise a separate probability distribution will be developed for each distinct type of set.

While the matrix elements are being varied in the probability calculation, the computer records the maximum and minimum ratios for various pairs of matrix elements. These ratios are used to test the predictions of the CVC theory and nuclear models. The matrix element w_0 is not included in the stepping process since the theoretical parameters are very insensitive to $m_{11}^{(0)}$. A separate section of the program sets an upper limit on w_0. In all phases of the program, the statistical controls in the tests for agreement with the experimental data can be varied, and restrictions on agreement with the CVC theory can be imposed if this is desired.

IV. RESULTS

A. Rb^{86}

The experimental data used in the analysis of Rb^{86} is shown in Figs. 1, 2, and 3. The directional-correlation data was obtained from the work discussed in the preceding paper. The anisotropy for Rb^{86} is unusually large. The circular-polarization data combines the results of Boehm and Rogers[11] with the results given in the preceding paper. Both experiments clearly show that there are contributions from the $P_3(\theta)$ term as well as the $P_1(\theta)$ term in the polarization correlation. The shape-correction factor was measured by Robinson and Langer.[12] Two curves are shown in each figure which fit the experimental data equally well. The sets of matrix-element parameters from which these curves were calculated are given in Table IV. The results of Eichler and Wahlborn[4] are included for comparison. Their parameters V and Y have been divided by ξ so they can be compared to the parameters used here. The probability distributions for the matrix-element parameters V, y_0, x_0, and u_0 ($z_0=1.0$) are shown in Figs. 4 and 5. One can see from the probability distributions that the matrix elements which change the total spin by 0 or 1 units are suppressed compared to the B_{ij} ($\Delta J=0$) matrix element.

The two sets of matrix-element parameters given in Table IV were selected to illustrate the application of the CVC theory to this transition. From Eq. 14 it is seen that for Rb^{86}

$$(D'y)/x=+0.323.$$

It should be remembered that the value of $(D'y_0)/x_0$ given in Table IV is most probably too high. If $x'\approx x$ and $u'\approx u$, then the ratio would be 16% smaller. Thus set 1 is in good agreement with the prediction of the CVC theory and set 2 gives a ratio which is at least 30% too large. When all of the sets of matrix elements which were found are considered, the following limits can be set on the ratio of $(D'y_0)$ to x_0:

$$0.28\leq(D'y_0)/x_0\leq 0.58.$$

The uncertainty in estimating the ratios x'/x and u'/u has been included in the lower limit. Thus it is seen that the results are in very good agreement with the calculation of Fujita, but it is not possible to set good limits on the ratio.

One can see from Figs. 1 and 3 that it would be very difficult to improve the experimental data on $\epsilon(W)$ or

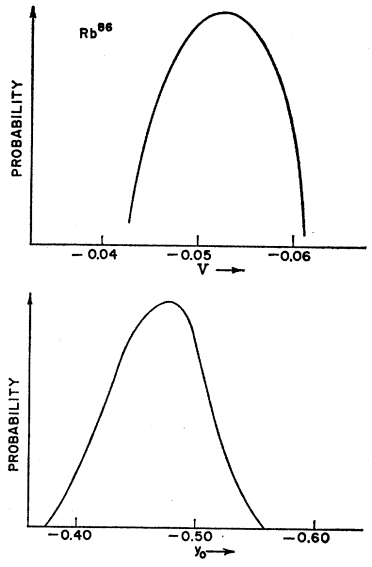

FIG. 5. The probability function [defined in Eq. (49)] for the matrix-element parameters V and y_0 of Rb^{86}. The scale on the horizontal axis gives the ratio of the parameter to z_0.

[11] F. Boehm and J. Rogers, Nucl. Phys. **45**, 392 (1963).
[12] R. L. Robinson and L. M. Langer, Phys. Rev. **112**, 481 (1958).

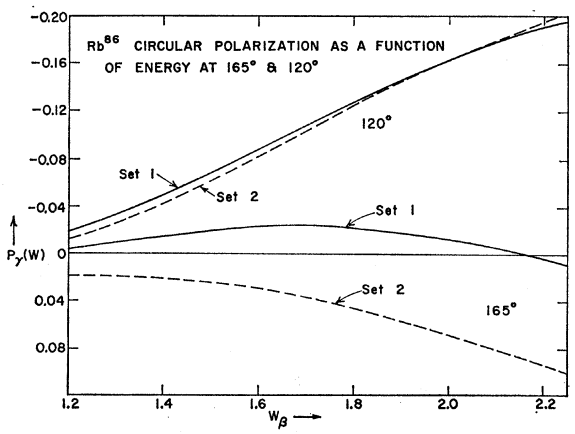

FIG. 6. Theoretical prediction for the energy dependence of the Rb^{86} beta-gamma circular polarization.

$C(W)$ to the point that a distinction could be made between sets 1 and 2. However, Fig. 2 shows that there is some difference in P_γ for the two sets when $\theta_{\beta\gamma}$ is large. The experimental data at 180° seems to favor set 2 which does not agree with the CVC ratio. However, since set 1 fits the data at 161 and 137° better than set 2, it is impossible to distinguish between the two sets using the present experimental results. In Fig. 6 the energy dependence of P_γ is plotted for $\theta_{\beta\gamma}=165$ and 120°. For the types of experiments considered in this paper, it is evident that the best way to set better

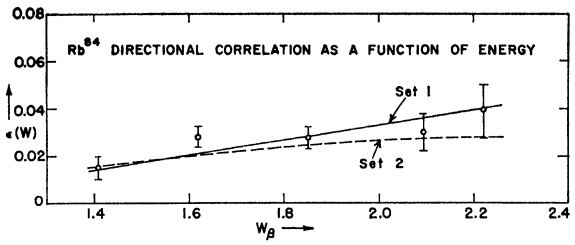

FIG. 7. Rb^{84} beta-gamma directional-correlation parameter as a function of beta energy.

limits on the matrix elements would be to measure the energy dependence of P_γ at an angle close to 180°.

In order to determine the absolute magnitude of the matrix elements and make final conclusions about the accuracy of the results, it is necessary to calculate the standard matrix element η. The half-life of Rb^{86} is 19 days, and the fraction of the transitions which go to the first excited state is 9%.[13] The results for η and $\log ft$ are:

$$\eta=0.78\pm0.15, \quad \log ft=7.94.$$

One should remember that the matrix-element parameters and η were defined so that it is clear physically that the product of these two quantities cannot be much greater than one. Thus it is easily seen that, since $\eta z_0=0.78$, no significant error is made in setting $z_0=z$. (The contribution of $az'\eta$ is of the order of a—i.e., 3%.) Therefore:

$$\int i\frac{B_{ij}}{\rho}=0.65\pm0.13.$$

Since all of the other matrix elements are considerably smaller than unity, the corresponding higher-order matrix elements cannot be neglected without further consideration. It is quite probable that the matrix elements are suppressed by an angular-momentum selection rule. (See Sec. V.) If this is the case, the higher order and the first-order matrix elements would be affected to the same degree, and it is reasonable to assume that the higher-order matrix elements cannot be

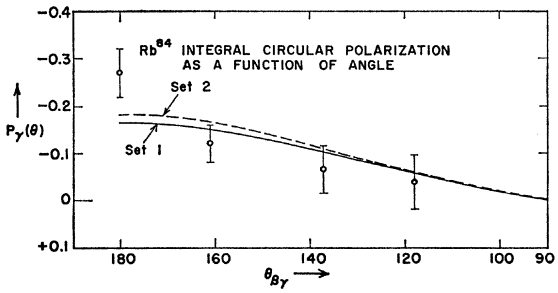

FIG. 8. Rb^{84} beta-gamma circular polarization as a function of angle. Integral pulse-height selection was used for the beta detector.

larger than the first-order matrix elements. Thus the uncertainty introduced by x' in determining x from x_0 would be small, and the probability distributions in Fig. 4 have been used to set limits on x and u.

$$-0.18\leq\int i\frac{\mathbf{r}}{\rho}\leq-0.078, \quad 0.15\leq\int\frac{\boldsymbol{\sigma}\times\mathbf{r}}{\rho}\leq0.23.$$

The values of y_0 given in Table IV and Fig. 5 were based on the assumption that the contribution of x' and u' to Y was zero. Even if x' and u' are no larger than x and u, this is not a valid assumption because the coefficient of x' and u' is large [$d\approx0.16$—see Eq. (17)]. The uncertainty in estimating x' and u' must be combined with the uncertainty for y_0 shown in Fig. 5 to set limits on y.

$$-0.063\leq\int\alpha\leq-0.034.$$

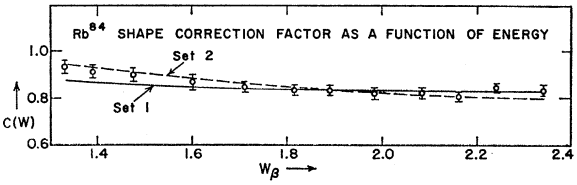

FIG. 9. Rb^{84} shape-correction factor as a function of beta energy. The verticle scale was chosen arbitrarily.

[13] *Nuclear Data Sheets*, compiled by K. Way *et al.* (Printing and Publishing Office, National Academy of Sciences–National Research Council, Washington, D. C., 1960), NCR 60-3-45.

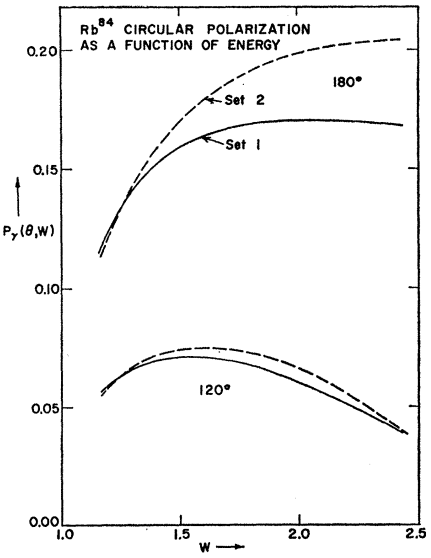

FIG. 10. Theoretical prediction for the energy dependence of the Rb^{84} beta-gamma circular polarization.

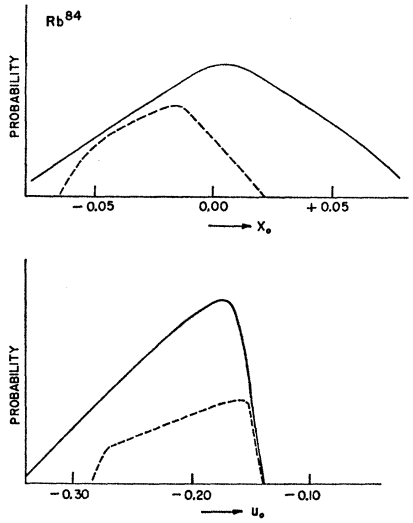

FIG. 11. The probability function [defined in Eq. (49)] for the matrix-element parameters x_0 and u_0 of Rb^{84}. The scale on the horizontal axis gives the ratio of the parameter to z_0.

There are additional complications in setting limits on v and w. Although the experimental data is sufficiently accurate to set good limits on V, it is difficult to set limits on w_0 because the experimental results are very insensitive to this parameter. Only an upper limit on the magnitude of w_0 could be determined. $|w_0| \leq 0.1$. This means that only upper limits are possible for the matrix elements in V.

$$\left|\int \gamma_5\right| \leq 0.02, \quad \left|\int i\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{\rho}\right| \leq 0.08.$$

B. Rb^{84}

The experimental data used in analyzing Rb^{84} is shown in Figs. 7, 8, and 9. The data for $\epsilon(W)$ were obtained from the work reported in the preceding paper, $P_\gamma(\theta)$ was measured by Boehm and Rogers,[10] and $C(W)$ was measured by Langer, Spejewski, and Wortam.[14] The experimental parameters for Rb^{84} are very different from those of Rb^{86}. There is a $P_3(\theta)$ contribution to $P_\gamma(\theta)$, but it is not nearly as evident as in the case of Rb^{86}. Also, $\epsilon(W)$ is much smaller for Rb^{84} than for Rb^{86}. Typical sets of matrix elements are shown in Table V for two very different types of solutions. For Rb^{84}, Eq. (14) predicts that $D'y/x = -0.256$. In set 1, the B_{ij} matrix element dominates and there is agreement with the CVC theory. In set 2, the matrix elements which change the total spin by one unit are not small compared to B_{ij} and the absolute magnitude of the CVC ratio is 50% too small.

One can see that it would be very difficult to distinguish experimentally between these two sets. The angular dependence and the energy dependence of P_γ are very similar for the two sets (see Figs. 8 and 10). It would be very difficult to measure the absolute magnitude of the circular polarization accurately enough to distinguish between the two sets. There is some difference in $C(W)$ and also $\epsilon(W)$. It would seem that the best way to distinguish experimentally between the two sets would be to improve the measurement of $\epsilon(W)$ at the upper end of the beta spectrum.

It should be emphasized however that for all sets of type 2, the absolute magnitude of the ratio of $\int \boldsymbol{\alpha}$ to $\int i\mathbf{r}/\rho$ is at least 50% smaller than the prediction of Fujita. If it is assumed that this prediction is even moderately accurate, the present experimental data exclude solutions of type 2. Therefore, the probability densities were calculated only for solutions of type 1. Even though some sets of type 1 agree with the CVC ratio, there are many others which do not. The solid

TABLE V. Matrix-element parameters for Rb^{84}.

Set	z_0	V	w_0	Y	y_0	x_0	u_0	$D'y_0/x_0$
1	1.0	+0.189	−0.122	+0.154	+0.0615	−0.0269	−0.189	−0.282
2	1.0	−0.295	−0.122	−0.215	−1.44	+1.09	+0.548	−0.161
E. & W.[a]	1.0	+0.13±0.031	−0.35±0.25	+0.19±0.037		−0.07±0.07	−0.30±0.25	

[a] See Ref. 4.

[14] L. M. Langer, E. H. Spejewski, and D. E. Wortam, Phys. Rev. 133, B1145 (1964).

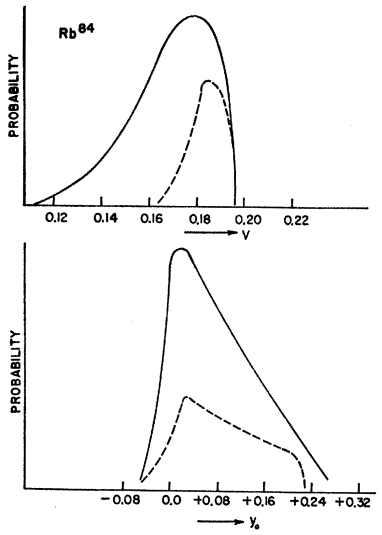

FIG. 12. The probability function [defined in Eq. (49)] for the matrix-element parameters V and y_0 of Rb^{84}. The scale on the horizontal axis gives the ratio of the parameter to z_0.

curves shown in Figs. 11 and 12 for the probability distributions represent all of the type 1 solutions. The dashed curves were obtained by requiring that the value of $(D'y)/x$ be within 50% of the predicted ratio. This restriction changes the limits on V and x_0 much more than it does the limits on y_0 and u_0.

For the final results for Rb^{84}, it is assumed that the prediction of Fujita is accurate to within 50%. Then

$$\eta=0.61\pm0.25, \quad ft=7.2,$$

when the half-life is 33 days and the relative probability for a $\beta+$ transition to the first excited state is 11.7%.[13] Since the matrix elements of the Rb^{84} are similar to those of Rb^{86}, the comments given in Sec. IV A about the accuracy of the results are also applicable here. One should notice however that in Rb^{84} there is no cancellation in Y. The primary reason that the experimental observables of Rb^{86} are very different from those of Rb^{86} is that V and Y are larger for Rb^{84}.

$$\int i\frac{B_{ij}}{\rho}=+0.51\pm0.20,$$

$$-0.04\leq\int i\frac{\mathbf{r}}{\rho}\leq0.015, \quad 0.06\leq\int\frac{\boldsymbol{\sigma}\times\mathbf{r}}{\rho}\leq0.15,$$

$$-0.01\leq\int\boldsymbol{\alpha}\leq0.03, \quad \left|\int\gamma_5\right|\leq0.05, \quad \left|\int i\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{\rho}\right|\leq0.2.$$

V. DISCUSSION

The major features of the results can be understood on the basis of simple shell-model considerations. Both $_{37}Rb_{49}^{86}$ and $_{37}Rb_{47}^{84}$ have neutron and proton numbers close to the closed-shell configurations at nucleon numbers 38 and 50. The orbitals available in the 28–50 major shell are $p_{3/2}$, $f_{5/2}$, $p_{1/2}$, and $g_{9/2}$. One would expect that the odd proton would be in a $f_{5/2}$ or $p_{3/2}$ orbital while the odd neutron would be in a $g_{9/2}$ orbital. Only the $f_{5/2}$ and $g_{9/2}$ can couple to give the 2^- state. Thus one would expect that in either a β^+ or β^- transition the total angular momentum must change by at least two units, and the B_{ij} matrix element would dominate the transition. The analysis shows that the B_{ij} matrix element is indeed large as this picture suggests. The presence of the $\Delta J=0$ and $\Delta J=1$ matrix elements must be explained in terms of admixtures of other states.

Since it is impossible to have a parity change and a total spin change of less than 2 in the 28–50 shell, the admixture must come from adjacent major shells. It is unlikely that the admixtures are large enough so that the contribution of transitions between one admixed component and another admixed component would be significant. Thus all transitions would involve at least one of the orbitals $f_{5/2}$ or $g_{9/2}$. Transitions could occur between the $g_{9/2}$ orbital and the $f_{7/2}$ from the 20–28 shell or the $h_{11/2}$ from the 50–82 shell, and transitions could occur between the $f_{5/2}$ orbital and the $d_{3/2}$, $d_{5/2}$, or $g_{7/2}$ of the 50–82 shell. There are many more possibilities for $\Delta J=1$ transitions than for $\Delta J=0$ transition. For the case of Rb^{86} where one would expect these arguments to be strongest, the analysis shows that the $\Delta J=1$ matrix elements are larger than the $\Delta J=0$ matrix elements. Wahlborn[15] has considered the mechanisms by which this configuration mixing could occur. He concludes that the 2^+ states of Sr^{86} and Kr^{84} cannot be understood in terms of two-particle excitations, or in terms of a purely collective picture. An intermediate description is required to get agreement with the experimental results.

One would not expect that the results obtained in this analysis would be very different from the results obtained by Eichler and Wahlborn[4] using the formulas of Kotani.[7] The higher order matrix elements were not used in the present analysis except to estimate their contribution to the limits of error. Small terms which contain the first-order matrix elements were retained, and the electron wave functions of Bhalla and Rose were used, but it is impossible to make a general statement about the improvement in accuracy which results from these steps. The results are quite similar except for one important point. (See Tables IV and V.) The parameter Y obtained here for Rb^{86} is much smaller than that obtained by Eichler and Wahlborn. This changes the ratio of $\int\boldsymbol{\alpha}$ to $\int\mathbf{r}$ so that the agreement with CVC theory is much better.

The calculations of Wahlborn[15] concerning the configuration mixing will not be affected seriously. He calculates the parameters x, u, and w. The two methods of analysis are in good agreement for x and u. However, the limits of error quoted by Eichler and Wahlborn for w are certainly unrealistic. In both isotopes w can be zero and in Rb^{86} it can be positive, as shown in Table IV. Wahlborn[15] uses the argument that w cannot be zero in his discussion of the configuration mixing. Since

[15] S. Wahlborn, Nucl. Phys. 58, 209 (1964).

V is not zero it is certainly unlikely that w is exactly zero, but values of w outside the limits of error given by Eichler and Wahlborn are in perfect agreement with the experimental data.

A further question arises about the accuracy of the formulas presented here compared to those of Kotani[7] because solutions of type 2 in Table V were missed completely by Eichler and Wahlborn. An analysis of the data was performed using the procedure discussed in Sec. III and the formulas of Kotani.[7] Solutions of type 2 were found which agreed with the data. Thus it would seem that these solutions were missed because of the technique used in the analysis rather than the formulas that were used.

In order to analyze transitions where all of the matrix elements are abnormally small, it will be essential to include the higher-order matrix elements. If the matrix elements are small because of cancellations inside the overlap integral, the higher-order matrix elements could be much larger than the first-order matrix element since they have a different radial dependence. This type of cancellation has been suggested by Kisslinger[16] as an explanation of the small matrix elements of Sb^{124}. Four isotopes, Sb^{124}, La^{140}, Eu^{152}, and Eu^{154}, that have abnormally large ft values are now being analyzed with the higher-order matrix elements included.

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[16] L. S. Kisslinger and C. Wu, Phys. Rev. **136**, B1254 (1964).

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Low-Lying Excited States of Sc^{46} Populated in the Thermal-Neutron Capture Reaction $Sc^{45}(n,\gamma)Sc^{46}$ †

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The thermal-neutron capture reaction $Sc^{45}(n,\gamma)Sc^{46}$ was utilized to populate low-lying states in Sc^{46} by means of γ-ray cascades from the compound-nucleus capture level. Scintillation singles and coincidence γ-ray spectroscopy techniques were used to study the gamma-ray decay characteristics of the levels observed. A rather complete set of data has been obtained for transitions between levels at excitation energies up to 0.675 MeV. Levels at 0.142, 0.225, 0.289, 0.445, 0.585, and 0.675 MeV have been inferred, and the decay characteristics of each of these states have been determined. In addition, many levels up to an excitation energy of 3.62 MeV have been observed and the principal γ-ray decay modes of these states are described. Conventional and time-to-pulse-height delayed-coincidence techniques were employed to establish the "prompt" ($t_{1/2} \lesssim 2\times10^{-9}$ sec) lifetimes of the 0.225-, 0.289-, 0.445-, 0.585-, and 0.675-MeV states. The isomeric character ($t_{1/2}=20$ sec) of the state at 0.142 MeV has been confirmed. A complete decay scheme for states up to an excitation energy of 0.675 MeV is proposed, and the observed characteristics of these levels are compared with previous (d,p) stripping studies and recent theoretical calculations. In addition, a decay scheme is also presented which involves states at higher excitation energies ($\leqslant 3.62$ MeV) and includes the γ-ray decay modes observed in the present work.

I. INTRODUCTION

EXPERIMENTAL knowledge and theoretical understanding of the characteristic properties of excited states of odd-odd nuclei have not grown as rapidly as for other classes of nuclei. Various nuclear models have achieved some degree of success in explaining many features associated with even-even and odd-A nuclei, for which an abundance of experimental data has, for the most part, been available. This success has served to focus added experimental and theoretical attention upon them. On the other hand, the experimental and theoretical difficulties and complexities associated with odd-odd nuclei have led to the relative neglect of these nuclides.

Relatively little detailed experimental data concerning the low-lying levels in odd-odd nuclei is available. Experimental studies of the levels in odd-odd nuclei are disadvantaged by the fact that, with relatively few exceptions, the excited states of these nuclei are not populated by β decay. Application of ordinarily powerful γ- and β-ray spectroscopic techniques are restricted to those few (usually one or two) excited states fed from relatively long-lived isomeric levels. Since odd-odd

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