Coupled-Channel Scattering with Complex Angular Momentum*

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A study of some problems in nonrelativisitic multichannel scattering is made in the context of complex angular momentum. A generalization of the Mandelstam symmetry is obtained. It is shown that the well-known "cusp" behavior in the elastic cross section when an inelastic channel opens up is not only present for S waves, but is a persistent feature as the (real) angular momentum is varied continuously from $-\frac{3}{2}$ to $+\frac{1}{2}$. The existence of the "indeterminacy points" is indicated. Using the factorizability condition on the residues of the S matrix, it is explicitly exhibited how a part of the S matrix decouples when a resonance pole drifts to the real axis.

I. INTRODUCTION

THE analytic properties of nonrelativistic scattering amplitudes have been studied by several authors in the context of complex angular momentum, both for single-channel[1] and multichannel[2,3] potential scattering. The purpose of the present paper is to make explicit some of the consequences of these properties for the multichannel case. Most pertinent features of the single-channel Regge trajectories are well known and have been verified by numerical computation.[1]

In Sec. II various quantities are defined and the generalization of the Mandelstam symmetry is derived and translated into trajectory language. The factorization of residues and the decoupling of channels when a pole crosses the positive energy axis is discussed in Sec. III. In Sec. IV the cusp behavior in the elastic cross section when an inelastic channel opens up is shown to persist as the angular momentum is varied from $-\frac{3}{2}$ to $+\frac{1}{2}$. Indeterminacy points for $\lambda=-\frac{1}{2}$ and $\lambda=-1$ are exhibited in the Appendix.

II. MANDELSTAM SYMMETRY FOR COUPLED CHANNELS AND INDETERMINACY POINTS

Assume an $n\times n$ potential matrix $V_{ij}(r)$ of the Yukawa type

$$V_{ij}(r)=\int_{\mu_0}^{\infty}\sigma_{ij}(\mu)\frac{e^{-\mu r}}{r}d\mu\,,\quad \mu_0>0 \tag{2.1}$$

with all absolute moments of $\sigma_{ij}(\mu)$ existing; i.e., for any integer $p\geq 0$,

$$\int_{\mu_0}^{\infty}|\sigma(\mu)|\mu^p<\infty\,. \tag{2.2}$$

The conditions (2.1) and (2.2) imply the existence of $rV_{ij}(r)$ and all its derivatives. The radial Schrödinger equation for this n-channel problem reads

$$\frac{d^2}{dr^2}\psi(r)+K^2\psi(r)-\frac{\lambda^2-\frac{1}{4}}{r^2}\psi(r)=V(r)\psi(r)\,, \tag{2.3}$$

where K is a diagonal matrix with k_i the incident channel momenta:

$$K_{ij}=k_i\delta_{ij}\,,\quad k_i=(k^2-k_{i0}^2)^{1/2}\,. \tag{2.4}$$

The wave function $\psi(r)$ is an $n\times n$ matrix whose rows indicate the n-channel components and whose columns differ by their boundary conditions; i.e., the ith column has an incident wave only in the ith channel for the scattering solution. One also defines, in the standard way, matrix solutions of (2.3) by the boundary conditions at infinity and at the origin:

$$f_{ij}(\lambda,K,r)\underset{r\to\infty}{\longrightarrow}f_i^0(\lambda,K,r)\delta_{ij}=\left(\frac{\pi k_i r}{2}\right)^{1/2}e^{-\frac{1}{2}i\pi(\lambda+\frac{1}{2})}H_\lambda^{(2)}(k_i r)\delta_{ij}\,, \tag{2.5}$$

and

$$\phi_{ij}(\lambda,K,r)\underset{r\to 0}{\longrightarrow}\phi_i^0(\lambda,K,r)\delta_{ij}=2^\lambda\Gamma(1+\lambda)r^{1/2}k_i^{-\lambda}J_\lambda(k_i r)\delta_{ij}\,. \tag{2.6}$$

The solutions f and φ satisfy the following matrix integral equations:

$$f(\lambda,K,r)=f^0(\lambda,K,r)+\int_0^\infty dr'g(\lambda,K;r,r')V(r')f(\lambda,K,r')\,, \tag{2.7}$$

and

$$\phi(\lambda,K,r)=\phi^0(\lambda,K,r)+\int_0^r dr'g(\lambda,K;r,r')V(r')\phi(\lambda,K,r')\,, \tag{2.8}$$

where the Green matrix function, $g(\lambda,K;r,r')$ is a diagonal matrix with elements

$$g_i(\lambda,K;r,r')=-\tfrac{1}{2}\pi(rr')^{1/2}\times[J_\lambda(k_i r')Y_\lambda(k_i r)-J_\lambda(k_i r)Y_\lambda(k_i r')]\,. \tag{2.9}$$

* Supported in part by the U. S. Atomic Energy Commission.
[1] R. G. Newton, *The Complex j Plane* (W. A. Benjamin Inc., New York, 1963).
[2] A. M. Jaffe and Y. S. Kim, Phys. Rev. **127**, 2261 (1962). Our definition of Jost matrix differs from the one in this reference and that of Ref. 3. J. M. Charap and E. J. Squires, Ann. Phys. (N. Y.) **20**, 145 (1962); **21**, 8 (1963); **25**, 143 (1963).
[3] Hong-Mo Chan, J. Math. Phys. **4**, 1042 (1963).

Here J_λ and Y_λ are the Bessel functions of the first and second kind. The various symmetries of g_i, φ, and f may be listed as

$$g_i(\lambda,-k_i;r,r')=g_i(\lambda,k_i;r,r'), \tag{2.10}$$

$$g_i(-\lambda,k_i;r,r')=g_i(\lambda,k_i;r,r'), \tag{2.11}$$

$$\phi(\lambda,-K,r)=\phi(\lambda,K,r), \tag{2.12}$$

$$f(-\lambda,K,r)=f(\lambda,K,r). \tag{2.13}$$

In order to remove the singularity of φ at negative integer and half-integer λ's, we define a function $\bar{\phi}$ in analogy with the single case[1]

$$\bar{\phi}(\lambda,K,r)\equiv\phi(\lambda,K,r)/\Gamma(1+2\lambda). \tag{2.14}$$

The Wronskians[4] of $\bar{\phi}(\lambda,K,r)$ and $f(\lambda,K,r)$ are given by

$$W[f(\lambda,K,r),f(\lambda,-K,r)]=2iK,$$

$$W[\bar{\phi}(\lambda,K,r),\bar{\phi}(-\lambda,K,r)]=-\frac{1}{\pi}\mathbf{1}\sin 2\pi\lambda. \tag{2.15}$$

The solution $\bar{\phi}$ can now be expressed in terms of f:

$$\bar{\phi}(\lambda,K,r)=\frac{1}{2i}\{f(\lambda,-K,r)K^{-1}D^T(\lambda,-K) -f(\lambda,K,r)K^{-1}D^T(\lambda,K)\}, \tag{2.16}$$

where the Jost matrix $D(\lambda,\pm K)$ is given by

$$D(\lambda,\pm K)=W[\bar{\phi}(\lambda,K,r),f(\lambda,\mp K,r)] \tag{2.17}$$

and satisfies the relationship

$$D(\lambda,K)K^{-1}D^T(\lambda,-K)=D(\lambda,-K)K^{-1}D^T(\lambda,K). \tag{2.18}$$

A scattering solution can now be defined

$$\psi(\lambda,K,r)\equiv\bar{\phi}(\lambda,K,r)D^{-1T}(\lambda,K)K^{1/2}$$
$$=\frac{1}{2i}\{f(\lambda,-K,r)K^{-1/2}[K^{-1/2}D^T(\lambda,-K)D^{-1T}(\lambda,K)K^{1/2}]-f(\lambda,-K,r)K^{-1/2}\}, \tag{2.19}$$

which gives as a definition of the S matrix

$$S(\lambda,K)=S^T(\lambda,K)=K^{1/2}D^{-1}(\lambda,K)D(\lambda,-K)K^{-1/2}e^{i\pi(\lambda-1/2)}. \tag{2.20}$$

The definition of $D(\lambda,K)$ leads to the relation

$$W(\bar{\phi}_1,\bar{\phi}_2)=\tfrac{1}{4}D(\lambda,-K)K^{-1}W(f_1,f_2)K^{-1}D^T(-\lambda,K)-\tfrac{1}{4}D(\lambda,K)K^{-1}W(f_1,f_2)K^{-1}D^T(-\lambda,-K). \tag{2.21}$$

Substituting from (2.15) for the Wronskians, we obtain

$$D(\lambda,-K)K^{-1}D^T(-\lambda,K)-D(\lambda,K)K^{-1}D^T(-\lambda,-K)=(2i/\pi)\sin 2\pi\lambda\mathbf{1}. \tag{2.22}$$

The symmetry (2.22) can be written in terms of the S matrix to read[5]

$$S(\lambda,K)e^{-i\pi\lambda}-S(-\lambda,K)e^{i\pi\lambda}=(2/\pi)(\sin 2\pi\lambda)K^{1/2}D^{-1}(\lambda,K)D^{-1T}(-\lambda,K)K^{1/2}, \tag{2.23}$$

which gives the matrix analog of the Mandelstam symmetry

$$S(-\tfrac{1}{2}N,K)=(-)^N S(\tfrac{1}{2}N,K) \tag{2.24}$$

for integer and half-integer λ's.

We can understand the implication of the symmetry (2.23) better if we rewrite our S matrices for the two-channel case now in terms of their eigenvalues and a mixing parameter $\epsilon(\lambda,K)$:

$$S(\lambda,K)=\begin{pmatrix}(\cos^2\epsilon^+)S_1+(\sin^2\epsilon^+)S_2 & \sin\epsilon^+\cos\epsilon^+(S_1-S_2)\\ \sin\epsilon^+\cos\epsilon^+(S_1-S_2) & (\sin^2\epsilon^+)S_1+(\cos^2\epsilon^+)S_2\end{pmatrix}. \tag{2.25}$$

A pole of S_1 or S_2 therefore generally appears in all matrix elements of the S matrix.

Similarly

$$S(-\lambda,K)=\begin{pmatrix}(\cos^2\epsilon^-)S_1(-\lambda,K)+(\sin^2\epsilon^-)S_2(-\lambda,K) & \sin\epsilon^-\cos\epsilon^-[S_1(-\lambda,K)-S_2(-\lambda,K)]\\ \sin\epsilon^-\cos\epsilon^-[S_1(-\lambda,K)-S_2(-\lambda,K)] & (\sin^2\epsilon^-)S_1(-\lambda,K)+(\cos^2\epsilon^-)S_2(-\lambda,K)\end{pmatrix}. \tag{2.26}$$

[4] The Wronskian of two matrix functions $A(r)$, $B(r)$ is defined by $W[A(r),B(r)]=A^T(r)B'(r)-A^{T\prime}B(r)$. Here T means "transpose," and primes mean differentiation with respect to r.

[5] Analogous symmetry satisfied by a generalized Jost function for the case of elastic scattering of two particles of spin $\frac{1}{2}$ was shown by B. R. Desai and R. G. Newton, Phys. Rev. **129**, 1437 (1963).

The Mandelstam symmetry (2.24) then implies that when $\lambda=\frac{1}{2}N$

$$\begin{aligned} &\epsilon^{-}=\epsilon^{+}+\tfrac{1}{2}m\pi \\ &S_1(\tfrac{1}{2}N,K)=(-)^N S_1(-\tfrac{1}{2}N,\,K)\,, \quad m \text{ even} \qquad (2.27)\\ &S_1(\tfrac{1}{2}N,K)=(-)^N S_2(-\tfrac{1}{2}N,\,K)\,, \quad m \text{ odd}. \end{aligned}$$

We see from (2.27) that when $S_1(\frac{1}{2}N,K)$ has a pole, i.e., a trajectory passing through a positive integer or half-integer λ, then either $S_1(-\frac{1}{2}N,\,K)$ or $S_2(-\frac{1}{2}N,\,K)$ must have a pole, or a corresponding trajectory passes through the negative integer or half-integer λ. An S_2-type trajectory with $\epsilon+\frac{1}{2}\pi$ is quite indistinguishable from an S_1-type with mixing parameter ϵ, except possibly by its history, if one can define $\sin\epsilon$ as being zero in the high-energy uncoupled limit.

A trajectory can also pass through negative integer and half-integer values through the existence of indeterminacy points. An indeterminacy point is defined by the vanishing of the residue $\beta(\lambda,K)$ when the S matrix in the neighborhood of a pole is written from (2.25) as

$$S(\lambda,K)=\frac{\beta(\lambda,K)}{\lambda-\alpha(K)}\begin{pmatrix}\cos^2\epsilon(\lambda,K) & \cos\epsilon(\lambda,K)\,\sin\epsilon(\lambda,K)\\ \sin\epsilon(\lambda,K)\,\cos\epsilon(\lambda,K) & \sin^2\epsilon(\lambda,K)\end{pmatrix}. \qquad (2.28)$$

Indeterminacy points for $\lambda=-\frac{1}{2}$ and $\lambda=-1$ are explicitly derived in the Appendix.

III. FACTORIZABILITY OF THE RESIDUES AND RESONANCE POLES

The S matrix can be expressed in the neighborhood of one of the poles of its eigenvalues S_n by a Regge-pole form:

$$S_{ij}^{n}=[\lambda-\alpha_n(K)]^{-1}\beta_{ij}^{n}(K)\,. \qquad (3.1)$$

The factorization of residues gives[1]

$$\beta_{ij}^{n}=\beta_i^{n}\beta_j^{n}\,. \qquad (3.2)$$

In the two-channel problem this factorization is manifest because we may write the nth pole term for $S_1(\lambda,K)$ from (2.28)

$$S_1^{n}(\lambda,K)=\frac{\beta_n'(\lambda,K)}{\lambda-\alpha_n'(\lambda,K)}\begin{pmatrix}\cos^2\epsilon & \sin\epsilon\,\cos\epsilon\\ \sin\epsilon\,\cos\epsilon & \sin^2\epsilon\end{pmatrix}. \qquad (3.3)$$

Consider now the T matrix defined by[6]

$$S(\lambda,K)=1+2iT(\lambda,K)\,, \qquad (3.4)$$

$$T(\lambda,K)=[K^{-\lambda}M(\lambda,K)K^{-\lambda}+e^{-i\pi\lambda}/\sin\pi\lambda]^{-1} \qquad (3.5)$$

$$=\sum_n\sum_{i=1}^{N}\frac{K^{2\lambda}\bar{\beta}_n{}^{i}}{\lambda-\alpha_n{}^{i}(K)}+K^{2\lambda}\tilde{T}(\lambda,K)\,. \qquad (3.6)$$

As usual $\bar{\beta}_n{}^i$ and $\alpha_n{}^i$ give the residue and position of the nth "resonance pole" in the ith eigenchannel. $\tilde{T}$ serves to define the nonresonant part of the scattering.

The trajectories $\lambda\equiv\alpha_n{}^i(K)$ are defined by the solutions of

$$\mathrm{Det}[M(\lambda,K)+K^{2\lambda}(e^{-i\pi\lambda}/\sin\pi\lambda)]=0\,. \qquad (3.7)$$

For simplicity consider a 2-channel problem. The argument that follows holds for any $N>1$. On a trajectory, then, we have explicitly

$$[M_{11}(\alpha,K)+k_1^{2\alpha}e^{-i\pi\alpha}/\sin\pi\alpha]\times[M_{22}(\alpha,K)+k_2^{2\alpha}e^{-i\pi\alpha}/\sin\pi\alpha]=M_{12}^2(\alpha,K)\,. \qquad (3.8)$$

Let us look at this expression below the threshold of the second channel, but above the first. That is, keep $k_1^2>0$ and $k_2=iK_2$, K_2 real. Then, *if* α were to pass through a *real* value, since M's are real analytic functions of k_i^2 and λ,[6] we have

$$[M_{11}(\alpha,k_1^2)+k_1^{2\alpha}\cot\pi\alpha-ik_1^{2\alpha}]\times\left[M_{22}(\alpha,K_2^2)+K_2^{2\alpha}\frac{1}{\sin\pi\alpha}\right]=M_{12}^2(\alpha,K)\,. \qquad (3.9)$$

Taking the imaginary part, we obtain

$$M_{22}(\alpha,K_2^2)+K_2^{2\alpha}(\sin\pi\alpha)^{-1}=0\,. \qquad (3.10)$$

Of course, from (3.6) this implies that

$$\bar{\beta}_{11}=0\,, \qquad (3.11)$$

and hence that

$$\bar{\beta}_{12}=\bar{\beta}_{21}=0\,. \qquad (3.12)$$

In other words, for such a situation, the channels 1 and 2 are decoupled, since the trajectory has a zero residue for $1\rightarrow1$ and $1\rightarrow2$ channels. This corresponds to the case of a "bound state" of channel 2 (and in general cases of more channels, to a "bound state" of *all* channels having thresholds above the bound state energy).[7]

IV. CUSP BEHAVIOR AND ANGULAR MOMENTUM

Consider λ *real* and to be specific, a two-channel problem. We are interested then in the behavior of T_{11} near the threshold $k_2^2\approx0$.

[6] Y. N. Srivastava (to be published).

[7] For a corresponding discussion in the context of S-matrix theory, see C. E. Jones, Princeton University, 1964 (unpublished).

Case I: $\lambda>0$

From Eq. (3.5), we can write

$$k_1^{2\lambda}T_{11}^{-1}=\left[M_{11}(k_1^2)+k_1^{2\lambda}\frac{e^{-i\pi\lambda}}{\sin\pi\lambda}\right]-M_{12}^2\left[M_{22}(k_2^2)+k_2^{2\lambda}\frac{e^{-i\pi\lambda}}{\sin\pi\lambda}\right]^{-1}. \quad (4.1)$$

Since M_{22} can be written as

$$M_{22}(k_2^2)\underset{k_2^2\to 0}{\approx} M_{22}^{(0)}+k_2^2M_{22}^{(1)}+\cdots, \quad (4.2)$$

for $\lambda>1$, the kinematical factor $k_2^{2\lambda}e^{-i\pi\lambda}/\sin\pi\lambda$ would not be important compared to $M_{22}^{(1)}$, for small k_2^2. So, let us restrict ourselves to $0<\lambda<1$. There are two cases.

(*a*) k_2^2 above threshold: Then, since in this region (i.e., $0<\lambda<1$), $(\sin\pi\lambda)^{-1}$ is finite, let us write, to lowest power in k_2^2

$$k_1^{2\lambda}T_{11}^{-1}\underset{k_2^2\to 0+}{\approx}(M_{11}+k_1^{2\lambda}\cot\pi\lambda-ik_1^{2\lambda})-\frac{M_{12}^2}{M_{22}+k_2^{2\lambda}\cot\pi\lambda-ik_2^{2\lambda}}. \quad (4.3)$$

Defining

$$N_{11}\equiv M_{11}+k_1^{2\lambda}\cot\pi\lambda, \quad (4.4)$$

we have

$$k_1^{2\lambda}T_{11}^{-1}\approx N_{11}-ik_1^{2\lambda}-\frac{M_{12}^2}{M_{22}^{(0)}}\left(1-\frac{k_2^{2\lambda}e^{-i\pi\lambda}}{M_{22}^{(0)}\sin\pi\lambda}\right)$$
$$\approx\left\{\left(N_{11}-\frac{M_{12}^2}{M_{22}^{(0)}}\right)+\frac{M_{12}^2}{M_{22}^{(0)2}}(\cot\pi\lambda)k_2^{2\lambda}\right\}-i\left(k_1^{2\lambda}+\frac{M_{12}^2}{M_{22}^{(0)2}}k_2^{2\lambda}\right).$$

After some algebra, one obtains

$$|T_{11}|^2\approx\frac{k_1^{4\lambda}}{(N_{11}-M_{12}^2/M_{22}^{(0)})^2+k_1^{4\lambda}}\left\{1-2k_2^{2\lambda}\frac{(N_{11}-M_{12}^2/M_{22}^{(0)})(M_{12}^2/M_{22}^{(0)2})\cot\pi\lambda+k_1^{2\lambda}M_{12}^2/M_{22}^{(0)2}}{(N_{11}-M_{12}^2/M_{22}^{(0)})^2+k_1^{4\lambda}}\right\}. \quad (4.5)$$

This already shows that

$$\left.\frac{d}{dk_1^2}|T_{11}|^2\right|_{k_2^2\to 0^+}\to\infty$$

for $0<\lambda<1$.

(*b*) k_2^2 *below threshold*:

$$k_2\to +iK_2=e^{i\pi/2}K_2, \quad k_2^{2\lambda}\to e^{i\pi\lambda}K_2^{2\lambda}. \quad (4.6)$$

Then,

$$k_1^{2\lambda}T_{11}^{-1}\underset{k_1^2\to 0-}{\approx}M_{11}+k_1^{2\lambda}\cot\pi\lambda-ik_1^{2\lambda}-\frac{M_{12}^2}{M_{22}^0+K_2^{2\lambda}/\sin\pi\lambda}$$
$$\approx\left(N_{11}-\frac{M_{12}^2}{M_{22}^{(0)}}\right)+\frac{M_{12}^2}{M_{22}^{(0)}\sin\pi\lambda}K_2^{2\lambda}-ik_1^{2\lambda}. \quad (4.7)$$

Similar to (4.5) one again obtains

$$|T_{11}|^2\approx\frac{k_1^{4\lambda}}{(N_{11}-M_{12}^2/M_{22}^{(0)})^2+k_1^{4\lambda}}\left\{1-2K_2^{2\lambda}\frac{(N_{11}-M_{12}^2/M_{22}^{(0)})M_{12}^2/M_{22}^{(0)2}\sin\pi\lambda}{(N_{11}-M_{12}^2/M_{22}^{(0)})^2+k_1^{4\lambda}}\right\}. \quad (4.8)$$

Again, the quantity

$$\left.\frac{d}{dk_1^2}|T_{11}|^2\right|_{k_2^2\to 0-}$$

becomes infinite for $\lambda<1$. Also, one sees that the coefficients of the lowest term in k_2 (K_2) from above (below) the threshold are different—explicitly showing the discontinuity in the first derivative (with respect to energy of σ_{11} at $k_2^2=0$).

The expressions (4.5) and (4.7) for *S waves* become

$$|T_{11}|^2\underset{k_1^2\to 0+}{\sim}\frac{k_1^2}{(M_{11}-M_{12}^2/M_{22}^{(0)})^2+k_1^2}\left\{1-\frac{k_1M_{12}^2/M_{22}^{(0)2}}{(M_{11}-M_{12}^2/M_{22}^0)^2+k_1^2}2k_2\right\} \quad (4.9a)$$

$$\underset{k_1^2\to 0-}{\sim}\frac{k_1^2}{(M_{11}-M_{12}^2/M_{22}^{(0)})^2+k_1^2}\left\{1-2K_2\frac{(M_{11}-M_{12}^2/M_{22}^{(0)})^2M_{12}^2/M_{22}^{(0)2}}{(M_{11}-M_{12}^2/M_{22}^{(0)})^2+k_1^2}\right\}. \quad (4.9b)$$

Let

$$C\equiv\frac{k_1}{M_{11}-M_{12}^2/M_{22}^0}$$

and

$$B\equiv\frac{M_{12}^2/M_{22}^{(0)2}}{(M_{11}-M_{12}^2/M_{22}^{(0)})^2+k_1^2}. \qquad (4.10)$$

Then, (4.9) takes the familiar form

$$|T_{11}|^2=c^2/(1+c^2)(1-2Bk_2), \quad \text{above threshold}$$
$$=c^2/(1+c^2)(1-2(B/c)K_2), \quad \text{below threshold} \qquad (4.11)$$

which shows explicitly the discontinuity and the infinite derivative (with respect to k^2) at the threshold. [Equation (4.11) compares exactly with Eq. (22) of Ref. 8.]

Case II: $\lambda<0$ (and $\neq -N$).

As before, we write

$$T_{11}^{-1}=k_1^{-2\lambda}M_{11}+\frac{e^{-i\pi\lambda}}{\sin\pi\lambda}-\frac{(k_1k_2)^{-2\lambda}M_{12}^2}{k_1^{-2\lambda}M_{22}+e^{-i\pi\lambda}/\sin\pi\lambda}. \qquad (4.12)$$

To lowest power in k_2:

$$\approx N_{11}'-i-k_1^{-2\lambda}M_{12}^2e^{i\pi\lambda}k_2^{-2\lambda}\sin\pi\lambda, \qquad (4.13)$$

where

$$N_{11}'\equiv k_1^{-2\lambda}M_{11}+\cot\pi\lambda. \qquad (4.14)$$

Again, there are two cases.

(*a*) *k_2 real* (*above threshold*). From Equation (4.13), we obtain

$$T_{11}^{-1}\approx(N_{11}'-k_1^{-2\lambda}M_{12}^2k_2^{-2\lambda}\cos\pi\lambda\sin\pi\lambda)-i(1+k_1^{-2\lambda}k_2^{-2\lambda}M_{12}^2\sin^2\pi\lambda),$$

which after some algebra reduces to

$$|T_{11}|^2\approx\frac{1}{1+N_{11}'^2}\left\{1-2k_1^{-2\lambda}M_{12}^2\sin\pi\lambda\times\frac{\sin\pi\lambda-N_{11}'\cos\pi\lambda}{1+N_{11}'^2}k_2^{-2\lambda}\right\}. \qquad (4.15)$$

Hence, $(\partial/\partial k_1^2)|T_{11}|^2$ would blow up at $k_2^2=0^+$, unless $\lambda<-1$. The restriction imposed above about $\lambda\neq-N$ really does not hinder us, therefore, since the interesting region is only till $\lambda>-1$.

(*b*) $k_2=e^{i\pi/2}K_2$ (*below threshold*). Repeating the procedure, one obtains

$$T_{11}^{-2}\approx[N_{11}'-k_1^{-2\lambda}M_{12}^2(\sin\pi\lambda)K_2^{-2\lambda}]-i, \qquad (4.16)$$

which gives us finally

$$|T_{11}|^2\approx\frac{1}{1+N_{11}'^2}\times\left\{1+2K_2^{-2\lambda}\frac{k_1^{-2\lambda}N_{11}'M_{12}^2\sin\pi\lambda}{1+N_{11}'^2}\right\}. \qquad (4.17)$$

[8] M. H. Ross and G. L. Shaw, Ann. Phys. (N. Y.) 9, 391 (1960).

From the above analysis, we obtain for $0>\lambda>-1$ that there exists a discontinuity in the first derivative of σ_{11} at the new threshold.

Case III. $\lambda=0$.

The behavior of the amplitude near this point is much more complicated.[9] It was shown in Ref. 6 that as $\lambda\to0$, near any threshold, the elements T_{ij} have the "threshold poles" and also the diagonal elements T_{ii} have "zeros," at slightly displaced positions. Thus, we may conclude that the discontinuity in the first derivative of $|T_{11}|^2$ found above for cases I and II at the second threshold is still there for $\lambda=0$, since the arrival of the two sets of threshold poles (viz., $k_2^2\to0$, $K_2^2\to0$) is different.

The above analysis shows that the "cusp" behavior in the (elastic) cross section as a new channel opens up is not only present for S waves, as known earlier, but that this behavior persists as one varies continuously the (real) angular momentum from $l=-\frac{3}{2}$ to $l=+\frac{1}{2}$.

APPENDIX: INDETERMINACY POINTS FOR $\lambda=-\frac{1}{2}$ AND $\lambda=-1$

Consider the Schrödinger equation (2.3) for a two-channel problem with a potential of the form (2.1) given by

$$V_{ij}(r)=\begin{pmatrix}V(r) & U(r)\\ U(r) & W(r)\end{pmatrix}. \qquad (A1)$$

Let $\bar{\varphi}(r)$ be represented by a power series:

$$\bar{\varphi}_{ij}(r)=\sum_n a_{ij}{}^n r^{n+\lambda+1/2},$$

where

$$a_{ij}{}^0=\begin{pmatrix}\Gamma^{-1}(1+2\lambda) & 0\\ 0 & \Gamma^{-1}(1+2\lambda)\end{pmatrix}. \qquad (A2)$$

Expanding the potential in a power series, we write

$$rV_{ij}(r)=\sum_n V_{ij}{}^n r^n$$

with

$$V_{11}{}^n=v_n,\quad V_{22}{}^n=w_n,\quad V_{12}{}^n=V_{12}{}^n=u_n. \qquad (A3)$$

Dropping the matrix subscripts, we obtain the recursion relation for the coefficient a^n

$$a^p=\left(\sum_{m=0}^{p-1}V^ma^{p-m-1}-K^2a^{p-2}\right)/p(p+2\lambda). \qquad (A4)$$

A necessary condition that a trajectory for $-K$ coincides with a trajectory for K at the same λ is the condition for an indeterminacy point. This is equivalent

[9] The point near $\lambda=0$ and near thresholds has been investigated in detail by Dr. W. Carnahan (private communication, to be published).

to the conditon[10]

$$\mathrm{Det}D(\lambda,K)=\mathrm{Det}D(\lambda,-K)=0,$$

and hence (A5)

$$\mathrm{Det}\,\bar{\varphi}(\lambda,K,r)=0.$$

Because of the boundary condition (A2) this condition may be met at negative integer and half-integer $\lambda=-\frac{1}{2}N$, but because of the singular denominator in (A4) only if Det a^n vanishes in the limit $\lambda\rightarrow-\frac{1}{2}N$.

$$\mathrm{Det}\ a^N=\mathrm{Det}\left\{\frac{1}{N(N+2\lambda)}\times\left[\sum_{m=0}^{p-1}V^m a^{N-m-1}-K^2a^{N-2}\right]\right\}\xrightarrow[\lambda\to-\frac{1}{2}N]{}0. \quad \text{(A6)}$$

Otherwise only the first N coefficients a^p vanish, and the solution becomes a solution with finite determinant with all matrix elements having the leading term $r^{N/2+\frac{1}{2}}$ and nonvanishing determinant.

In order to apply the condition (A6) at $\lambda=-\frac{1}{2}$ and $\lambda=-1$, we note:

$$a^1=V^0a^0/(1+2\lambda),$$
$$a^2=[V^{02}/(1+2\lambda)+V^1-K^2]a^0/2(2+2\lambda). \quad \text{(A7)}$$

The indeterminacy condition is then satisfied when

$$\lambda=-\tfrac{1}{2}:\quad \mathrm{Det}V^0=v_0w_0-u_0^2=0; \quad \text{(A8)}$$

$$\lambda=-1:\quad \mathrm{Det}[K^2+V^{02}-V']=\{k^2-k_{10}^2+v_0^2-v_1+u_0^2\}\cdot\{k^2-k_{20}^2+w_0^2-w_1+u_0^2\}-[u_0(v_0+w_0)+u_1]^2=0, \quad \text{(A9)}$$

where k_{10} and k_{20} are the thresholds of the two channels. Letting

$$k_v^2=k_{10}^2-v_0^2+v_1,$$
$$k_w^2=k_{20}^2-w_0^2+w_1, \quad \text{(A10)}$$

which are the $\lambda=-1$ indeterminacy in the limit of no coupling, we have

$$\lambda=-1:\quad k^2=\tfrac{1}{2}[k_v^2+k_w^2]\pm\tfrac{1}{2}[(k_v^2-k_w^2)^2+4(u_0v_0+u_0w_0+u_1)^2]^{1/2}. \quad \text{(A11)}$$

The first condition (A8) shows that there are no indeterminacy points at $\lambda=-\frac{1}{2}$. As in the single-channel case every trajectory which goes through $\lambda=-\frac{1}{2}$ must have a compensating trajectory at $\lambda=\frac{1}{2}$.

If, however, $v_0w_0=u_0^2$, then every point is an indeterminacy point. This means a trajectory "lies down[11]" at $\lambda=-\frac{1}{2}$ just as it does in the single-channel case if $v_0=0$. In that case, the Mandelstam symmetry (2.24) does not hold at $\lambda=-\frac{1}{2}$, because the right-hand side of (2.23) will not vanish and trajectories can pass through $\lambda=-\frac{1}{2}$ without any particular restriction on the energy.

The indeterminacy points at $\lambda=-1$ as given by (A11) are both on the real energy axis. (In the single-channel case, there is one on the real energy axis.) For weak coupling,

$$k_1^2=k_v^2+\frac{[u_0(v_0+w_0)+u_1]^2}{(k_v^2-k_w^2)}.$$
$$k_2^2=k_w^2-\frac{[u_0(v_0+w_0)+u_1]^2}{(k_v^2-k_w^2)}. \quad \text{(A12)}$$

In other words, the lower-indeterminacy point is lowered and the higher one raised by the coupling $U(r)$, regardless of the sign of the coupling potential. Since the k_v^2 is likely to be lower than k_w^2, because $k_{10}^2<k_{20}^2$ by definition, this is in accordance with the statement that the coupling always appears "attractive" for the channel with the lower threshold. Obviously, such a statement is not rigorous and depends on the detailed nature of the potentials.

[10] Actually due to the Mandelstam symmetry condition at integer and half-integer λ's, viz., Eq. (2.24), this is also a sufficient condition for an indeterminacy point.

[11] P. Kaus, Nuovo Cimento **29**, 598 (1963).