($\kappa^2=0$). On the other hand, $(j^\mu+m\partial^\mu\phi+\partial^\mu G-m^2A^\mu)$ does not allow for any realistic physical interpretation as a local current density. The third term $\partial^\mu G$ is not a physical quantity. The fourth term $-m^2A^\mu$ is actually the mass term of the vector particle; to look at it as a part of a current density is just too artificial. Thus, we have to conclude that when the scalar field acquires a bare mass, no physically meaningful conserved local current density can be constructed (when $\mathcal{L}_{\rm int}=0$, $\partial_\mu j^\mu=0$ follows).

(A3) is modified into

$$(\partial^2-\kappa^2)\phi+(\kappa^2/m)G+J=0\,,$$

where

$$J\equiv(\delta\mathcal{L}_{\rm int})/(\delta\phi)\,.$$

Together with (A1) and (A2), we have

$$\partial_\mu j^\mu=mJ\,,$$

which is the counterpart of (10) in the massless case. We can define

$$J^\mu(x)=j^\mu(x)+m\partial^\mu\int D(x-x')J(x')\,,$$

$$-\partial^2D=1\,,$$

as a conserved current density:

$$\partial_\mu J^\mu=0;$$

but it is nonlocal.

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Multichannel Effective-Range Theory from the N/D Formalism*

Pran Nath†
Department of Physics, University of California, Riverside, California

AND

G. L. Shaw
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California

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An effective-range theory for systems of many coupled two-body channels is given using the N/D formalism. The effective-range expansion is carried out in the amplitudes M_{ij} (where M is essentially the matrix T^{-1} with the right-hand cut removed). Quite in analogy with the single-channel effective-range theory, the diagonal elements M_{ii} are given by an expression quadratic in k_i, the relative momentum in channel i. The effective ranges R_{ii} are given by certain principal-value integrals which depend on the position of the left-hand singularities in the corresponding channels and can be taken to be energy-independent to the same extent as in the one-channel theory. The nondiagonal elements M_{ij}, in general, have a weak energy dependence and can approximately be treated as constants. A two-channel computer experiment is performed to test these proposals in detail. Three different situations for the left-hand cut are considered: (i) a set of monopoles, (ii) a set of dipoles, and (iii) the left-hand cut produced by the exchange of scalar particles in the "crossed" t reactions. For a large number of situations considered, the simple features proposed for the multichannel effective range theory were found to exist. The above formalism is similar to the multichannel effective-range theory of Ross and Shaw in the potential model.

I. INTRODUCTION

We wish to discuss the energy dependence of the partial-wave amplitudes for many coupled two-body channels. We assume that these amplitudes satisfy the coupled N/D partial-wave dispersion relations. The purpose is to obtain a simple effective-range theory by removing the right-hand cuts explicitly and approximating the rest of the scattering matrix that contains only the left-hand singularities. We assume that only n channels need be considered explicitly. The inverse of the $n\times n$ scattering matrix $T(=ND^{-1})$ can be written from unitarity as $T^{-1}=M(s)-i\rho(s)$ (s is the square of the total energy in the center-of-mass system), where $\rho(s)$ is a diagonal right-hand cut function. The matrix M is both real and symmetric to the right of the left-hand singularities. Over an approximately small physical region of energy, therefore, an effective-range expansion in M can be carried out.

We propose an effective range formula for M, $M(s)=B^{-1}(s)-P(s)$, where $B(s)$ contains the unphysical singularities of the scattering amplitude T and $P(s)$ is a diagonal matrix and contains most of the energy dependence of $M(s)$. If the left-hand singularities carried by $M(s)$ do not lie very close to the energy region of interest, a further simplification in the effective-range formula for M follows. The nondiagonal effective ranges in this situation are small (the effective-range matrix R is approximately diagonal), and for small value of k^2 (see the computer experiment in Sec. III) we have the linear relationship $M=M(0)+\frac{1}{2}R[k^2-k^2(0)]$. The diagonal effective ranges can be

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related to the position of the singularities in the corresponding channels. These features of the multichannel effective-range theory obtained from the N/D formalism are common to the multichannel effective-range theory of Ross and Shaw[1] obtained in the potential model.

In order to test our proposals for the multichannel effective-range theory we perform a two-channel computer experiment. The calculations described in Sec. III are done for the case when both the channels are considered to be in relative P waves and three different situations for the left-hand cut are considered. The three different situations considered for the left-hand cut are (i) the left-hand cut is replaced by a set of monopoles, (ii) it is replaced by a set of dipoles, and (iii) it is produced by the exchange of scalar particles in the crossed t reactions of the corresponding channels. For a large number of situations investigated, the simple features of the multichannel effective-range theory were supported by the two-channel computer experiment.

In a many-coupled channel problem where a full analysis of the problem, numerical or otherwise, is in general very complicated, effective-range approximation can be of considerable value. Effective-range theory provides a useful tool for studying the general features of multichannel reactions as well as for an efficient parametrization of reaction cross sections. It may be stressed that all channels near the energy region of interest must be considered explicitly in order that the effective-range theory be accurate.

II. MULTICHANNEL EFFECTIVE-RANGE THEORY

Let us consider the usual N/D equations for a system of n strongly coupled two-body channels. The invariant partial-wave amplitude T is defined in terms of the S matrix by[2]

$$T\equiv ND^{-1}=\rho^{-1/2}\frac{S-1}{2i}\rho^{-1/2}, \qquad (2.1)$$

where the numerator function N has only left-hand cuts and the denominator function D has only the right-hand cuts. ρ is a diagonal matrix.

$$\rho_{ij}=\delta_{ij}\frac{k_i^{2l_i+1}}{\sqrt{s}}, \qquad (2.2)$$

k_i and l_i are the momentum and orbital angular momentum in channel i. The N and D equations are[3]

$$N(s)=B(s)+\frac{1}{\pi}\int_0^\infty\left[B(s')-\frac{s-s_0}{s'-s_0}B(s)\right]\times\theta\rho(s')N(s')\frac{ds'}{s'-s}, \qquad (2.3)$$

$$D(s)=1-\frac{s-s_0}{\pi}P\int_0^\infty\theta\rho(s')N(s')\frac{ds'}{(s'-s)(s'-s_0)}-i\theta\rho(s)N(s), \qquad (2.4)$$

where the "generalized potential"[4] $B(s)$ is regular in the physical region and the function θ ensures that the right-hand cuts in T^{-1} start at the appropriate thresholds s_i:

$$\theta_{ij}=\delta_{ij}\theta(s-s_i). \qquad (2.5)$$

Note that the solutions T are independent of the subtraction point s_0 in D.[5] Moreover, $A(s)$ is symmetric if the input $B(s)$ is symmetric as required by time-reversal invariance.[6] In order that a unique solution to (2.3) may exist, the kernel of the integral equation should be L^2; then (2.3) is an inhomogeneous Fredholm equation of the second kind.

Now we introduce the M matrix defined by

$$T^{-1}\equiv M(s)-i\rho(s) \qquad (2.6)$$

so that

$$M(s)=\mathrm{Re}(D)N^{-1}-i(\theta-1)\rho(s), \qquad (2.7)$$

where θ is given by (2.5). Thus, M is symmetric, real to the right of the singularities in $B(s)$, and due to the second term in (2.7) remains an even function of all the momenta k_i as one continues in energy below a threshold s_i. In the one-channel case, $M=k^{2l+1}/s^{1/2}\cot\delta$.

Solutions to coupled integral equations are in general complicated. However, if the left-hand cut is replaced by a set of poles, the solution to the integral equation (2.3) reduces to quadrature.

Let us consider first the simplest situation in which the left-hand cut is replaced by $g/(s+m)$, where g is an $n\times n$ matrix of constants,[7]

$$B(s)=\frac{g}{(s+m)}. \qquad (2.8)$$

Now choose the subtraction point $s_0=-m$. The kernel $\mathcal{K}(s,s')=0$ so that solution to (2.3) and (2.4) can immediately be written down;

$$M(s)=g^{-1}(s+m)-\frac{(s+m)^2}{\pi}P\int_0^\infty\frac{\theta\rho(s')ds'}{(s'+m)^2(s'-s)}-i(\theta-1)\rho(s), \qquad (2.9)$$

where P implies a principal value integral in (2.9). The integrals in (2.9) can be explicitly evaluated and the relativistic result for the S-wave case and particles of equal mass m_i in channel i is

$$M_{ii}(s)=(g^{-1})_{ii}(s+m)+\frac{1}{2\pi}\left\{\frac{m+s}{m}+\frac{s_i(s+m)+2m(s_i+m)}{2m[m(m+s_i)]^{1/2}}\ln\frac{(m+s_i)^{1/2}-m^{1/2}}{(m+s_i)^{1/2}+m^{1/2}}+\left(\frac{s-s_i}{s}\right)^{1/2}\ln\frac{s^{1/2}+(s-s_i)^{1/2}}{s^{1/2}-(s-s_i)^{1/2}}\right\} \qquad (2.10)$$

[1] M. H. Ross and G. L. Shaw, Ann. Phys. (N. Y.) **13**, 147 (1961).
[2] We use units $\hbar=c=m_\pi=1$.
[3] J. Uretsky, Phys. Rev. **123**, 1459 (1961); D. Y. Wong, *ibid.* **126**, 1220 (1962).
[4] G. Chew and S. Frautschi, Phys. Rev. **124**, 264 (1961).
[5] A. W. Martin, Phys. Rev. **135**, B967 (1964).
[6] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).
[7] See, e.g., W. Frazer, S. Patil, and N. Xuong, Phys. Rev. Letters **12**, 178 (1964).

for $s>s_i$. For $i\neq j$,

$$M_{ij}(s)=(g^{-1})_{ij}(s+m). \tag{2.11}$$

A simple (and familiar for the single-channel case) nonrelativistic result can be obtained (where we define $4k_{i0}^2\equiv s_i+m$):

$$M_{ii}=\left(4(g^{-1})_{ii}k_{i0}^2-\frac{k_{i0}}{2m_i}\right)+\left(4(g^{-1})_{ii}+\frac{1}{2m_ik_{i0}}\right)k_i^2. \tag{2.12}$$

The relativistic P-wave scattering produced by (2.8) is given by

$$M_{ij}(s)=(g^{-1})_{ij}(s+m)-\frac{1}{8\pi}\delta_{ij}\left\{\frac{s}{ms_i}(s+m)(m+s_i)\right.$$
$$+\tfrac{1}{2}\left(\frac{m+s_i}{m}\right)^{1/2}\frac{s(s_i+m)+3m(s_i-s)}{m}\ln\frac{(m+s_i)^{1/2}-m^{1/2}}{(m+s_i)^{1/2}+m^{1/2}}$$
$$\left.+\left(\frac{(s-s_i)^3}{s}\right)^{1/2}\ln\frac{s^{1/2}-(s-s_i)^{1/2}}{s^{1/2}+(s-s_i)^{1/2}}\right\}, \tag{2.13}$$

$s>s_i$. If one considers higher order poles for the left-hand cut, an explicit solution to M can, in principle, be written down. However, the solutions become increasingly cumbersome. Furthermore for any given potential $B(s)$, the kernel of the integral equation (2.3) would in general be nondegenerate, and one would be required to solve the coupled integral equations (2.3) numerically. Effective-range theory, on the other hand, is known to provide a useful tool for the purpose of studying the general features of the many coupled channel problems as well as for a purely phenomenological analysis of multichannel reactions.[1] Such a theory in multichannel N/D formalism would be equally interesting. We have just seen that for the situation in which the left-hand cut is replaced by a matrix of monopoles, of the form $g/(s+m)$, a simple effective-range expansion in M can be written down. We would therefore like to examine the effective-range expansion in M for more general situations. Let us assume that the potential $B(s)$ has the form

$$B(s)=g/f(s), \tag{2.14}$$

where again g is an $n\times n$ matrix of coefficients. We solve the N/D equations using the Fulton-Shaw approximation.[8,9] This approximation has the same degree of simplicity as the determinantal method, but avoids the subtraction-point dependence and the lack of symmetry of the determinantal method.[10] Define

$$N(s)=B(s)C(s). \tag{2.15}$$

We substitute (2.15) in (2.3) and (2.4) and replace $C(s')$ by $C(s)$ in all integrals over $N(s')$ to get

$$M(s)=B^{-1}(s)-\left[\frac{1}{\pi}P\int_0^\infty\frac{f^2(s)\theta\rho(s')ds'}{f^2(s')(s'-s)}+i(\theta-1)\rho(s)\right]. \tag{2.16}$$

When the potential $B(s)$ does not have the form (2.14) a more general form for $M(s)$ is expected[11]:

$$M(s)=B^{-1}(s)-P(s), \tag{2.17}$$

where

$$P_{ij}(s)=\delta_{ij}\left[\frac{1}{\pi}P\int_{s_i}^\infty\frac{\rho_i(s')B_{ij}^2(s')ds'}{(s'-s)B_{ij}^2(s)}+i(\theta(s-s_i)-1)\rho_i(s)\right]. \tag{2.18}$$

We equate certain principal-value integrals in obtaining (2.17) and (2.18). Equations (2.17) and (2.18) are expected to provide a useful effective-range theory for a large class of problems with relatively simple left-hand singularities. The energy dependence of the scattering matrix $M(s)$ is given in terms of certain principal value integrals which can be easily determined by numerical integration.

A very simple effective-range theory results if one makes a further simplifying assumption. We note that the matrix $P(s)$ is diagonal and contains most of the energy dependence of $M(s)$ if $B(s)$ itself has a weak energy dependence. In such situations one may reasonably assume that the effective-range matrix R in the expansion of $M(s)$ is diagonal and for small values of k^2 (see the computer experiment in Sec. III) we have the linear relation

$$M(s)=M(s_0)+\tfrac{1}{2}R[k^2-k^2(s_0)], \tag{2.19}$$

where

$$R_{ij}=\delta_{ij}R_i=-2\frac{d}{dk_i^2}P_{ij}\bigg|_{s_0}. \tag{2.20}$$

If the left-hand singularities are very close to the energy region of interest, $B(s)$ may significantly dominate the energy behavior of M such that it would necessitate the use of a full effective-range matrix R. The effective-range formulas (2.19) and (2.20) allow us to relate the slope of the scattering amplitudes M_{ii}'s (and hence the effective ranges R_i's), directly to the position of the singularities. In simple cases like (2.12), one can directly see the analytic dependence of the slope on the position of the singularities. In more complicated situations, however, the slope is given by the derivative of a principal-value integral (see 2.20) and this dependence may not be immediately clear.

[8] T. Fulton, in *Elementary Particle Physics and Field Theory* (W. A. Benjamin, Inc., New York, 1963), Vol. I, p. 55.
[9] G. Shaw, Phys. Rev. Letters **12**, 345 (1964).
[10] M. Baker, Ann. Phys. (N. Y.) **4**, 271 (1958).
[11] P. Nath, Ph.D. thesis, Stanford University, 1964 (unpublished).

III. NUMERICAL TEST OF THE MULTICHANNEL EFFECTIVE-RANGE APPROXIMATION

We have proposed in Sec. II a simple effective-range formula for systems of coupled two-body channels with angular momenta l_i. The familiar scattering matrix M (which carries only the left-hand singularities and is thus real and symmetric for real energy to the right of the left-hand singularities) is used for an effective-range expansion and we find that, quite in analogy with the one-channel effective-range theory, the diagonal elements M_{ii} are given by expressions quadratic in momenta k_i and the effective ranges R_{ii} can be taken to be energy-independent to the same degree as in the one-channel theory. The nondiagonal elements M_{ij} are, to a good approximation, expected to be energy-independent.

In this section we test these proposals in detail for the case of two coupled channels. In the present calculation we consider P-wave scattering ($l_1=l_2=l=1$) and the approximations (2.19)–(2.20) are tested under a variety of different conditions. Later in this section we shall demonstrate our results by some typical cases. The quantity of interest to us to compute numerically is the matrix M, Eq. (2.7). To compute $M(s)$ we solve the following set of coupled N/D integral equations

$$N_{ij}(s)=B_{ij}(s)+\frac{1}{\pi}\sum_{k=1}^{2}\int_0^\infty\left[B_{ik}(s')-\frac{s-s_0}{s'-s_0}B_{ik}(s)\right]\times\theta(s'-s_k)\rho_k(s')\frac{N_{kj}(s')ds'}{(s'-s)} \quad (3.1)$$

and

$$\mathrm{Re}D_{ij}(s)=\delta_{ij}-\frac{s-s_0}{\pi}P\int_{s_i}^\infty\frac{\rho_i(s')N_{ij}(s')ds'}{(s'-s)(s'-s_0)}. \quad (3.2)$$

For a given potential $B(s)$, the coupled integral equations (3.1) are solved by matrix inversion[11] (after replacing the s' integration by a sum over a discrete set of values of s') to obtain the N functions. Once the N functions are known, the D functions are given by the principal-value integral (3.2). To test the accuracy of our solution, we check the stability of the solutions against variations on the mesh size and symmetry as well as subtraction-point independence of the $N(\mathrm{Re}D)^{-1}$ solutions. The accuracy of the calculations is verified by substituting the values of N and D thus obtained in the equation

$$N(s)=B(s)\ \mathrm{Re}D(s)+\frac{P}{\pi}\int_0^\infty B(s')\theta\rho(s')N(s')\frac{ds'}{s'-s}. \quad (3.3)$$

In addition, the solutions to case (i) below were obtained by quadrature and checked against the full computer program.

We consider three different situations for the potential $B(s)$:

(i) $B(s)$ is a 2×2 matrix of monopoles with elements

$$B_{ij}(s)=g_{ij}/(s+m_{ij}); \quad (3.4)$$

(ii) $B(s)$ is a 2×2 matrix of dipoles with elements

$$B_{ij}(s)=g_{ij}/(s+m_{ij})^2; \quad (3.5)$$

(iii) $B_{ij}(s)$ is produced by the exchange of a scalar particle (whose mass-squared is m_{ij}) in the "crossed" t reaction of the scattering channel ij, i.e.,

$$B_{ij}(s)=\tfrac{1}{2}g_{ij}k_i^{-2}k_j^{-2}Q_1\left(\frac{\frac{1}{2}s-m_i^2-m_j^2+m_{ij}}{2k_ik_j}\right). \quad (3.6)$$

To perform the numerical test we first choose a particular $B(s)$ and fix the kinematical conditions: the masses in channels 1 and 2, the orbital momenta (we choose $l_1=l_2=l=1$) as well as the coupling constants g_{ij} and the constants m_{ij}. Equations (3.1)–(3.2) are then solved to compute $M(s)$. Such calculations are performed for each $B(s)$ (3.4) through (3.6).

Typical results are shown in Figs. 1–5. The matrix elements M_{11}, M_{22}, and M_{12} ($=M_{21}$) are plotted against k_1^2. We find the relation (linear in k^2)

$$M=M(s_0)+\tfrac{1}{2}R[k^2-k^2(s_0)] \quad (3.7)$$

for significantly large values of k^2. The nondiagonal element M_{12} ($=M_{21}$) is almost constant, so that the nondiagonal effective range $R_{12}\approx0$. The effective-range matrix R is thus diagonal and, in the region investigated, independent of energy as expected. The effective-range formula (3.7) was found to be good even for the diagonal effective ranges being significantly different. In addition, we note that the theoretical estimate of the diagonal

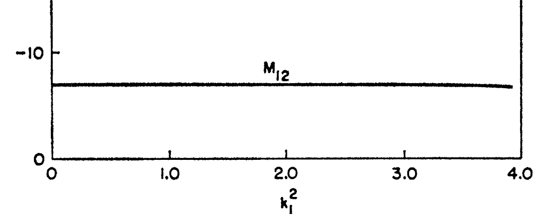

FIG. 1. Energy dependence of the M matrix using monopoles for the left-hand cut. The relative orbital angular momentum (in all the examples considered) is taken equal to one in both channels. The masses of the particles in channel 1 are 6.6 and 6.6; those in channel 2 are 7.0 and 7.0. The position of the monopoles is 5.0.

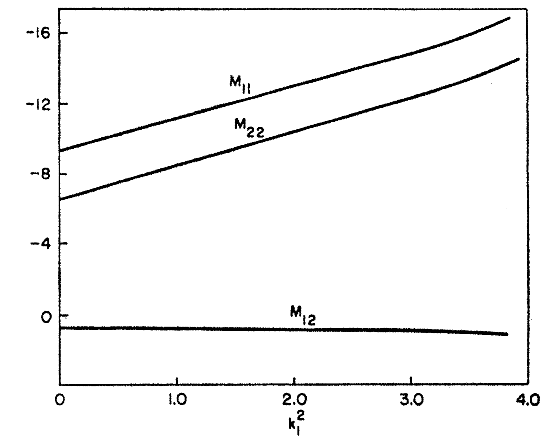

FIG. 2. Energy dependence of the M matrix using dipoles for the left hand.cut. The masses in channel 1 are 6.0 and 6.0; those in channel 2 are 6.1 and 6.1. The position of the dipoles is 20.0.

effective ranges as given by Eqs. (2.18) and (2.20) is reasonably good for a wide range of cases. For example, the "range" R_2 obtained from the slope of M_{22} in Fig. 5 is ≈ -0.6, whereas the theoretical estimate of R_2 is ≈ -0.5.

In conclusion, we can say that for relatively simple left-hand singularities, the effective-range formulas (2.19)–(2.20) is a respectable approximation at least for low l_i. If the left-hand singularities of M are not very close to the energy region of interest, the nondiagonal elements of M exhibit a weak energy dependence relative to the diagonal ones, and the effective-range matrix, therefore, is approximately diagonal. Moreover, under the same conditions the diagonal effective ranges are independent of energy over significantly large values of the relative momenta. Under more general circumstances, the use of relation (2.17) may be more appropriate.

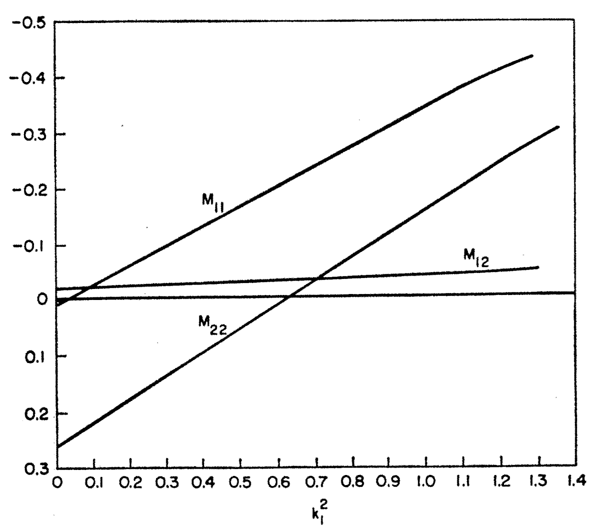

FIG. 3. Energy dependence of the M matrix with the left-hand cut given by the exchange of scalar particles. The masses of the particles in channel 1 are 1.0 and 1.0; those in channel 2 are 1.05 and 1.05. The masses of the scalar particles exchanged are $m_{11}=3.1$, $m_{22}=3.9$, and $m_{12}=7.1$.

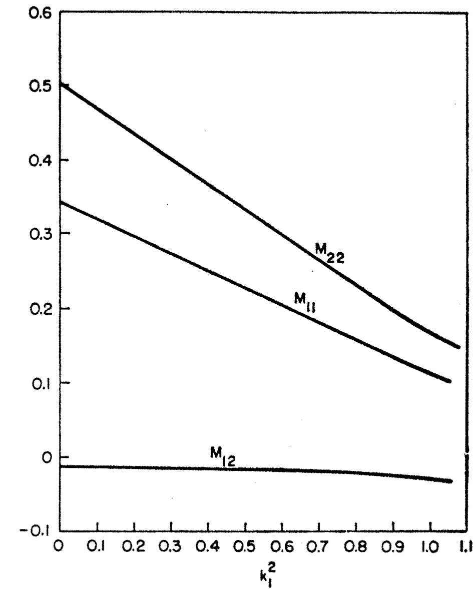

FIG. 4. Same as Fig. 3 (except for the values of the coupling constants g_{ij}).

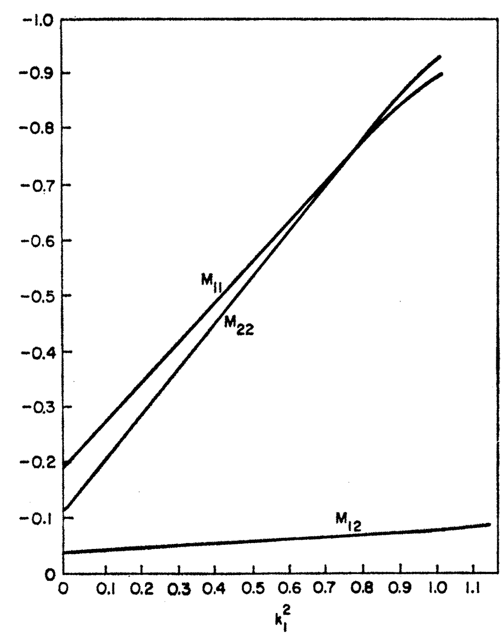

FIG. 5. Same as Fig. 3 except $m_{22}=3.1$. The effective range R_2, obtained from the slope of M_{22} is ≈ -0.6. The theoretical estimate of R_2, calculated using Eqs. (2.18) and (2.20), is ≈ -0.5.