General Gauge Invariance of Massive Electrodynamics and the Existence of a Massless Scalar Particle

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We have formulated a theory of massive electrodynamics which admits gauge invariance of the second kind. In so doing, a massless scalar field is needed. However, if the conserved-current condition $\partial_\mu j^\mu = 0$ is satisfied, this scalar field has no dynamical consequences. This theory then reduces to the conventional theory. When the conserved-current condition is not satisfied, a generalized current must still be conserved in order to have consistency of the theory. In an addendum, a brief discussion of the generalized Stueckelberg formulation is given.

THE connection between gauge invariance[1,2] of the second kind and the observed mass of a vector particle has been a subject of discussion recently. There are essentially two schools of thought:

(1) It is proposed that a (bare) massless vector field can generate a nonvanishing observed mass through interaction.[3,4] In this program, explicit gauge invariance[5] is never a problem. The situation is exactly the same as in quantum electrodynamics. We have nothing to add to this approach, except to point out the probable computational difficulty.

(2) This applies only to a conserved-vector-current theory.[6,7] There is the view that since only the (four-dimensional) transverse components of a vector field are coupled to a conserved vector current, one may as well confine one's attention to gauge invariance of these components only. Whereas physically this is the correct picture, nonetheless, some modifications are necessary for our purpose later.

It is our intention to give another approach to this problem here. We shall see that it is indeed possible to present a formulation, such that the vector field can have a nonvanishing bare mass and that all observables in the theory will be explicitly gauge-independent. In so doing, it is necessary to introduce a massless scalar field as a *vehicle* to maintain gauge invariance. This scalar field can be gauged away when we have a conserved (conventionally defined) vector current. We shall prove that, in this case, our theory is exactly the same as the conventional massive electrodynamics. Indeed, its gauge invariance is trivial. In general, when such a vector current does not exist, a generalized vector current, as we shall define below, must still be conserved in order to have consistency of the theory.

In a separate note, we shall discuss the implication of this conserved current concept on elementary interactions.[8] In particular, we shall show how this massless scalar field makes contact with a model recently proposed by Lee[9] to explain the apparent CP violation.[10]

MASSIVE ELECTRODYNAMICS

Gauge invariance of the second kind of massive electrodynamics has been discussed previously by several authors.[6,7] When one deals with a conserved (conventionally defined) vector current, which acts as the agent that generates these massive photons, one can still bypass the gauge problem by explicitly coupling the current to the 4-dimensional transverse photons only. In order to have a local Lagrangian density, an indefinite metric has to be introduced to the longitudinal photons. Since the longitudinal photons are not coupled to the physical observables through the conserved current anyway, this process does not raise any difficulty in probability interpretation. On the other hand, when a nonconserved vector current is involved, this method obviously fails. We would like to give a different formulation of this problem here, in which no such difficulty may arise. We shall see that all physically meaningful quantities—Hamiltonian density, currents, etc.—are explicitly gauge invariant.

The Lagrangian density of a massive spin-1 field coupled to a spin-$\frac{1}{2}$ field is[11]

$$\mathcal{L}_1 = -\bar{\psi}(\gamma_\mu(1/i)\partial^\mu + M)\psi - \tfrac{1}{2}G^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}m^2 A^\mu A_\mu + A_\mu j^\mu,$$

where

$$j^\mu = e\bar{\psi}\gamma^\mu\psi$$

is the conventionally defined current. Except for the mass term, it is invariant under the gauge trans-

1 C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).
2 See also, R. Utiyama, Phys. Rev. **101**, 1597 (1956).
3 J. Schwinger, Phys. Rev. **125**, 397 (1962).
4 D. G. Boulware and W. Gilbert, Phys. Rev. **126**, 1563 (1962).
5 By this we mean that all physically relevant expressions must be gauge-independent.
6 G. Feldman and P. T. Matthews, Phys. Rev. **130**, 1633 (1963); **132**, 823 (1963).
7 V. I. Ogievetskii and I. V. Polubarinov, Zh. Eksperim. i Teor. Fiz. **41**, 247 (1961) [English transl.: Soviet Phys.—JETP **14**, 179 (1962)].
8 Y. P. Yao (to be published).
9 T. D. Lee (to be published).
10 J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters **13**, 138 (1964); A. Abashian, R. J. Abrams, D. W. Carpenter, G. P. Fisher, B. M. K. Nefkens, and J. H. Smith, Phys. Rev. Letters **13**, 243 (1964).
11 We use the metric $(-1,1,1,1)$. $\mu=0,1,2,3$. $k=1,2,3$. $\{\gamma_\mu,\gamma_\nu\} = -2g_{\mu\nu}$, $\gamma_k{}^\dagger = -\gamma_k$, and $\gamma_0{}^\dagger = \gamma_0$. Appropriate symmetrization and antisymmetrization over the Bose-Einstein and Fermi-Dirac fields, respectively, are implicitly assumed.

formation

$$\psi(x) \to e^{ie\Lambda(x)}\psi(x)\,,$$
$$A_\mu(x) \to A_\mu(x)+\partial_\mu\Lambda(x)\,, \qquad (1)$$

and

$$G_{\mu\nu}(x) \to G_{\mu\nu}(x)\,,$$

where $\Lambda(x)$ is an arbitrary function of space-time coordinates x.

The Lagrangian density of a spin 0 field is

$$\mathcal{L}_0=-\phi^\mu\partial_\mu\phi+\tfrac{1}{2}\phi^\mu\phi_\mu-\tfrac{1}{2}\mu^2\phi^2+\mathcal{L}_{\text{int}}'\,,$$

where $\mathcal{L}_{\text{int}}'$ is the interaction density of ϕ, A^μ, and ψ. It is seen that we can have a gauge-invariant Lagrangian density, if we choose

$$\mu=0\,,$$
$$\mathcal{L}_{\text{int}}'=mA^\mu\partial_\mu\phi+\mathcal{L}_{\text{int}}\,,$$

and, under a gauge transformation,

$$\phi(x) \to \phi(x)+m\Lambda(x)\,,$$
$$\phi^\mu(x) \to \phi^\mu(x)+m\partial^\mu\Lambda(x)\,. \qquad (2)$$

$\mathcal{L}_{\text{int}}$ is invariant under the gauge transformation (1) and (2). We then have the following gauge-invariant Lagrangian density,[12]

$$\mathcal{L}=-\bar{\psi}(\gamma_\mu(1/i)\partial^\mu+M)\psi-\tfrac{1}{2}G^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)$$
$$+\tfrac{1}{4}G^{\mu\nu}G_{\mu\nu}-\tfrac{1}{2}m^2A^\mu A_\mu-\phi^\mu\partial_\mu\phi+\tfrac{1}{2}\phi^\mu\phi_\mu$$
$$+A_\mu(j^\mu+m\partial^\mu\phi)+\mathcal{L}_{\text{int}}\,.$$

In order to have gauge invariance for a massive vector field, we have to introduce a massless scalar field.

We must hasten to point out, however, that this Lagrangian density cannot yield a consistent theory. The equation

$$\partial_k G^{0k}+m^2A^0=j^0+m\partial^0\phi$$

or

$$(-\partial^0\phi+mA^0)=-(1/m)(\partial_k G^{0k}-j^0)$$

is inconsistent with the fundamental commutation relation between ϕ and its conjugate momentum:

$$(1/i)[\phi(x),(-\partial^0\phi+mA^0)(x')]=\delta(\mathbf{x}-\mathbf{x}')\,.$$

The reason is that the commutator can also be written as

$$(1/i)[\phi,\,-(1/m)(\partial_k G^{0k}-j^0)]\,,$$

which must vanish, because ϕ, G^{0k}, and ψ are all different independent degrees of freedom.

The cause for this inconsistency is familiar. Due to the derivative coupling $mA_\mu\partial^\mu\phi$, we have made

$$A^0=(1/m^2)(j^0+m\partial^0\phi-\partial_k G^{0k})$$

an independent dynamical variable. (The time derivative ∂^0 appears explicitly on the right-hand side.) On the other hand, we have not assigned an independent conjugate momentum to it. The situation here resembles that in the Lorentz gauge electrodynamics. The solution is also well known[13]: We introduce a gauge operator G, which as the property that it annihilates all the gauge-invariant states Ψ; i.e.,

$$G\Psi=\partial_0 G\Psi=0\,.$$

These comprise of all the physically relevant states. To ensure that this condition be satisfied at all space-time points, we impose the condition

$$\partial^2 G=0$$

on G. Consequently, all higher time derivatives of G vanish when applied to a gauge-invariant state.

The extended Lagrangian density is

$$\mathcal{L}=-\bar{\psi}(\gamma_\mu(1/i)\partial^\mu+M)\psi-\tfrac{1}{2}G^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)$$
$$+\tfrac{1}{4}G^{\mu\nu}G_{\mu\nu}-\tfrac{1}{2}m^2A_\mu A^\mu-\phi^\mu\partial_\mu\phi+\tfrac{1}{2}\phi_\mu\phi^\mu$$
$$+A_\mu(j^\mu+m\partial^\mu\phi)-G\partial_\mu A^\mu\,.$$

We have let $\mathcal{L}_{\text{int}}=0$; we shall continue with this assumption for the rest of this paper.

We shall agree, under a gauge transformation,

$$G \to G\,.$$

Then,

$$\mathcal{L} \to \mathcal{L}-G\partial^2\Lambda\,.$$

By the action principle, the generator of an infinitesimal gauge transformation (1) and (2) is given by

$$G_{\delta\lambda}=\int(d^3x)(G\partial_0\delta\lambda-\delta\lambda\partial_0 G)\,.$$

Because we are dealing with Abelian gauge fields here, the order of performing two successive gauge transformations at a common time is immaterial. This implies the relation

$$[G_{\delta\lambda_1},G_{\delta\lambda_2}]=0\,,$$

which can be satisfied if

$$[G(x),G(x')]=[G(x),\partial_0 G(x')]$$
$$=[\partial_0 G(x),\partial_0 G(x')]=0\,.$$

The canonical equations

$$(1/i)[A_\mu,G_{\delta\lambda}]=\partial_\mu\delta\lambda\,,$$
$$(1/i)[\psi,G_{\delta\lambda}]=ie\delta\lambda\psi\,,$$

etc. give us the following nonvanishing equal-time commutation relations:

$$(1/i)[\psi(x),\partial_0 G(x')]=-ie\psi(x)\delta(\mathbf{x}-\mathbf{x}')\,,$$
$$(1/i)[\bar{\psi}(x),\partial_0 G(x')]=ie\bar{\psi}(x)\delta(\mathbf{x}-\mathbf{x}')\,,$$
$$(1/i)[A_0(x),G(x')]=\delta(\mathbf{x}-\mathbf{x}')\,,$$
$$(1/i)[A_k(x),\partial_0 G(x')]=-\partial_k\delta(\mathbf{x}-\mathbf{x}')\,,$$
$$(1/i)[\phi(x),\partial_0 G(x')]=-m\delta(\mathbf{x}-\mathbf{x}')\,,$$

[12] This Lagrangian resembles that of D. G. Boulware and W. Gilbert (Ref. 5), if we replace our $mA_\mu\partial^\mu\phi$ term by $mA_\mu\phi^\mu$. However, in so doing, the theory possesses gauge invariance only when $m=0$. We have the freedom of multiplying $\frac{1}{4}G^{\mu\nu}G_{\mu\nu}$ by a dimensionless number g^2 which can be used to vary the relative coupling strengths of $A_\mu j^\mu$ and $\mathcal{L}_{\text{int}}$. We shall assume minimal coupling $A_\mu j^\mu$ between A_μ and ψ. Hence, $\mathcal{L}_{\text{int}}\equiv\mathcal{L}_{\text{int}}[\psi,\bar{\psi},\phi]$.

[13] J. Schwinger, Phys. Rev. **130**, 402, 406 (1963).

and

$$(1/i)[\phi_0(x),G(x')]=m\delta(\mathbf{x}-\mathbf{x}').$$

G is the "conjugate momentum" to A^0.

It is interesting to observe that

$$G_{\delta\lambda}\Psi=0,$$

which shows that Ψ are indeed gauge invariant. We also have the following commutators:

$$\{\psi^\dagger(x),\psi(x')\}=\delta(\mathbf{x}-\mathbf{x}'),$$

$$(1/i)\left[\phi(x),\frac{\partial}{\partial t}\phi(x')\right]=\delta(\mathbf{x}-\mathbf{x}'),$$

and

$$(1/i)[G^{0k}(x),A_l(x')]=\delta_l{}^k\delta(\mathbf{x}-\mathbf{x}').$$

Let us emphasize that

$$[A^0,\psi]=[A^0,G^{0k}]=[A^0,A^k]=[A^0,\phi]=[A^0,\phi^0]=0.$$

That is, A^0 is truly an independent dynamical variable.

Euler's equations are

$$\delta\bar\psi:\ [\gamma_\mu((1/i)\partial^\mu-eA^\mu)+M]\psi=0, \tag{3}$$

$$\delta\psi:\ \bar\psi[\gamma_\mu(-(1/i)\overleftarrow{\partial}_\mu-eA^\mu)+M]=0, \tag{4}$$

$$\delta G^{\mu\nu}:\ G_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \tag{5}$$

$$\delta A_\nu:\ \partial_\mu G^{\mu\nu}-m^2A^\nu=-j^\nu-m\partial^\nu\phi-\partial^\nu G, \tag{6}$$

$$\delta\phi^\mu:\ \phi_\mu=\partial_\mu\phi, \tag{7}$$

$$\delta\phi:\ \partial^2\phi=m\partial_\mu A^\mu=0, \tag{8}$$

and

$$\delta G:\ \partial_\mu A^\mu=0. \tag{9}$$

In particular, let us consider once again the equation

$$\partial_k G^{k0}-m^2A^0=-j^0-m\partial^0\phi-\partial^0G.$$

It is now consistent with the commutation relations

$$(1/i)[\phi(x),(-\partial^0\phi+mA^0)(x')]=\delta(\mathbf{x}-\mathbf{x}')$$

since the left-hand side is also

$$-(1/i)(1/m)[\phi(x),\partial_0G(x')]=\delta(\mathbf{x}-\mathbf{x}').$$

By introducing the gauge operator G, we have been able to write down a consistent set of equations and commutation relations. To proceed further, a natural query is whether such an extended formalism is Lorentz invariant, or whether the generators of the inhomogeneous Lorentz group of such an extended system obey the group commutation relations. The easiest way to investigate this aspect is to look into the commutators of the energy-momentum densities.[13] After adding some irrelevant surface terms and making use of the Euler's equations, we find that the extended energy-momentum densities derived from the above Lagrangian density can be written as

$$\Theta^{00}=T^{00}-A^k\partial_kG-A^0(\partial_kG^{0k}+m^2A^0-j^0-m\partial^0\phi)$$

and

$$\Theta^{0k}=T^{0k}-A^0\partial_kG-A^k(\partial_lG^{0l}+m^2A^0-j^0-m\partial^0\phi),$$

where

$$T^{00}=\bar\psi\gamma_k((1/i)\partial^k-eA^k)\psi+M\bar\psi\psi+\tfrac14(G^{kl})^2+\tfrac12(G^{0k})^2+\tfrac12m^2(A^k-(1/m)\phi^k)^2+\tfrac12m^2(A^0-(1/m)\phi^0)^2$$

is gauge-independent and positive definite in the integral spin variables.

$$T^{0k}=\psi^\dagger(1/i\partial_k-A_k)\psi+G^{0l}G_{kl}+(\phi^0-mA^0)(\partial_k\phi-mA_k)+\partial^l(i/8\psi^\dagger[\gamma_k,\gamma_l]\psi)$$

are again gauge-independent.

Clearly, we have the reductions

$$\Theta^{00}\Psi=T^{00}\Psi$$

and

$$\Theta^{0k}\Psi=T^{0k}\Psi,$$

i.e., T^{00} and T^{0k} are the true energy-momentum densities of the realizable physical states.

It is now a matter of some algebraic manipulations to arrive at the equal-time commutation relations,

$$-i[\Theta^{00}(x),\Theta^{00}(x')]=-[\Theta^{0k}(x)+\Theta^{0k}(x')]\partial_k\delta(\mathbf{x}-\mathbf{x})$$

and

$$-i[T^{00}(x),T^{00}(x')]=-[T^{0k}(x)+T^{0k}(x')]\partial_k\delta(\mathbf{x}-\mathbf{x}).$$

Thus, the theory is Lorentz invariant.

We shall now show how this theory goes into the conventional theory. We define the new variables

$$\psi'=e^{-i(e/m)\phi}\psi$$

and

$$A_\mu'=A_\mu-(1/m)\partial_\mu\phi$$

(which are, incidentally, all gauge-invariant quantities). Then, Eqs. (3)–(8) simplify to

$$[\gamma_\mu((1/i)\partial^\mu-eA'^\mu)+M]\psi'=0, \tag{3'}$$

$$\bar\psi'[\gamma_\mu(-(1/i)\overleftarrow{\partial}_\mu-eA'^\mu)+M]=0, \tag{4'}$$

$$G_{\mu\nu}'=\partial_\mu A_\nu'-\partial_\nu A_\mu'=G_{\mu\nu}, \tag{5'}$$

$$\partial_\mu G'^{\mu\nu}-m^2A'^\nu+\partial^\nu G=-j'^\nu=-j^\nu, \tag{6'}$$

$$\phi_\mu=\partial_\mu\phi, \tag{7'}$$

and

$$\partial^2\phi=0. \tag{8'}$$

It is worth noting that we still have the gauge condition

$$\partial_\mu A'^\mu=0. \tag{9'}$$

The energy-momentum densities become

$$T^{00}=\bar{\psi}'\gamma_k((1/i)\partial^k-eA'^k)\psi'+M\bar{\psi}'\psi'+\tfrac{1}{4}(G'^{kl})^2+\tfrac{1}{2}(G'^{0k})^2+\tfrac{1}{2}m^2((A'^k)^2+(A'^0)^2)$$

and

$$T^{0k}=\psi'^\dagger((1/i)\partial_k-A_k')\psi'+G'^{0l}G_{kl}'+m^2A'^0A_k'+\partial^l(i/8\psi'^\dagger[\gamma_k,\gamma_l]\psi'),$$

in which expressions ϕ does not appear.

The fundamental commutation relations

$$\{\psi'^\dagger(x),\psi'(x')\}=\delta(\mathbf{x}-\mathbf{x}'),$$
$$(1/i)[G'^{0k}(x),A_l'(x')]=\delta_l{}^k\delta(\mathbf{x}-\mathbf{x}'),$$

and

$$(1/i)[A'^0(x),A_l'(x')]=-(1/m^2)\partial_l\delta(\mathbf{x}-\mathbf{x}')$$

are exactly like those in conventional massive electrodynamics.[14]

Equations (3′)–(6′), the gauge condition $\partial_\mu A'^\mu$, and $T^{0\mu}$ are all gauge-independent quantities, since ψ', A'^μ, and $G'^{\mu\nu}$ are. Consequently, we can drop $\partial^\nu G$ in (6′), as long as we neglect (7′) and (8′). (Remember G is a gauge operator. It has effects only on gauge-dependent quantities; but $[G,\psi']=[G,A'^\mu]=[G,G'^{\mu\nu}]=0$.) To put it differently, if we neglect (7′) and (8′), all the above expressions can be derived from

$$\mathcal{L}'=-\bar{\psi}'(\gamma_\mu(1/i)\partial^\mu+M)\psi'-\tfrac{1}{2}G'^{\mu\nu}(\partial_\mu A_\nu'-\partial_\nu A_\mu')+\tfrac{1}{4}G_{\mu\nu}'G'^{\mu\nu}-\tfrac{1}{2}m^2A_\mu'A'^\mu+j_\mu'A'^\mu,$$

which is the Lagrangian density we used all along in massive electrodynamics. Let us repeat what we have proved: The conventional massive electrodynamics, derivable[15] from $\mathcal{L}'$, is a general gauge-invariant theory, since all field variables, ψ', A'^μ, and $G'^{\mu\nu}$ are gauge invariant.

Now it is obvious how the scalar field is used only as an artifice to introduce gauge invariance of the second kind. Once this purpose has been served, we can decouple it from the rest of the system. The decoupling cannot be carried out in general, when $\mathcal{L}_{\text{int}}\neq 0$.

From (6) and (9), we have

$$\partial_\mu J^\mu=0, \tag{10}$$

where

$$J^\mu=j^\mu+m\partial^\mu\phi, \tag{11}$$

which we shall call generalized current. We would like to stress here that the conservation of charge is a consequence of the antisymmetric property of $G^{\mu\nu}$, just as in electrodynamics the conservation of electric charge is a result of the antisymmetry of the field intensities $F^{\mu\nu}$.[16] No dynamics can change this antisymmetry.

[14] E. g. K. Johnson, Nucl. Phys. **25**, 435 (1961).

[15] We also have $\partial_\mu j'^\mu=0$; which is to say that we have a conserved vector current theory. (See below.)

[16] This point has been emphasized by J. Schwinger in his Brandeis lectures (Summer, 1964) when he discussed baryon conservation. It may well be a fundamental property that all absolute conservation laws (of internal symmetries) share.

In this case, where we have assumed $\mathcal{L}_{\text{int}}=0$, Eq. (10) is consistent with Eq. (8) or (8′) only if

$$\partial_\mu j^\mu=\partial_\mu j'^\mu=0 \tag{12}$$

as we can also check directly by using the definition of $j^\mu=j'^\mu$ and Eqs. (3) and (4) or (3′) and (4′). We have here a conserved (conventionally defined) vector current theory. This is because we have taken $\mathcal{L}_{\text{int}}=0$. We shall show in a subsequent paper that there are examples in which (12) is not true.

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ADDENDUM

Upon completion of this work, it was brought to my attention that a generalized Stueckelberg formalism has been proposed.[17] We would like to modify that formulation to fit into our development. Thus, we replace $-G\partial_\mu A^\mu$ in $\mathcal{L}$ by $-G[\partial_\mu A^\mu-(\kappa^2/m)\phi]$. In so going, the gauge condition (9) becomes

$$\partial_\mu A^\mu-(\kappa^2/m)\phi=0, \tag{A1}$$

and the gauge operator G has to satisfy the equation

$$(\partial^2-\kappa^2)G=0. \tag{A2}$$

The other changes are that Eq. (8) is replaced by

$$\partial_\mu\phi^\mu-m\partial_\mu A^\mu+(\kappa^2/m)G=0,$$

or

$$(\partial^2-\kappa^2)\phi+(\kappa^2/m)G=0, \tag{A3}$$

upon using (A2).[18] Θ^{00} acquires an additional term $-(\kappa^2/m)G\phi$, but T^{00} remains unchanged.

The consistency of canonical quantization in the main text still holds. Lorentz invariance of the theory can be checked as before.

We would like to make a few remarks when $\mathcal{L}_{\text{int}}[\psi,\phi]\neq 0$. From (6), differentiating both sides by ∂_ν, we have

$$\partial_\mu(j^\mu+m\partial^\mu\phi+\partial^\mu G-m^2A^\mu)=0. \tag{A4}$$

In general, no combination of the terms inside the parentheses can be made as a conserved local current density; we must take the whole object. This is, of course, a consequence of the gauge conditions (A1) and (A2). In other words, $\partial^\mu G$ and A^μ are not 'conserved' separately, in contradistinction to the massless case

[17] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, 299 (1938); Y. Fujii, Progr. Theoret. Phys. (Kyoto) **21**, 232 (1959); S. Bonometto, Nuovo Cimento **28**, 1855 (1963); Y. Fujii and S. Kamefuchi, Nuovo Cimento **33**, 1639 (1964); Y. Fujii, Stanford Report (to be published). See also references quoted in these papers.

[18] κ^2 can therefore be identified as the mass of the scalar field.

($\kappa^2=0$). On the other hand, $(j^\mu+m\partial^\mu\phi+\partial^\mu G-m^2A^\mu)$ does not allow for any realistic physical interpretation as a local current density. The third term $\partial^\mu G$ is not a physical quantity. The fourth term $-m^2A^\mu$ is actually the mass term of the vector particle; to look at it as a part of a current density is just too artificial. Thus, we have to conclude that when the scalar field acquires a bare mass, no physically meaningful conserved local current density can be constructed (when $\mathcal{L}_{\rm int}=0$, $\partial_\mu j^\mu=0$ follows).

(A3) is modified into

$$(\partial^2-\kappa^2)\phi+(\kappa^2/m)G+J=0\,,$$

where

$$J\equiv(\delta\mathcal{L}_{\rm int})/(\delta\phi)\,.$$

Together with (A1) and (A2), we have

$$\partial_\mu j^\mu=mJ\,,$$

which is the counterpart of (10) in the massless case. We can define

$$J^\mu(x)=j^\mu(x)+m\partial^\mu\int D(x-x')J(x')\,,$$

$$-\partial^2D=1\,,$$

as a conserved current density:

$$\partial_\mu J^\mu=0;$$

but it is nonlocal.

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Multichannel Effective-Range Theory from the N/D Formalism*

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An effective-range theory for systems of many coupled two-body channels is given using the N/D formalism. The effective-range expansion is carried out in the amplitudes M_{ij} (where M is essentially the matrix T^{-1} with the right-hand cut removed). Quite in analogy with the single-channel effective-range theory, the diagonal elements M_{ii} are given by an expression quadratic in k_i, the relative momentum in channel i. The effective ranges R_{ii} are given by certain principal-value integrals which depend on the position of the left-hand singularities in the corresponding channels and can be taken to be energy-independent to the same extent as in the one-channel theory. The nondiagonal elements M_{ij}, in general, have a weak energy dependence and can approximately be treated as constants. A two-channel computer experiment is performed to test these proposals in detail. Three different situations for the left-hand cut are considered: (i) a set of monopoles, (ii) a set of dipoles, and (iii) the left-hand cut produced by the exchange of scalar particles in the "crossed" t reactions. For a large number of situations considered, the simple features proposed for the multichannel effective range theory were found to exist. The above formalism is similar to the multichannel effective-range theory of Ross and Shaw in the potential model.

I. INTRODUCTION

WE wish to discuss the energy dependence of the partial-wave amplitudes for many coupled two-body channels. We assume that these amplitudes satisfy the coupled N/D partial-wave dispersion relations. The purpose is to obtain a simple effective-range theory by removing the right-hand cuts explicitly and approximating the rest of the scattering matrix that contains only the left-hand singularities. We assume that only n channels need be considered explicitly. The inverse of the $n\times n$ scattering matrix $T(=ND^{-1})$ can be written from unitarity as $T^{-1}=M(s)-i\rho(s)$ (s is the square of the total energy in the center-of-mass system), where $\rho(s)$ is a diagonal right-hand cut function. The matrix M is both real and symmetric to the right of the left-hand singularities. Over an approximately small physical region of energy, therefore, an effective-range expansion in M can be carried out.

We propose an effective range formula for M, $M(s)=B^{-1}(s)-P(s)$, where $B(s)$ contains the unphysical singularities of the scattering amplitude T and $P(s)$ is a diagonal matrix and contains most of the energy dependence of $M(s)$. If the left-hand singularities carried by $M(s)$ do not lie very close to the energy region of interest, a further simplification in the effective-range formula for M follows. The nondiagonal effective ranges in this situation are small (the effective-range matrix R is approximately diagonal), and for small value of k^2 (see the computer experiment in Sec. III) we have the linear relationship $M=M(0)+\frac{1}{2}R[k^2-k^2(0)]$. The diagonal effective ranges can be

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