Mass Splittings Within Spin-Degenerate Multiplets*

M. Y. HAN
Department of Physics, Syracuse University, Syracuse, New York
(Received 17 November 1964)

In connection with the problem of combining the relativistic invariance and internal symmetries, it has recently been shown by McGlinn and others that the mass splittings within a spin-degenerate multiplet are not possible via a theorem that states that, if an internal symmetry group commutes with the homogeneous Lorentz group, then it must commute with the translation group. Rather than considering the two subgroups of the inhomogeneous Lorentz group separately, if we consider the invariants of the whole group we find that it is not the generators $M_{\mu\nu}$ but rather the pseudovector $w_\mu=\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}M^{\nu\lambda}p^\sigma$ that is related to the spin of a particle. We show that if the set of generators of an internal symmetry group (assumed to be simple) commute with w_μ, then they have to commute with every generator of the inhomogeneous Lorentz group. Thus, the mass splittings are not possible. A few related theorems are also derived.

I. INTRODUCTION

THERE have recently appeared several papers[1] concerning the possibility of combining the inhomogeneous Lorentz group L and an internal symmetry group S into a larger invariance group G such that the mass splittings within a multiplet of particles are the manifestation of an exact invariance under the group G. Since the particles in a multiplet have same spin but different masses, the group G which contains the groups L and S as its subgroups must be such that the generators of S commute with the generators of L that are related to spin, but do not commute with the generators of L that are related to mass. The group G must at least be larger than a direct product of L and S.

However, McGlinn[2] has recently shown that G in fact can only be a direct product of L and S under such conditions. Under the assumption that the group G has a Lie algebra[3] G whose elements can be linearly expressed in terms of the elements of the algebras L and S, he has proved that, if all elements of S commute with the homogeneous Lorentz algebra (spin degeneracy), then they have to commute with the translation algebra (mass degeneracy). It has been further shown[4–6] that, if only a complete set of commuting elements of S commutes with the elements of L, the same conclusion is reached.

That a nontrivial coupling of L and S is at least mathematically possible has been shown by Sudarshan.[7] Under an assumption[8] that only one element of S commutes with all elements of L (corresponding to the existence of at least one absolutely conserved quantum number, namely, the electric charge), he has proved that the algebra G is a direct sum of S and $\tilde{L}$, where $\tilde{L}$ is a sum of L and a certain linear combination, which is isomorphic to L, of the elements of S. He has further shown that, however, if only the unitary representations of G are considered, then $\tilde{L}$ becomes identical with L and $G=L\oplus S$.

As is well known, the handling of the translation and the homogeneous Lorentz group separately may be misleading since the two invariants which can be constructed out of $M_{\mu\nu}$, namely, $M_{\mu\nu}M^{\mu\nu}$ and $\epsilon_{\mu\nu\lambda\sigma}M^{\mu\nu}M^{\lambda\sigma}$, have no clear physical meaning and, furthermore, are not invariants of L since they do not commute with p_μ. When the (proper) inhomogeneous Lorentz group is considered as a whole, there are in general four independent scalars[9]:

$$p_\mu p^\mu\,,\quad w_\mu w^\mu\,,\quad v_\mu v^\mu\,,\quad v_\mu w^\mu\,, \tag{1}$$

where

$$v_\mu=M_{\mu\nu}p^\nu\,, \tag{2}$$

$$w_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma}M^{\nu\lambda}p^\sigma\,. \tag{3}$$

Of these, the last two do not commute with p_μ and only the first two are the well-known invariants of the group L which are related to the mass and spin of an elementary relativistic system. The w_μ's satisfy the commutation relations

$$[M_{\mu\nu},w_\sigma]=i(\delta_{\nu\sigma}w_\mu-\delta_{\mu\sigma}w_\nu)\,, \tag{4}$$

$$[w_\mu,p_\nu]=0\,, \tag{5}$$

$$[w_\mu,w_\nu]=i\epsilon_{\mu\nu\lambda\sigma}w^\lambda p^\sigma\,. \tag{6}$$

In this paper, we investigate the consequences of replacing the generators $M_{\mu\nu}$ of the homogeneous Lorentz group by the pseudovector w_μ in the requirement of spin degeneracy of a multiplet. Keeping the

---

* Work supported by the U. S. Atomic Energy Commission.

[1] F. Lurçat and L. Michel, Nuovo Cimento **21**, 574 (1961); *Proceedings of the Coral Gables Conference on Symmetry Principles at High Energy*, edited by B. Kurşunoğlu and A. Perlmutter (W. J. Freeman Company, San Francisco, California, 1964); A. O. Barut, Nuovo Cimento **32**, 234 (1964); B. Kurşunoğlu, Phys. Rev. **135**, B761 (1964).

[2] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964).

[3] We shall use the same notation for a Lie group and corresponding Lie algebra.

[4] F. Coester, M. Hamermesh, and W. D. McGlinn, Phys. Rev. **135**, B451 (1964).

[5] M. E. Mayer, H. J. Schnitzer, E. C. G. Sudarshan, R. Acharya, and M. Y. Han, Phys. Rev. **136**, B888 (1964).

[6] A. Beskow and U. Ottoson, Nuovo Cimento **34**, 248 (1964).

[7] E. C. G. Sudarshan, J. Math. Phys. (to be published).

[8] U. Ottoson, A. Kihlberg, and J. Nilsson, Phys. Rev. **137**, B658 (1965).

[9] Iu. M. Shirokov, Zh. Eksperim. i Teor. Fiz. **33**, 861 (1957) [English transl.: Soviet Phys.—JETP **6**, 664 (1958)].

assumption that the algebra G is linearly closed under the algebras L and S, we show that, if the set of generators of S commute with w_μ, then they have to commute with every generator of L. Thus, the algebra G is a direct sum of the algebras L and S, and no mass splittings within spin-degenerate multiplets are possible. We further prove a few theorems that show that the same conclusion may be reached under much (mathematically) weaker conditions.

II. THEOREMS

Lemma: Let S_i be *any* generator of an internal symmetry group S. If

$$[S_i,w_\sigma]=0 \tag{7}$$

for all σ, then

$$[S_i,p_\mu]=0 \tag{8}$$

and

$$[S_i,M_{\mu\nu}]=\sum_\lambda b_{i\mu\nu\lambda}p_\lambda. \tag{9}$$

Proof: From the definition of w_μ, (3), the assumption (7) can be written as

$$\epsilon_{\sigma\mu\nu\lambda}[S_i,M_{\mu\nu}]p_\lambda+\epsilon_{\sigma\mu\nu\lambda}M_{\mu\nu}[S_i,p_\lambda]=0. \tag{7$'$}$$

By hypothesis, we can write most generally,

$$[S_i,p_\mu]=\sum_\lambda a_{i\mu\lambda}p_\lambda+\sum_{\alpha\beta} a_{i\mu\alpha\beta}M_{\alpha\beta}+\sum_k a_{i\mu k}S_k, \tag{10}$$

$$[S_i,M_{\mu\nu}]=\sum_\lambda b_{i\mu\nu\lambda}p_\lambda+\sum_{\alpha\beta} b_{i\mu\nu\alpha\beta}M_{\alpha\beta}+\sum_k b_{i\mu\nu k}S_k. \tag{11}$$

Substituting (10) and (11) into (7′), we see that the assumption (7) involves a linear combination of bilinear products of the generators $p_\mu p_\nu$, $M_{\mu\nu}M_{\alpha\beta}$, $M_{\mu\nu}p_\lambda$, S_ip_μ, and $S_iM_{\mu\nu}$. These bilinear products are elements of a tensor (enveloping) algebra of the original Lie algebra G and they are linearly independent by a theorem concerning an enveloping algebra of a Lie algebra.[10] From the linear independence of these products, we see immediately that

$$a_{i\mu k}=0 \tag{12}$$

and

$$b_{i\mu\nu k}=0. \tag{13}$$

From (5), (7), and (10), we have

$$[w_\lambda,[S_i,p_\mu]]=-i\sum_{\alpha\beta} a_{i\mu\alpha\beta}(\delta_{\beta\lambda}w_\alpha-\delta_{\alpha\lambda}w_\beta)=0 \tag{14}$$

and

$$a_{i\mu\alpha\beta}=0. \tag{15}$$

Likewise from (4), (7), and (11), we have

$$[w_\lambda,[S_i,M_{\mu\nu}]]=-i\sum_{\alpha\beta} b_{i\mu\nu\alpha\beta}(\delta_{\beta\lambda}w_\alpha-\delta_{\alpha\lambda}w_\beta)=0 \tag{16}$$

and

$$b_{i\mu\nu\alpha\beta}=0. \tag{17}$$

The result (17), namely, that $[S_i,M_{\mu\nu}]$ does not contain $M_{\alpha\beta}$'s, then yields, by the linear independence of the bilinear products,

$$a_{i\mu\lambda}=0. \tag{18}$$

Thus, we have proved that

$$[S_i,p_\mu]=0, \tag{10$'$}$$

$$[S_i,M_{\mu\nu}]=\sum_\lambda b_{i\mu\nu\lambda}p_\lambda. \tag{11$'$}$$

It should be noted that this Lemma is proved for a fixed element S_i of the algebra S. No properties of the group S have been used in the proof. From this Lemma, it is now easy to prove the main theorem.

Theorem 1: Let S_i be the set of generators of an internal symmetry group S. If S has no Abelian factor groups and if

$$[S_i,w_\sigma]=0 \tag{19}$$

for all σ, then

$$[S_i,p_\mu]=0 \tag{20}$$

and

$$[S_i,M_{\mu\nu}]=0. \tag{21}$$

Thus,

$$G=L\oplus S. \tag{22}$$

Proof: The first part of the theorem, (20), follows trivially from the Lemma. Now from (10′) and (11′) (in this case, the index i is not fixed), we have

$$[[S_i,S_j],M_{\mu\nu}]=0. \tag{23}$$

Therefore, if the group S is such that every element S_i can be expressed as a linear combination of commutators $[S_j,S_k]$, then

$$[S_i,M_{\mu\nu}]=0. \tag{24}$$

The groups which have the said property are shown to be those groups with no Abelian factor groups and this includes the class of semi-simple groups.[11] The equations (10′) and (11′) show that, even for those cases in which S_i's do not commute with $M_{\mu\nu}$, they may still commute with p_μ leading to no mass splittings. This supplements the McGlinn theorem.[2]

We now prove that the conclusion $G=L\oplus S$ may be obtained under much (mathematically) weaker assumptions. It will be shown that it is sufficient to assume that a complete set of the Cartan subalgebra commutes with w_μ and still further that, *provided* only unitary representations of G are considered, only one member of the Cartan subalgebra commutes with w_μ. To this end, we first derive the following theorem:

Theorem 2: Let the Lie algebra S be simple and H_1 be a member of the Cartan subalgebra. If

$$[H_1,w_\sigma]=0 \tag{25}$$

[10] For the definition of a tensor algebra of a Lie algebra and the linear independence of products of the generators see, for example, N. Jacobson, *Lie Algebras* (Interscience Publishers, Inc., New York, 1962), pp. 155–156. More generally, this follows from the Poincaré–Birkhoff–Witt theorem concerning the universal enveloping algebra of a Lie algebra. (N. Jacobson, *ibid.*, pp. 156–163). I am grateful to Dr. F. Coester and Dr. W. D. McGlinn for their communications bringing the Poincaré–Birkhoff–Witt theorem and its relevance to my attention.

[11] See remarks following Theorem 1 in Ref. 5.

for all σ, then

$$[H_1,p_\mu]=0 \tag{26}$$

and

$$[H_1,M_{\mu\nu}]=0. \tag{27}$$

Proof: For a simple Lie algebra S, there exists a Cartan-Weyl basis H_l, E_α such that

$$[H_l,H_m]=0, \tag{28}$$

$$[H_l,E_\alpha]=r_l(\alpha)E_\alpha, \tag{29}$$

$$[E_\alpha,E_{-\alpha}]=\sum_l r_l(\alpha)H_l, \tag{30}$$

$$[E_\alpha,E_\beta]=N_{\alpha\beta}E_{\alpha+\beta};\quad r_l(\alpha)+r_l(\beta)\neq 0. \tag{31}$$

The proof here follows closely those of Refs. 5 and 7. From the Lemma, we have

$$[H_1,p_\mu]=0, \tag{32}$$

$$[H_1,M_{\mu\nu}]=\sum_\lambda b_{1\mu\nu\lambda}p_\lambda. \tag{33}$$

Let us write

$$[E_\alpha,p_\sigma]=\sum_\beta c_{\alpha\sigma\beta}E_\beta+\sum_m c_{\alpha\sigma m}H_m+\sum_\lambda c_{\alpha\sigma\lambda}p_\lambda+\sum_{\mu\nu} c_{\alpha\sigma\mu\nu}M_{\mu\nu}. \tag{34}$$

Since

$$[H_1,[E_\alpha,p_\sigma]]=r_1(\alpha)[E_\alpha,p_\sigma], \tag{35}$$

it follows that

$$\sum_\beta c_{\alpha\sigma\beta}\{r_1(\alpha)-r_1(\beta)\}E_\beta+\sum_m r_1(\alpha)c_{\alpha\sigma m}H_m+\sum_{\mu\nu} r_1(\alpha)c_{\alpha\sigma\mu\nu}M_{\mu\nu}+\sum_\lambda\{r_1(\alpha)c_{\alpha\sigma\lambda}-\sum_{\mu\nu} c_{\alpha\sigma\mu\nu}b_{1\mu\nu\lambda}\}p_\lambda=0. \tag{36}$$

For those E_α's for which $r_1(\alpha)\neq 0$, comparing coefficients, we obtain

$$[E_\alpha,p_\sigma]=\sum_\beta c_{\alpha\sigma\beta}\delta\{r_1(\alpha)-r_1(\beta)\}E_\beta. \tag{37}$$

If, on the other hand, $r_1(\alpha)=0$ for some E_α, we can find β and γ such that $r_1(\beta)=-r_1(\gamma)\neq 0$, and using (31), we have

$$N_{\beta\gamma}[E_\alpha,p_\sigma]=[E_\beta,[E_\gamma,p_\sigma]]+[[E_\beta,p_\sigma],E_\gamma]. \tag{38}$$

Using (37) for $[E_\beta,p_\sigma]$ and $[E_\gamma,p_\sigma]$, since $r_1(\beta)\neq 0$ and $r_1(\gamma)\neq 0$ by construction, we can deduce a general expression

$$[E_\alpha,p_\sigma]=\sum_\beta c_{\alpha\sigma\beta}\delta\{r_1(\alpha)-r_1(\beta)\}E_\beta+\sum_m c_{\alpha\sigma m}\delta\{r_1(\alpha)\}H_m. \tag{39}$$

Considering the Jacobi identity

$$[H_1,[E_\alpha,M_{\mu\nu}]]=r_1(\alpha)[E_\alpha,M_{\mu\nu}]+\sum_\lambda b_{1\mu\nu\lambda}[E_\alpha,p_\lambda], \tag{40}$$

together with the expression (39) and the expansion of $[E_\alpha,M_{\mu\nu}]$ similar to (34), we can deduce that generally

$$[E_\alpha,M_{\mu\nu}]=\sum_\beta d_{\alpha\mu\nu\beta}\delta\{r_1(\alpha)-r_1(\beta)\}E_\beta+\sum_m d_{\alpha\mu\nu m}\delta\{r_1(\alpha)\}H_m. \tag{41}$$

Now, from the identity

$$[[E_\alpha,E_{-\alpha}],M_{\mu\nu}]=\sum_l r_l(\alpha)[H_l,M_{\mu\nu}], \tag{42}$$

we can then deduce that

$$b_{1\mu\nu\lambda}=0. \tag{43}$$

Corollary 1: Let the Lie algebra S be simple and H_l be the complete set of the Cartan subalgebra. If

$$[H_l,w_\sigma]=0 \tag{44}$$

for all σ, then

$$[H_l,p_\mu]=0,\quad [H_l,M_{\mu\nu}]=0 \tag{45}$$

and

$$[E_\alpha,p_\mu]=0,\quad [E_\alpha,M_{\mu\nu}]=0. \tag{46}$$

Thus,

$$G=L\oplus S. \tag{47}$$

Proof: The first part (45) follows trivially from the Theorem 2. The second part, (46), follows from the first part by the Theorem 2 of Ref. 5 (also Refs. 4 and 6).

This corollary, of course, covers the Theorem 1 above, but it serves to show that the assumption (19) may be replaced by the assumption (44).

As mentioned earlier in the Introduction, if only one element of S commutes with all elements of L, then $G=L\oplus S$, *provided* one restricts oneself to only unitary representations of G. Since no generality is lost by taking the one element to be H_1, Theorem 2 directly leads to Corollary 2.

Corollary 2: Let the Lie algebra S be simple and H_1 be a member of the Cartan subalgebra. If

$$[H_1,w_\sigma]=0 \tag{48}$$

for all σ, then

$$G=L\oplus S, \tag{49}$$

provided only unitary representations of G are considered.[7]

The results of this work, along with those of other authors mentioned above, show extreme difficulties encountered in an attempt to combine the space-time and internal properties of the members of an internal multiplet.

ACKNOWLEDGMENTS

The author takes great pleasure in thanking Professor E. C. G. Sudarshan for his encouragement and many useful discussions. Interesting discussions with Professor A. J. Macfarlane and Professor L. S. O'Raifeartaigh, as well as the critical reading of the manuscript by the former, are gratefully acknowledged. He also wishes to thank Professor F. Coester and Professor W. D. McGlinn for their helpful communications pertaining to the content of this paper.