Electromagnetic Solutions of the Field Equations of General Relativity

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An investigation of the electromagnetic field in general relativity has been undertaken. This study was partially motivated by the recent interest in the intense quasistellar sources and gravitational collapse and relativistic models for such phenomena. Metrics and fields as functions of two or three independent variables are considered. The equations for the case of two independent variables are presented in a simple form, and certain special solutions of these equations are derived, which may have some bearing on gravitational collapse. The major result of this paper may be stated as follows. Suppose given a set of metric coefficients g_{ik} which are functions of no more than three independent variables and which satisfy the vacuum field equations. We also suppose it possible to express this metric in diagonal form. Then, from this metric, one can find another metric, plus nonzero electromagnetic field, which satisfy the field equations with electromagnetic sources plus Maxwell's equations.

I. INTRODUCTION

THERE has been a resurgence of interest in theories of gravitation in the last few years, in particular, in general relativity. Furthermore, the recent discovery of intense quasistellar radio sources[1] has caused much speculation about gravitational collapse, which some feel to be a possible cause of the intense radiation from these sources.[2] We thus have a very good reason for investigating general relativity—to see what sort of cosmological consequences it may predict, which consequences may be used either to verify or disprove the theory or to provide models which may contribute to understanding of such cosmological phenomena as the quasistellar sources.

This paper is an outgrowth of discussions by the author with M. A. Melvin concerning the latter's electric and magnetic "geons."[3] Investigation of possible time dependence of such "geons" led to the present paper. Most of the results of this paper, however, go beyond Melvin's geon and are quite general.

The field equations of general relativity with an electromagnetic stress tensor as source and Maxwell's equations in curved space are investigated. The metric coefficients g_{ik} and the electromagnetic fields F_{ik} are restricted to be functions of three independent variables only. It is then shown how introduction of potentials, plus a "duality rotation,"[4] simplifies the equations.

The case in which g_{ik} and F_{ik} are functions of only two independent variables is considered in some detail. Simplification *à la* Weyl and Einstein–Rosen[5] leads to simple quasilinear wave equations for a metric coefficient and an electromagnetic potential. Certain special solutions—mainly similarity solutions—of these equations are found and discussed; these solutions may have some bearing on the problem of gravitational collapse.

It is then shown that, if one assumes a functional relationship between the two dependent variables mentioned in the last paragraph, that one can find a solution of the coupled gravitational-electromagnetic equations in terms of a solution of the vacuum equations. This result also holds for the more general equations with three independent variables if a similar functional relationship is assumed. The final result is: Given any solution (metric) of the vacuum field equations which is a function of no more than three variables, one can generate from this a solution of the coupled Einstein–Maxwell equations with nonzero electromagnetic field. The proof given in this paper assumes a diagonal metric, so that one must restrict the above-mentioned vacuum metric to be diagonal or diagonalizable. (Metrics which are functions of three or fewer variables can, in theory, be diagonalized, unless there is an asymmetry such as a rotation in the nonoccurring coordinate.)

The assumption of functional dependence of certain dependent variables has been made by several authors before.[6] However, the full generality of the above result does not seem to have been completely stated and proved.

II. SIMPLIFICATION OF THE EINSTEIN–MAXWELL EQUATIONS

We work with the field equations of general relativity plus electromagnetism,

$$R_{ij}-\tfrac{1}{2}g_{ij}R=kT_{ij}, \tag{2.1}$$

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[1] G. R. Burbidge, Paris Symposium on Radio Astronomy (Stanford University, Palo Alto, California, 1959); C. K. Seyfert, Astrophys. J. **97**, 28 (1963); F. Hoyle and W. A. Fowler, Nature **197**, 533 (1963).

[2] *Quasi-Stellar Sources and Gravitational Collapse*, edited by Ivor Robinson, Alfred Schild, and E. L. Schucking (University of Chicago Press, Chicago, 1965); P. G. Bergmann, Phys. Rev. Letters **12**, 139 (1964).

[3] Mael A. Melvin, Phys. Rev. Letters **8**, 65 (1964); M. A. Melvin, Phys. Rev. (to be published).

[4] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525–603 (1957), reprinted in J. A. Wheeler, *Geometrodynamics* (Academic Press, New York, 1962).

[5] H. Weyl, Ann. Physik **54**, 117 (1917); A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).

[6] H. Weyl, Ann. Physik **54**, 117 (1917); S. D. Majumdar, Phys. Rev. **72**, 390 (1947); A. Papapetrou, Proc. Roy. Irish Acad. **A51**, 191 (1947); W. B. Bonner, Proc. Phys. Soc. **A66**, 145 (1953) and **A67**, 225 (1954); J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), pp. 367–71; M. Misra and L. Radhakrishna, Proc. Nat. Inst. Sci. India **A28**, 632 (1962).

with

$$k=8\pi Gc^{-4} \tag{2.2}$$

and

$$T_{ij}=(4\pi)^{-1}(F_{il}F_j{}^l-\tfrac{1}{4}F_{lm}F^{lm}g_{ij}). \tag{2.3}$$

F_{ij} is the electromagnetic-field tensor, g_{ij} is the metric tensor, and conventions as to metric signature, Riemann tensor, etc., are as in Landau and Lifshitz.[7] Maxwell's equations in curved space are[4]

$$[ijkl]\partial F_{jk}/\partial x^l=0 \tag{2.4}$$

and

$$(\partial/\partial x^k)[(-g)^{1/2}g^{ij}g^{kl}F_{jl}]=0. \tag{2.5}$$

$\{[ijkl]=(+1,-1)$ for (even, odd) permutation of i, j, k, l; $=0$ if any two of i, j, k, l are equal.$\}$

Contraction on Eq. (2.1) yields

$$R=-kT, \tag{2.6}$$

where $R=R_i{}^i$, $T=T_i{}^i$. Since $T=0$, we have $R=0$. If we put

$$G_{ij}=[k(8\pi)^{-1}]^{1/2}F_{ij} \tag{2.7}$$

and

$$U_{ij}=kT_{ij}, \tag{2.8}$$

then Eqs. (2.1) and (2.3) become

$$R_{ij}=U_{ij}=2(G_{il}G_j{}^l-\tfrac{1}{4}G_{lm}G^{lm}g_{ij}). \tag{2.9}$$

In Eqs. (2.4) and (2.5), F_{ij} is replaced by G_{ij}.

We now assume the metric to be diagonal (for justification see the next paragraph):

$$g_{ij}=\delta_{ij}e_i\exp(2f_i), \tag{2.10a}$$

with

$$e_0=-1,\quad e_1=e_2=e_3=1. \tag{2.10b}$$

(We now drop the summation convention). Equations (2.9), (2.4), and (2.5) now become[8]

$$\sum_{l\neq i,k}(f_{l,k}f_{k,i}+f_{i,k}f_{l,i}-f_{l,i}f_{l,k}-f_{l,ik})=2\sum_l G_{il}G_{kl}e_l\exp(-2f_l)\quad(i\neq k), \tag{2.11}$$

$$\sum_{l\neq i}[f_{i,i}f_{l,i}-f_{l,i}{}^2-f_{l,ii}+e_ie_l\exp(2f_i-2f_l)\times(f_{l,l}f_{i,l}-f_{i,l}{}^2-f_{i,ll}-f_{i,l}\sum_{m\neq i,l}f_{m,l})]=\sum_{l\neq i}(G_{il})^2e_l\exp(-2f_l)-\tfrac{1}{2}e_i\exp(2f_i)\times\sum_{l,m\neq i}(G_{lm})^2e_le_m\exp(-2f_l-2f_m), \tag{2.12}$$

$$\sum_{jkl}[ijkl]G_{jk,l}=0, \tag{2.13}$$

$$\sum_i[e_i\exp(f_0+f_1+f_2+f_3-2f_i-2f_j)G_{ij}]_{,i}=0, \tag{2.14}$$

where commas denote ordinary differentiation.

We now assume that the f_i and the G_{ij} are independent of x^3. This partially justifies the previous assumption of a diagonal metric, since a metric can be diagonalized if

(a) it is independent of one variable (say x^3), and

(b) it is invariant under the reflection $x^3\to -x^3$.

The first requirement makes it possible to diagonalize the part of the metric involving $g_{ij}(i,j\neq 3)$ by a general coordinate transformation involving three functions (new coordinates), for then one can require $g_{ij}=0$ $(i\neq j;\ i,j\neq 3)$. The second requirement then makes possible the setting of the g_{i3} $(i\neq 3)$ equal to zero. If x^3 is an azimuthal angle, (a) requires axial symmetry and (b) requires no rotation.

The independence of x^3 now brings about the following results. The equations for $i=0$, 1 or 2 in Eq. (2.13) each reduce to two terms; the same thing happens for $j=0$, 1, or 2 in Eq. (2.14). This circumstance makes it very easy to satisfy these equations by means of potentials. We define A and B by

$$G_{i3}=-A_{,i}\quad(i=0,1,2), \tag{2.15}$$

$$G_{ij}=e_kB_{,k}\exp(f_i+f_j-f_k-f_3)\quad(i,j,k=0,1,2\text{ in cyclic order}). \tag{2.16}$$

(A is a component of the usual vector potential; B is not.) We now find that Eqs. (2.11) become

$$\sum_{l\neq i,k}(f_{l,k}f_{k,i}+f_{i,k}f_{l,i}-f_{l,i}f_{l,k}-f_{l,ik})=2\exp(-2f_3)(A_{,i}A_{,k}+B_{,i}B_{,k})\quad(i,\,k,\,3\neq) \tag{2.17}$$

and

$$B_{,i}A_{,k}=B_{,k}A_{,i}\quad(i,k,3\neq). \tag{2.18}$$

Equations (2.12) yield

$$\sum_{l\neq i}[f_{i,i}f_{l,i}-f_{l,i}{}^2-f_{l,ii}+e_ie_l\exp(2f_i-2f_l)\times(f_{l,l}f_{i,l}-f_{i,l}{}^2-f_{i,ll}-f_{i,l}\sum_{m\neq i,l}f_{m,l})]=\exp(-2f_3)\{(A_{,i})^2+(B_{,i})^2-\sum_{l\neq i,3}e_ie_l\exp(2f_i-2f_l-2f_3)\times[(A_{,l})^2+(B_{,l})^2]\}\quad(i\neq 3) \tag{2.19a}$$

$$=\sum_{l\neq 3}e_l\exp(-2f_l)[(A_{,l})^2+(B_{,l})^2]\quad(i=3). \tag{2.19b}$$

Equations (2.13) and (2.14) reduce to

$$\sum_{i\neq 3}e_i[A_{,i}\exp(-f_i+f_j+f_k-f_3)]_{,i}=0\quad(i,j,k=0,1,2\text{ in cyclic order}) \tag{2.20}$$

and exactly the same equation for B. Equation (2.20) is a wave equation for A; were the absent independent variable x^0 instead of x^3, this equation would be Laplace's equation.

It should be noted that there are alternative ways of defining potentials. We could write

$$G_{ij}=A_{i,j}-A_{j,i}\quad(i,j=0,1,2) \tag{2.21}$$

[7] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Cambridge, Mass., 1962), 2nd ed., Chaps. 10 and 11.

[8] B. Kent Harrison, Phys. Rev. **116**, 1285 (1959).

and

$$G_{i3}=e_i \exp(f_i-f_j-f_k+f_3)(B_{j,k}-B_{k,j})$$
$$i,j,k=0,1,2 \text{ in cyclic order}), \quad (2.22)$$

or we could take Eqs. (2.15) and (2.21) (the usual vector potential) or Eqs. (2.16) and (2.22). The potentials in Eqs. (2.15) and (2.16) seem to be the most convenient for quantities depending on only three variables.

We now note that Eqs. (2.18) imply that there exists a functional relationship between A and B:

$$h(A,B)=0. \quad (2.23)$$

We can now use Eq. (2.20) (for both A and B) to further elicit the relationship of A and B. We find that, in general,

$$A=C\cos\alpha \quad (2.24a)$$

$$B=B\sin\alpha \quad (2.24b)$$

for constant α, where C is a new potential. There is another case, in which B is an essentially arbitrary function of A; however, in this case, A is restricted further by a first-order differential equation. This second case is too special for our purposes and will not be considered.

In either of these two cases, we may introduce a new potential C which takes the place of A and B. It may be defined by the equation

$$C_{,i}C_{,ij}=A_{,i}A_{,j}+B_{,i}B_{,j} \quad (\text{all } i,j) \quad (2.25)$$

and it satisfies the wave equation (2.20).

The final set of equations is now the set Eqs. (2.17), (2.19), and (2.20), with $A_{,i}A_{,k}+B_{,i}B_{,k}\rightarrow C_{,i}C_{,k}$ in Eq. (2.17), $(A_{,l})+(B_{,l})^2\rightarrow(C_{,l})^2$ in Eq. (2.19), and $A\rightarrow C$ in Eq. (2.20). There are eight equations and five unknowns.

We note that the transformation from A and B to C is apparently a "duality rotation" of Misner and Wheeler.[4] If the fields were independent of x^0 instead of x^3, we could choose B to be zero, for example, with $A=C$, and have a pure electric field; or if $A=0$, $B=C$, we would have a magnetic field. However, in the present case, we will have both electric and magnetic fields present.

One other comment should be made. The same equations given here will hold for the independence of x^0, provided proper changes in signs and notations are made in them.

III. TWO-VARIABLE DEPENDENCE ONLY

We obtain considerable simplification if we assume all quantities independent of x^2. In this case, we may transform the metric so that $f_0=f_1$. We write the f_i as

$$f_0=f_1=\gamma-\psi,$$
$$f_2=\psi, \quad (3.1)$$
$$f_3=\ln R-\psi.$$

If one substitutes these expressions into the final equations of the last section and suitably combines the equations, one finds that R satisfies

$$R_{,00}-R_{,11}=0. \quad (3.2)$$

We must distinguish three cases.

(a) $R=\text{const}$. This case reduces to a flat, vacuum space.

(b) $R=f(x^1\pm x^0)$, $f'\neq 0$. This case is completely soluble, and after some reduction, we find a metric of the form

$$-ds^2=dudv\exp\left\{-2\psi+\int[u\psi'^2+u^{-1}C'^2\exp(2\psi)]du\right\}$$
$$+\exp(2\psi)(dx^2)^2+u^2\exp(-2\psi)(dx^3)^2, \quad (3.3)$$

where $u=x^1\pm x^0$, $v=x^1\mp x^0$, and C and ψ are arbitrary functions of u. This is clearly a function of one variable, u. The coefficient of $dudv$ may be absorbed into du by making a coordinate transformation, but this probably does not contribute to an understanding of the metric. This metric may represent a combination of plane electromagnetic and gravitational waves; such a conclusion, however, must be made cautiously because of the uncertainty of the meaning of x^0 and x^1 and because it may be possible to transform away the u dependence. Some sort of analysis, such as that carried out on vacuum plane gravitational waves[9] is clearly needed.

(c) $R=f(x^1+x^0)+g(x^1-x^0)$, $f'g'\neq 0$. It is easily shown[5] that in this case we may define a new $x^1=R$. If we assume the new x^1 to be used already and put

$$R=x^1, \quad (3.4)$$

and also

$$\gamma=\xi+\ln x^1-2\ln V, \quad (3.5)$$

$$\psi=\ln x^1-\ln V, \quad (3.6)$$

we get the following equations:

$$V_{,11}+(V_{,1}/x^1)-V_{,00}$$
$$=(1/V)(V_{,1}{}^2-V_{,0}{}^2+C_{,0}{}^2-C_{,1}{}^2), \quad (3.7)$$

$$C_{,11}+(C_{,1}/x^1)-C_{,00}=(2/V)(C_{,1}V_{,1}-C_{,0}V_{,0}), \quad (3.8)$$

$$\xi_{,0}=2x^1V^{-2}(V_{,0}V_{,1}-C_{,0}C_{,1}), \quad (3.9)$$

$$\xi_{,1}=x^1V^{-2}(V_{,0}{}^2+V_{,1}{}^2+C_{,0}{}^2+C_{,1}{}^2), \quad (3.10)$$

$$\xi_{,11}-\xi_{,00}=V^{-2}(V_{,0}{}^2-V_{,1}{}^2+C_{,0}{}^2-C_{,1}{}^2). \quad (3.11)$$

The metric is now

$$-ds^2=V^{-2}(\exp 2\xi)[-(dx^0)^2+(dx^1)^2]$$
$$+(x^1)^2V^{-2}(dx^2)^2+V^2(dx^3)^2. \quad (3.12)$$

[9] F. A. E. Pirani, articles in *Recent Developments in General Relativity* (Pergamon Press, Inc., New York, 1962); *Gravitation: An Introduction to Current Research*, edited by Louis Witten (John Wiley & Sons, Inc., New York, 1962).

Equation (3.11) is derivable from Eqs. (3.7)–(3.10); $\xi_{,01}$ as calculated from Eq. (3.9) is identical with that calculated from Eq. (3.10), with Eqs. (3.7) and (3.8) assumed. Thus we can find ξ from V and C, and Eqs. (3.7) and (3.8) are the main equations. They are seen to be quasilinear wave equations for V and C. If we write

$$V=\exp U\,, \tag{3.13}$$

they take the slightly simpler form

$$U_{,11}+(U_{,1}/x^1)-U_{,00}=\exp(-2U)(C_{,0}{}^2-C_{,1}{}^2)\,, \tag{3.14}$$

$$C_{,11}+(C_{,1}/x^1)-C_{,00}=2(C_{,1}U_{,1}-C_{,0}U_{,0})\,. \tag{3.15}$$

The author has tried various ways of simplifying Eqs. (3.7) and (3.8) further. The most interesting way, which involves assumption of a functional relationship between C and V, is given in the next section. In this section we shall discuss a number of special solutions.

Case I.

We first note that if

$$V=k\sqrt{x^1}\,, \tag{3.16a}$$

then

$$C=f(u)\quad (u=x^1\pm x^0)\,, \tag{3.16b}$$

and

$$\xi=2\int[f'(u)]^2du\,. \tag{3.16c}$$

Again we have a "wave-like" solution, but without any justification for a true wave nature.

If we assume

$$C=F(x^1)+G(x^0)\,, \tag{3.17}$$

and

$$U=H(x^1)+J(x^0)\,, \tag{3.18}$$

we get two cases which are, after some simplification:

Case II.

$$C=\lambda x^0+\mu\int dx^1(x^1)^{-1}\exp(2U)\,, \tag{3.19a}$$

$$U=U(x^1)\,, \tag{3.19b}$$

with

$$U''+\frac{U'}{x^1}=\exp(-2U)\left[\lambda^2-\frac{\mu^2}{(x^1)^2}\exp(4U)\right], \tag{3.19c}$$

$$\xi=2\lambda\mu x^0+\int dx^1\left[x^1(U'^2+\lambda^2e^{-2U})+\frac{\mu^2}{x^1}\exp(2U)\right], \tag{3.19d}$$

where λ and μ are constants.

Case III.

$$C=\tfrac{1}{2}\lambda(x^1)^2\,, \tag{3.20a}$$

$$V=\lambda x^1\cosh x^0\,, \tag{3.20b}$$

$$\xi=2\ln\cosh x^0+\ln x^1+\tfrac{1}{2}(x^1)^2\,. \tag{3.20c}$$

There are also other solutions which, however, fall in the category of cases discussed in the next section.

Also considered were a class of similarity solutions. To discuss these, it is convenient to define quantities σ and η by

$$\sigma=\operatorname{sech}\eta=x^1(x^0)^{-1},\ \eta\geq 0\,. \tag{3.21}$$

Then, if we assume

$$C=F(r)+(x^1)^kG(\eta) \tag{3.22}$$

and

$$U=\ln V=H(r)+J(\eta)\,, \tag{3.23}$$

which form is convenient for calculation, we get two more cases:

Case IV.

$$C=(x^1)^kG(\eta)\,, \tag{3.24a}$$

$$V=(x^1)^kK(\eta)\,, \tag{3.24b}$$

with

$$G'^2+K'^2=k^2G^2-mK^2\,, \tag{3.24c}$$

$$G''-k^2G=2K'K^{-1}(G'-kG\coth\eta)\,, \tag{3.24d}$$

$$\xi=(k^2+m)\ln x^1+2\int d\eta K^{-2}[k(GG'+KK')+(mK^2-k^2G^2)\coth\eta]\,, \tag{3.24e}$$

where k and m are constants.

Case V.

$$C=\ln x^1-\int Q(\eta)d\eta\,, \tag{3.25a}$$

$$V=V(\eta)\,, \tag{3.25b}$$

with

$$Q'=2V'V^{-1}(Q+\coth\eta)\,, \tag{3.25c}$$

$$Q^2+V'^2+mV^2=1\,, \tag{3.25d}$$

$$\xi=m\ln x^1+2m\ln|\sinh\eta|+2\int d\eta\, V^{-2}(Q+\coth\eta)\,, \tag{3.25e}$$

where m is a constant. Again some solutions have not been presented because they are included in other cases in this paper.

Another interesting sloution can be obtained from case V by putting $m=0$, $G_{02}=G_{12}=0$, and

$$G_{03}=-x^1C_{,1}V^{-2}=-V^{-2}(Q\coth\eta+1)\,, \tag{3.26a}$$

$$G_{13}=-x^1C_{,0}V^{-2}=V^{-2}Q\operatorname{csch}\eta\,. \tag{3.26b}$$

We term this case VI. This can be obtained in a more straightforward manner by employing the alternate potentials defined in Eqs. (2.21) and (2.22). It cannot be obtained directly with the potentials of Eqs. (2.15) and (2.16).

A special case of VI can be obtained if one takes

$$Q=-1, \tag{3.27a}$$

$$V=1. \tag{3.27b}$$

Then we get

$$\exp\xi=[1+(1-\sigma^2)^{1/2}]^2(1-\sigma^2)^{-1}, \tag{3.27c}$$

$$G_{03}=-1+(1-\sigma^2)^{1/2}, \tag{3.27d}$$

and

$$G_{13}=-\sigma(1-\sigma^2)^{-1/2}. \tag{3.27e}$$

Why consider similarity solutions? The motivation can be found in the hope that such solutions may represent gravitational collapse in some simplified model. Indeed, they may represent an approximation to the time-dependent approach to Melvin's static solution,[3] which solution can be written

$$V=\rho(\rho^2+1)^{-1}, \tag{3.28a}$$

$$C=\pm(\rho^2+1)^{-1}, \tag{3.28b}$$

where $\rho=x^1/a$ and a is a constant (characteristic length). If x^1 is the radial cylindrical coordinate r, this represents a bundle of electric or magnetic flux held together by its own gravitational attraction. For $x^1\gg a$, a similarity solution may well approximate an approach to this static configuration.

Undoubtedly, further analysis may serve to simplify the above solutions. In this paper we shall content ourselves with a few remarks about their nature.

To understand the solutions better, we need to look at the electromagnetic fields associated with them. We first combine Eqs. (2.7), (2.15), (2.16), and (2.24), obtaining

$$F_{ij}=(8\pi k^{-1})^{1/2}e_l C_{,l}(\sin\alpha)\exp(f_i+f_j-f_l-f_3) \quad (i,j,l=0,1,2 \text{ in cyclic order}) \tag{3.29a}$$

and

$$F_{i3}=-(8\pi k^{-1})^{1/2}C_{,1}\cos\alpha \quad (i=0,1,2). \tag{3.29b}$$

We note that the F_{ij} are the covariant components of the electromagnetic-field tensor and that they are *not* the physical fields in a curved space. To get the physical fields, we use a prescription of Melvin.[3] The physical electric field is

$$E_i=F_{0i}(-g_{00}g_{ii})^{-1/2} \quad (i=1,2,3) \tag{3.30}$$

and the magnetic field is

$$B_i=F_{jk}(g_{jj}g_{kk})^{-1/2} \quad (i,j,k=1,2,3 \text{ in cyclic order}). \tag{3.31}$$

Thus, if

$$P_i=(8\pi k^{-1})^{1/2}C_{,j}\exp(-f_j-f_3) \tag{3.32}$$

(with $j=3-i$, $i=1$, 2, or 3), we get

$$E_1=P_1\sin\alpha,\quad E_2=-P_2\sin\alpha,\quad E_3=-P_3\cos\alpha \tag{3.33}$$

and

$$B_1=-P_1\cos\alpha,\quad B_2=P_2\cos\alpha,\quad B_3=-P_3\sin\alpha. \tag{3.34}$$

α is an arbitrary constant. Thus, we need investigate only the quantities P_i.

For the case of independence of x^2, we find that

$$P_1=0 \tag{3.35a}$$

$$P_2=(8\pi k^{-1})^{1/2}C_{,1}(V^2/x^1)\exp(-\xi) \tag{3.35b}$$

$$P_3=(8\pi k^{-1})^{1/2}C_{,0}(V^2/x^1)\exp(-\xi). \tag{3.35c}$$

Suppose now that we calculate the physical fields for the special case of VI, Eqs. (3.27). Here $\alpha=0$. We find ($r=x^1$, $T=x^0$)

$$B_2=r^{-2}(1-\sigma^2)[1-(1-\sigma^2)^{-1/2}][1+(1-\sigma^2)^{1/2}]^{-2} \tag{3.36a}$$

and

$$E_3=-(rT)^{-1}(1-\sigma^2)^{1/2}[1+(1-\sigma^2)^{1/2}]^{-2}. \tag{3.36b}$$

Both B_2 and E_3 are well-behaved at $r=T$. At $r=0$, B_2 is finite, but E_3 is not. This is probably to be expected, since the solution was derived from a similarity assumption. As previously mentioned, for large r, the solution may be a good approximation to a possible physical situation. One fly in the ointment remains: the singularity in $\exp\xi$ at $r=T$ [see Eq. (3.27c)]. This may be a coordinate singularity. Such singularities may be tested for phyiscal reality by computing the differential invariants of the metric.[10]

We may examine case V near $r=T$ by making a power-series expansion in η for Q and V. [Equation (3.25d) seems to indicate that Q and V are bounded for small η.] Calculation with power series in η indicates the following facts (if we assume V and Q not constant): (1) Q is odd in η, V is even; (2) $m>0$; (3) the coefficients in the power series are functions of m only; (4) the metric coefficients and the electromagnetic fields are all finite and nonzero at $r=T(\eta=0)$; (5) the radial derivatives of the electromagnetic fields at $r=T$ are not zero.

Inspection of the equations near $r=0$ ($\eta=\infty$) shows that the metric is quite singular there. However, the metric may have possible physical meaning for large r.

It is apparent that similar treatments can be carried out for other solutions. Furthermore, coordinate transformations may simplify some of the metrics. A likely candidate for this type of simplification is solution III.

IV. DERIVATION OF ELECTROMAGNETIC SOLUTIONS FROM VACUUM SOLUTIONS

The form of Eqs. (3.7), (3.8) suggests many possible techniques for simplifying the equations. One of these is to attempt to define new dependent variables as functions of the old such that one equation involves only one dependent variable; solution of that equation would then yield something like a source function in the other equation for the second variable. However, it is

[10] J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by Louis Witten (John Wiley & Sons, Inc., New York, 1962).

easily shown that this sort of simplification is not possible for the given equations.

However, another sort of technique does yield rich rewards. It has the disadvantage that a general solution of Eqs. (3.7)–(3.8) is not obtained; however, the insight gained more than compensates for this.

We assume a functional dependence between C and V, in the form

$$C=C(H)\,, \tag{4.1a}$$

$$V=V(H)\,, \tag{4.1b}$$

where H is a new function of x^1 and x^0. We obtain, by substitution into Eqs. (3.7) and (3.8),

$$V'(H_{,11}+H_{,1}/x^1-H_{,00}) =(H_{,1}{}^2-H_{,0}{}^2)((V'^2-C'^2)/V-V'')\,, \tag{4.2}$$

$$C'(H_{,11}+H_{,1}/x^1-H_{,00}) =(H_{,1}{}^2-H_{,0}{}^2)(2C'V'/V-C'')\,, \tag{4.3}$$

where the prime denotes differentiation with respect to H. If $H_{,0}{}^2-H_{,1}{}^2$ and $H_{,11}+(x^1)^{-1}H_{,1}-H_{,00}$ are both nonzero, we find that

$$V=\lambda\cos\theta\,, \tag{4.4a}$$

$$C=\lambda\sin\theta\,, \tag{4.4b}$$

$$\theta=\theta(H)\,, \tag{4.4c}$$

$$H_{,11}+H_{,1}/x^1-H_{,00} =(H_{,0}{}^2-H_{,1}{}^2)-(\theta''/\theta'+\theta'\tan\theta)\,. \tag{4.4d}$$

We will be more interested in the only other nontrivial case, in which we assume

$$H_{,11}+(H_{,1}/x^1)-H_{,00}=0\,, \tag{4.5a}$$

$$H_{,1}{}^2-H_{,0}{}^2\neq 0\,. \tag{4.5b}$$

Then we get

$$V''=V^{-1}(V'^2-C'^2)\,, \tag{4.5c}$$

$$C''=2C'V'V^{-1}. \tag{4.5d}$$

The solution of these equations, slightly simplified, is

$$V=\lambda\,\mathrm{sech}H\,, \tag{4.5e}$$

$$C=\lambda\tanh H\,, \tag{4.5f}$$

where λ is a constant. Here we have a linear wave equation for H, so that we may get a readily obtained solution of Eqs. (3.7) and (3.8) with two arbitrary functions; or, alternatively, we can impose two boundary conditions on our solution.

If we substitute Eqs. (4.5e) and (4.5f) into Eqs. (3.9)–(3.11), we find another surprise—the equations

$$\xi_{,0}=2x^1H_{,0}H_{,1}\,, \tag{4.6a}$$

$$\xi_{,1}=x^1(H_{,0}{}^2+H_{,1}{}^2)\,, \tag{4.6b}$$

$$\xi_{,11}-\xi_{,00}=H_{,0}{}^2-H_{,1}{}^2. \tag{4.6c}$$

But if we took the vacuum case with $C=0$ and put $V=\exp H$, we would obtain exactly Eqs. (4.5a) and (4.6). In other words, we can do the following: choose any solution H of the wave equation (4.5a). Find the corresponding ξ from Eqs. (4.6a, b). The functions ξ and $V=\exp H$ now form a solution of the vacuum equations. However, the functions ξ, $V=\lambda\,\mathrm{sech}H$, $C=\lambda\tanh H$ also form a solution of the nonvacuum, coupled Einstein–Maxwell equations. Thus, from any vacuum solution, we may obtain a solution for the case with electromagnetic fields.

This result can be extended to the equations with three independent variables, Eqs. (2.17), (2.19), and (2.20) (with A and B replaced by C as previously outlined). We write

$$f_0=u-y\,, \tag{4.7a}$$

$$f_1=v-y\,, \tag{4.7b}$$

$$f_2=w-y\,, \tag{4.7c}$$

$$f_3=y. \tag{4.7d}$$

Then, if we put

$$C=C(H)\,, \tag{4.8a}$$

$$y=\ln Z(H)\,, \tag{4.8b}$$

and require the vacuum equations to hold [Eqs. (2.17), (2.19), (2.20), and (4.7), with $C\to 0$, $y\to H$], we get again equations for $C(H)$ and $Z(H)$ [Eqs. (4.5c) and (4.5d) with $V\to Z$] which can be solved to give

$$Z=\lambda\,\mathrm{sech}H\,, \tag{4.9a}$$

$$C=\lambda\tanh H. \tag{4.9b}$$

Thus, any (diagonalizable) solution of the vacuum equations in not more than three variables yields a corresponding solution of the combined Einstein–Maxwell equations.

We may write this result in a very simple form. Suppose the metric

$$-ds^2=V^{-2}[-e^{2u}(dx^0)^2+e^{2v}(dx^1)^2+e^{2w}(dx^2)^2] +V^2(dx^3)^2\,, \tag{4.10}$$

with u, v, w, V functions of x^0, x^1, and x^2, satisfies the vacuum equations. Then the metric

$$-ds^2=((V^2+1)^2/4\lambda^2V^2)[-e^{2u}(dx^0)^2+e^{2v}(dx^1)^2 +e^{2w}(dx^2)^2]+(4\lambda^2V^2/(V^2+1)^2)(dx^3)^2 \tag{4.11}$$

and the potential

$$C=\lambda(V^2-1)/(V^2+1)=\lambda(1-2/(V^2+1)) \tag{4.12}$$

satisfy the Einstein–Maxwell equations. (A similar theorem holds with x^0 and x^3 and appropriate signs interchanged. When this is done, the definitions of the potentials must be appropriately modified.)

The idea of assuming a functional relation between certain variables, as in Eqs. (4.1), is apparently not new. Several authors have derived solutions in this manner.[6] However, none completely present the general result that to every vacuum solution in three variables there corresponds an electromagnetic solution. (Misra and Radhakrishna do state this, but show it only for functions of two variables.)

Two obvious possible generalizations present themselves. The first is to extend this result to any metric, diagonalizable or not, in not more than three variables. This is only a slight generalization and quite possibly can be done. The second is to extend the result to any metric whatsoever. The restriction to three independent variables makes possible the singling out of two special functions $V[=(g_{33})^{1/2}$ in Eq. (4.10)$]$ and C (the electromagnetic potential)—which can then be related. In the general case, no such special functions exist, and the assumptions to be made to get a similar result are unknown.

The most immediate utility of the results above is in the generation of electromagnetic metrics from vacuum metrics. However, there is, perhaps, a more fundamental conclusion which can be drawn from these findings: There exists a hitherto undiscovered intimate relationship between the electromagnetic and gravitational fields, implicit in the Einstein–Maxwell system of equations. These results may be nothing more than trivial consequences of known theorems, or they may be a completely new relationship of profound significance. It is at least certain that this relationship will bear some close connection to the "already unified theory" of Rainich, Misner, and Wheeler.[4]

So far we have no clue as to such a possible relationship except the correspondence of the metrics (4.10) and (4.11). We can see a few facts from this correspondence which may shed some light on the problem.

First, we note that, if $V^2 \ll 1$, the metric (4.11) is essentially the same as that of (4.10), the only difference being scale changes due to the constant λ. For large V^2, this is not the case. In connection with this, we see that singularities $V=0$ in Eq. (4.10) yield the same singularities in Eq. (4.11), but that if $|V|=\infty$ in Eq. (4.10), the expression $V^2(V^2+1)^{-2}$ in Eq. (4.11) equals 0. The only other singularities in either metric are those (if any) resulting from u, v, or $w=\pm\infty$, which singularities occur in both metrics. There is thus a one-to-one correspondence between the singularities of both metrics; they all produce the same result in both metrics except when $|V|=\infty$.

The fact that there exists a special symmetry in the given metrics—independence of x^3 and consequent invariance under a one-parameter group of transformations—prompts another observation. It may be that the looked-for relation between gravitation and electromagnetism is a consequence of this special symmetry in the fields. If such is the case, we might hope to find different relationships, corresponding to different symmetries in the fields.

Finally, we note that we can view—in some sense—metric (4.11) as metric (4.10) with a special electromagnetic field introduced into it. If there were a plausible physical reason for introducing this particular field, we might derive some insight from it. A hypothetical example follows. Suppose the vacuum metric (4.10) to arise from a system of point or line masses, occurring at singularities of the metric. Then suppose the masses to be suddenly converted entirely into electromagnetic radiation, which then distributes itself until it reaches equilibrium with the curved space around it. Then the resulting metric plus field might be represented by Eqs. (4.11) and (4.12). This example is in no way proved; it is merely hypothetical. One defect with the example is that the singularities at which there were masses in Eq. (4.10) would continue to be singularities in Eq. (4.11). However, it might still be possible to reinterpret these singularities in the electromagnetic-field metric, particularly if they had changed their nature (as in the $V=\infty$ case). Such an example need not be the case, as with Melvin's static metric.[3] If we take

$$u=v=w=\ln x^1, \tag{4.13a}$$

$$V=x^1/a, \tag{4.13b}$$

$$\lambda=\tfrac{1}{2}, \tag{4.13c}$$

(a=constant) in Eqs. (4.10)–(4.12), we get a flat space from Eq. (4.10) (cylindrical coordinates) but Melvin's solution from Eqs. (4.11)–(4.12). Thus the electromagnetic field in Melvin's case does not arise from line or point masses in the corresponding vacuum metric. However, Thorne[11] notes that Melvin's "geon," or "universe," is the most diffuse distribution of electromagnetic gravitational energy possible under the circumstances. A more condensed, concentrated system may represent exploded or converted masses, as discussed above.

A perhaps obvious comment should be made in closing: Not all solutions of the equations can be obtained in this way. The solutions given in Sec. III cannot be obtained by the methods of Sec. IV. One wonders whether or not there is some fundamental physical distinction between solutions obtainable by the methods of Sec. IV and those not so obtainable.

V. CONCLUSIONS

It is hoped that the results given in this paper will lead to a deeper understanding of the relations between electromagnetic and gravitational fields—or between electromagnetic fields and curved space. Failing that, we can at least hope that some physical insight can be gained from the solutions of the Einstein–Maxwell equations given here. Future work is contemplated on all of these fronts.

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[11] K. S. Thorne, Phys. Rev. 138, B251 (1965).