that

$$H(\mathbf{x},\mathbf{k}) = 1\omega(\mathbf{x},\mathbf{k}); \tag{96}$$

then Eq. (90) gives

$$\frac{\partial f}{\partial t} = 2i\omega_A f + \frac{\partial \omega_H}{\partial \mathbf{k}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \omega_H}{\partial \mathbf{x}} \frac{\partial f}{\partial \mathbf{k}}. \tag{97}$$

If we identify ω_A with $-i\omega_i$ and ω_H with ω_r we have the desired result. The assumption contained in Eq. (96) seems to be difficult to justify in general.

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Statistical Descriptions of Free Boson Fields

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Several serious mathematical deficiencies in Sudarshan's probability-functional approach to the statistical description of light beams are demonstrated. In particular, it is shown that all the correlation functions of the beam do not necessarily determine its density matrix.

I. INTRODUCTION

RECENTLY, Sudarshan[1,2] has developed a probability-functional approach for describing all free boson fields. He concludes that "the description of statistical states of a quantum-mechanical system with an arbitrary (countably infinite) number of degrees of freedom is completely equivalent to the description in terms of classical probability distributions in the same (countably infinite) number of degrees of freedom." This conclusion and the methods introduced by Sudarshan have been used in several discussions of the statistical properties of light beams including that of an optical maser.[3–5]

The purpose of this note is to demonstrate several serious mathematical deficiencies in Sudarshan's probability functional approach. In particular, we will show that all the correlation functions of the beam do not necessarily determine its density matrix.

II. SUDARSHAN'S PROBABILITY FUNCTIONAL

The most general form taken by the density matrix of a free boson field is

$$\rho = \sum_{\{n_{\mathbf{k}}\},\{n_{\mathbf{k}}'\}} |\{n_{\mathbf{k}}\}\rangle \rho(\{n_{\mathbf{k}}\},\{n_{\mathbf{k}}'\}) \langle \{n_{\mathbf{k}}'\}|\,, \tag{1}$$

where

$$\rho = \rho^\dagger, \quad \mathrm{Tr}\rho = 1,$$

and[6]

$$|\{n_{\mathbf{k}}\}\rangle = \prod_{\mathbf{k}} |n_{\mathbf{k}}\rangle.$$

Sudarshan[1,2] has argued that all density matrices of the form given by Eq. (1), i.e., *every* free field boson density matrix, can be put into a special form in a unique way which allows the conclusion that "there is a one-to-one correspondence between density matrices of a quantized (free boson) field and classical probability functions." We shall now review for a single mode the demonstration which precedes this conclusion.

The most general density matrix for an isolated oscillator (field mode) is

$$\rho = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} |n\rangle \rho(n,n') \langle n'|\,, \tag{2}$$

and the expectation value of the normal ordered product $(b^\dagger)^\lambda (b)^\mu$ for this statistical state is[7]

$$\begin{aligned} \xi_{\lambda;\mu} &= \mathrm{Tr}\{\rho (b^\dagger)^\lambda (b)^\mu\} \\ &= \sum_{l=0}^{\infty} \rho(l+\mu, l+\lambda)(1/l!)[(l+\lambda)!(l+\mu)!]^{1/2}. \end{aligned} \tag{3}$$

[1] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[2] E. C. G. Sudarshan, *Proceedings of the Symposium on Optical Masers* (Brooklyn Polytechnic Press, New York and John Wiley & Sons, Inc., New York, 1963), pp. 45–50.
[3] L. Mandel, Phys. Letters **7**, 117 (1963).
[4] L. Mandel, Phys. Rev. **134**, A10 (1964).
[5] L. Mandel, Phys. Letters **10**, 166 (1964).

[6] $|n_{\mathbf{k}}\rangle$ is the occupation number state describing n bosons in the **k**th mode.
[7] b and b^+ are the annihilation and creation operators, respectively, for the bosons of the oscillator: $b|n\rangle = (n)^{\frac{1}{2}}|n-1\rangle$, $b^+|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$.

Now consider the "density matrix"

$$\rho_s=\int d^2z|z\rangle\Phi(r,\theta)\langle z|\,, \quad (4)$$

where[8]

$$|z\rangle=\exp(-|z|^2/2\hbar)\sum_{n=0}^{\infty}\frac{1}{(n!)^{1/2}}(z/\hbar^{1/2})^n|n\rangle\,,$$
$$b|z\rangle=(z/\hbar^{1/2})|z\rangle\,, \quad (5)$$

and the "classical probability functional" $\Phi(r,\theta)$ is given by[9]

$$\Phi(r,\theta)=\frac{1}{2\pi r\hbar^{1/2}}\sum_{n=0}\sum_{n'=0}\frac{\rho(n,n')(n!n'!)^{1/2}}{(n+n')!}\times\exp[(r^2/\hbar)+i(n'-n)\theta](-1)^{n+n'}\times(\hbar)^{(n+n')/2}\delta^{(n+n')}(r)\,,\quad z=re^{i\theta}. \quad (6)$$

Using Eq. (5), the properties of $\delta^{(n)}(r)$, and manipulating without worrying about the mathematical properties of $\Phi(r,\theta)$, we see that

$$\mathrm{Tr}\{\rho_s(b^\dagger)^\lambda(b)^\mu\}=\hbar^{-(\lambda+\mu)/2}\int d^2z\,\Phi(r,\theta)(z^*)^\lambda(z)^\mu=\hbar^{-(\lambda+\mu)/2}\hbar^{-1/2}\int_0^\infty dr\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\frac{\rho(n,n')(n!n'!)^{1/2}}{(n+n')!}$$
$$\times\exp(r^2/\hbar)(-1)^{n+n'}(\hbar)^{(n+n')/2}r^{\lambda+\mu}\delta^{(n+n')}(r)\delta_{n'-n,\lambda-\mu}=\sum_{l=0}^{\infty}\rho(l+\mu,l+\lambda)\frac{1}{l!}[(l+\lambda)!(l+\mu)!]^{1/2}=\xi_{\lambda;\mu}. \quad (7)$$

This result, according to Sudarshan,[1,2] implies that there is an equivalence between the classical and quantum-statistical descriptions of light beams and that all the moments $\xi_{\lambda;\mu}$ determine the density matrix uniquely.[10]

Having reviewed Sudarshan's demonstration, we will now show that:

(1) All the moments $\xi_{\lambda;\mu}$ do not determine the density matrix uniquely, and therefore, exhibiting a "probability functional" $\Phi(r,\theta)$ that gives the moments $\xi_{\lambda;\mu}$ is not sufficient to prove a "one-to-one correspondence between density matrices of a quantized field and classical probability functions."

(2) The "functional" $r\Phi(r,\theta)$ is not a generalized function,[11] and, consequently, manipulating it as if it were can easily lead to meaningless mathematical expressions.

(3) Because of the properties of $\Phi(r,\theta)$, Sudarshan's equivalence theorem is mathematically meaningless and without physical content.

A well-known example from probability theory can be used to demonstrate that the density matrix is not uniquely specified by all the $\xi_{\lambda;\mu}$.[12,13] Consider two density matrices in the coherent state representation[14]:

$$\rho_1=\int d^2z|z\rangle P_1(z)\langle z|\,,$$
$$\rho_2=\int d^2z|z\rangle P_2(z)\langle z|\,, \quad (8)$$

where

$$P_1(z)=\frac{1}{2\pi r\hbar^{1/2}}\frac{1}{4!}\exp[-(r/\hbar^{1/2})^{1/4}]\,,$$

$$P_2(z)=\frac{1}{2\pi r\hbar^{1/2}}\frac{1}{4!}\{1+\sin[(r/\hbar^{1/2})^{1/4}]\}\times\exp[-(r/\hbar^{1/2})^{1/4}].$$

It is clear that $\rho_1\neq\rho_2$. The moments $\xi_{\lambda;\mu}$ computed from ρ_1 and ρ_2 are easily found:

$$\xi_{\lambda;\mu}^{(1)}=\mathrm{Tr}[\rho_1(b^\dagger)^\lambda(b)^\mu]=\delta_{\lambda,\mu}\frac{1}{4!}\int_0^\infty r^{2\lambda}\exp(-r^{1/4})dr=\delta_{\lambda,\mu}[(8\lambda+3)!/3!];$$
$$\xi_{\lambda;\mu}^{(2)}=\mathrm{Tr}[\rho_2(b^\dagger)^\lambda(b)^\mu]=\delta_{\lambda,\mu}\frac{1}{4!}\int_0^\infty r^{2\lambda}[1+\sin(r^{1/4})]\exp(-r^{1/4})dr=\delta_{\lambda,\mu}[(8\lambda+3)!/3!]=\xi_{\lambda;\mu}^{(1)}.\quad \text{Q.E.D.} \quad (9)$$

[8] The states $|8\rangle$ are the minimum-uncertainty state vectors for the harmonic oscillator. A discussion of their properties can be found, for example, in E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill Book Company, Inc., New York, 1962).

[9] $\delta^{(n)}(r)$ is so defined that $\int_0^\infty\delta^{(n)}(r)f(r)dr=(-1)^nf^{(n)}(0)$, where $f(r)$ is n-times differentiable at $r=0$. A rigorous justification of manipulations with the $\delta^{(n)}(r)$ is given by the theory of generalized functions (Ref. 11).

[10] All the $\xi_{\lambda;\mu}$ determine all the correlation functions.

[11] M. J. Lighthill, *Fourier Analysis and Generalized Functions*, (Cambridge University Press, Cambridge, 1960), students ed.

[12] E. Lukacs, *Characteristic Functions* (Hafner Publishing Company, New York, 1960), p. 19.

[13] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford University Press, London, 1948), 2nd ed., Sec. 11.9.

[14] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

We have used the result that

$$\int_0^\infty r^m \sin(r^{1/4}) \exp(-r^{1/4}) dr = 0\,,$$

$m=0, 1, 2, \cdots$. Since the above example shows that two different density matrices lead to the same $\xi_{\lambda;\mu}$, one cannot conclude by any argument that all the correlation functions always determine the density matrix uniquely. Consequently, a demonstration of a moment equivalence is not sufficient to prove a "one-to-one correspondence between density matrices of a quantized field and classical probability functionals." This example can easily be generalized to the case in which a countably infinite set of modes are excited.

The defining mathematical properties of the "probability functional" $\Phi(r,\theta)$, such as which class of functions it operates on, have been left unstated by Sudarshan.[1,2] It can be shown, however, that $r\Phi(r,\theta)$ is not a generalized function[11] for an important class of density matrices. Take a function $f(r)$ which is identical to $\exp(r^2/\hbar)$ for $-r_0 \leq r \leq r_0$, which is differentiable any number of times, and which goes to zero along with all its derivatives faster than $|r|^{-N}$ for all N as $|r| \to \infty$.[15] Now consider those $\Phi(r,\theta)$ for density matrices diagonal in the occupation number representation:

$$\Phi(r,\theta) = \frac{1}{2\pi r \hbar^{1/2}} \sum_{n=0}^{\infty} \frac{\rho(n,n)n!}{(2n)!} e^{r^2/\hbar} \hbar^n \delta^{(2n)}(r)\,. \quad (10)$$

If $r\Phi(r,\theta)$ is a generalized function, then the sequence of partial sums

$$S_M = \sum_{n=0}^{M} \int_0^\infty \int_0^{2\pi} \frac{1}{2\pi r \hbar^{1/2}} \frac{\rho(n,n)n!}{(2n)!} \times e^{r^2/\hbar} \hbar^n \delta^{(2n)}(r) f(r) d\theta r dr \quad (11)$$

has a limit as $M \to \infty$.[16] Since $f(r)$ is equal to $\exp(r^2/\hbar)$ for $0 \leq r \leq r_0$, we obtain

$$\int_0^\infty \int_0^{2\pi} \frac{1}{2\pi r \hbar^{1/2}} \frac{\rho(n,n)n!}{(2n)!} e^{r^2/\hbar} \hbar^n \times \delta^{(2n)}(r) f(r) d\theta r dr = 2^n \rho(n,n)\,, \quad (12)$$

from which it follows that

$$S_M = \sum_{n=0}^{M} 2^n \rho(n,n)\,. \quad (13)$$

There are an infinite number of density matrices for which $\lim_{M\to\infty} S_M \to \infty$. Take, for example, an oscillator in thermal equilibrium at such a temperature that $\bar{n}=1$:

$$\rho(n,n') = \delta_{n,n'} 2^{-(n+1)}\,. \quad (14)$$

Then we have

$$S_M = \tfrac{1}{2} \sum_{n=0}^{M} 1 = \tfrac{1}{2} M\,, \quad (15)$$

which becomes infinite as $M \to \infty$. Consequently, $r\Phi(r,\theta)$ is not a generalized function for a wide class of density matrices including the physically important case of an harmonic oscillator in thermal equilibrium. If $r\Phi(r,\theta)$ is not a generalized function, then it is certainly not a function of r and θ in the ordinary sense[17] with

$$\int_0^\infty \int_0^{2\pi} \Phi(r,\theta) d\theta r dr = 1\,.$$

It then follows that $\Phi(r,\theta)$ cannot be interpreted as a probability-density function or a phase-space distribution function as has recently been done.[3–5] Without a detailed investigation of the properties of $\Phi(r,\theta)$ for each density matrix, such interpretations can easily lead to meaningless formulas.[18]

We showed above that Sudarshan's "probability functional" $r\Phi(r,\theta)$ was not a generalized function for a physically important class of density matrices. This implies that an expression of the form

$$\hbar^{-(\lambda+\mu)/2} \int d^2z\, \Phi(r,\theta)(z^*)^\lambda (z)^\mu$$

is, in general, meaningless. The fact that such an expression, which is similar to classical expressions in probability theory, can be made to yield the moments $\xi_{\lambda;\mu}$ is tautological, for one must follow *ad hoc* rules in using $\Phi(r,\theta)$ in contrast to well defined and general mathematical rules for using probability densities. Consequently, Sudarshan's equivalence theorem is mathematically meaningless and without physical content.

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[15] Such an $f(r)$ is called a good function in Lighthill's theory.
[16] M. J. Lighthill, Ref. 11, Sec. 2.6.
[17] M. J. Lighthill, Ref. 11, Sec. 2.3.
[18] The fact that unrestrained manipulations with infinite series of the form

$$\sum_{n=0}^{\infty} \alpha_n \delta^{(n)}(x),$$

where $\alpha_1, \alpha_2, \cdots \alpha_n, \cdots$ are a sequence of complex numbers, may lead to meaningless results cannot be emphasized too strongly.