Radiative Transfer in Dispersive Media*

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A radiative transfer equation is derived which is sufficiently general to apply to energy transfer by various types of wave motion, such as electromagnetic waves, sound waves, hydromagnetic waves, etc. Scattering of waves is neglected. The equation is compared to the equation found in the literature of radio astronomy, and the meaning of a term in this equation is clarified. The general solution of the equation is given. A more rigorous derivation of the transfer equation based on a method due to Wigner is given.

I. INTRODUCTION

IN problems of radiative transfer at optical frequencies, the index of refraction is usually very close to unity and the derivation of the equation of transfer is straightforward.[1] However, at radio frequencies the index of refraction may differ appreciably from unity and vary with position. In this case the derivation of the transfer equation involves subtleties which we feel have not been sufficiently appreciated.

In this paper we shall give a derivation of an equation of radiative transfer which is sufficiently general to apply to energy transfer by various types of wave motion, such as electromagnetic waves, sound waves, hydromagnetic waves, etc. There is a term in the equation which we derive which is apparently absent from the equation which occurs in the literature of radio astronomy.[2,3] This seems to be due to the fact that the meaning of a derivative which occurs in the conventional derivation is ambiguous. In any case this apparent omission is of no practical consequence since the solution given to the equation in the literature is the correct one.

In Sec. II we give a semi-intuitive derivation of the equation of transfer and compare the result with the equation found elsewhere.[2,3] In Sec. III the general solution of this equation is given. In Sec. IV a derivation is given for a spatially uniform system using a method due to Wigner[4] which has been successful in similar quantum-mechanical problems. In Sec. V we extend this derivation to spatially dependent media.

II. DERIVATION OF THE EQUATION OF TRANSFER

As our starting point we shall assume that the equations describing the propagation of waves of the form

$$\mathbf{A}(\mathbf{x},t)=\mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{1}$$

in some medium have been solved and a dispersion relation of the form

$$D(\omega,\mathbf{k})=D(\omega,k_x,k_y,k_z)=0 \tag{2}$$

has been found. If $\mathbf{k}$ is assumed to be a real vector, then Eq. (2) can be solved for ω to obtain

$$\omega_\alpha(\mathbf{k})=\omega_{\alpha r}(\mathbf{k})+i\omega_{\alpha i}(\mathbf{k})\,. \tag{3}$$

Generally there will be more than one solution and we distinguish between them by the subscript α. For instance, two solutions may represent transverse electromagnetic waves of different polarizations and a third solution may represent longitudinal plasma oscillations. The subscripts r and i denote the real and imaginary parts of ω_α.

We can also solve Eq. (2) for one component of $\mathbf{k}$, say k_x, as a function of real ω, k_y, and k_z to obtain

$$k_{x\alpha}(\omega,k_y,k_z)=k_{x\alpha r}+ik_{x\alpha i}\,. \tag{4}$$

If

$$|k_{x\alpha i}|\ll|k_{x\alpha r}|$$

and

$$|\omega_{\alpha i}|\ll|\omega_{\alpha r}|\,, \tag{5}$$

then a relation between $k_{x\alpha i}$ and $\omega_{\alpha i}$ can be found by substituting Eq. (4) into the argument of Eq. (3), keeping only the first term of a Taylor expansion and requiring the result to be real. We find

$$0=\omega_{\alpha i}(k_{xr},k_y,k_z)+(\partial\omega_{\alpha r}/\partial k_{xr})k_{x\alpha i}$$

or

$$\omega_{\alpha i}=-v_{gx\alpha}k_{x\alpha i}\,, \tag{6}$$

where $v_{gx\alpha}$ is the x component of the group velocity. Equations (4) and (5) will generally be valid under conditions for which transfer equations are valid.

Now in addition to the dependence on $\mathbf{k}$ which has been explicitly noted, ω will also depend on parameters of the medium such as density, temperature, magnetic field, etc. If these parameters depend on position and time, then we may write

$$\omega_\alpha=\omega_\alpha(\mathbf{k},\mathbf{x},t)\,. \tag{7}$$

If ω_α varies sufficiently slowly with respect to $\mathbf{x}$ and t (that is, if the relative change in a wavelength and in a period of the wave is small), then it is still sensible to describe disturbances in the medium in terms of wave

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[1] S. Chandrasekhar, *Radiative Transfer* (Dover Publications, New York, 1960).

[2] R. v. d. R. Wooley, Suppl. Aust. J. Sci. **10**, 1 (1947).

[3] R. v. d. R. Wooley and D. W. N. Stibbs, *The Outer Layer of a Star* (Clarendon Press, Oxford, England, 1953), p. 240.

[4] E. Wigner, Phys. Rev. **40**, 749 (1932).

packets. As is well known[5,6] the motion of such a wave packet is the same as that of a particle of momentum $\mathbf{k}$ whose Hamiltonian is $\omega_{\alpha r}(\mathbf{k},\mathbf{x},t)$. That is, the equations of motion of the packet are

$$\dot{\mathbf{x}}=\mathbf{v}_g=\partial\omega_{\alpha r}/\partial\mathbf{k} \tag{8}$$

$$\dot{\mathbf{k}}=-\partial\omega_{\alpha r}/\partial\mathbf{x}. \tag{9}$$

We now define $f_\alpha(\mathbf{x},\mathbf{k},t)d^3xd^3k$ as the number of wave packets of species α with $\mathbf{x}$ in d^3x and $\mathbf{k}$ in d^3k. Clearly this is a meaningful definition only for sufficiently slowly varying radiation fields. Having now sufficiently emphasized that there may be more than one species of wave in the medium, we will drop the subscript α in all that follows and refer to the wave packets as "photons." The distribution function f must obey a conservation equation in phase space.

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial\mathbf{x}}\cdot(\dot{\mathbf{x}}f)+\frac{\partial}{\partial\mathbf{k}}\cdot(\dot{\mathbf{k}}f)=\mathcal{E}-A\,, \tag{10}$$

where $\mathcal{E}(\mathbf{x},\mathbf{k},t)d^3xd^3k$ is the number of photons emitted per unit time with $\mathbf{x}$ in d^3x and $\mathbf{k}$ in d^3k, and $A(\mathbf{x},\mathbf{k},t)\times d^3xd^3k$ is the number of photons absorbed per unit time with $\mathbf{x}$ in d^3x and $\mathbf{k}$ in d^3k. We will assume that A and f are linearly related by an absorption coefficient:

$$A(\mathbf{x},\mathbf{k},t)=a(\mathbf{x},\mathbf{k},t)f(\mathbf{x},\mathbf{k},t)\,. \tag{11}$$

The quantities $\dot{\mathbf{x}}$ and $\dot{\mathbf{k}}$ are given by Eqs. (8) and (9). Equation (10) becomes

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial\omega_r}{\partial\mathbf{k}}\cdot\frac{\partial f}{\partial\mathbf{x}}-\frac{\partial\omega_r}{\partial\mathbf{x}}\cdot\frac{\partial f}{\partial\mathbf{k}}=\mathcal{E}-af. \tag{12}$$

The absorption coefficient may be found from the following considerations: Let f and ω be independent of $\mathbf{x}$ and let $\mathcal{E}=0$. Then

$$\partial f/\partial t=-af, \tag{13}$$

$$f\sim e^{-at}. \tag{14}$$

On the other hand, from Eq. (1)

$$|\mathbf{A}|^2\sim e^{+2\omega_i t}, \tag{15}$$

and since the number of photons present in a wave is proportional to the square of the amplitude we find by comparing Eqs. (14) and (15) that

$$a=-2\omega_i. \tag{16}$$

For an infinite uniform plasma in thermal equilibrium f and ω are independent of $\mathbf{x}$ and t; hence

$$\mathcal{E}_0(\mathbf{k})=a(\mathbf{k})f_0(\mathbf{k})\,, \tag{17}$$

where $f_0(\mathbf{k})$ is the blackbody photon distribution. It is given by

$$f_0(\mathbf{k})=\frac{1}{(2\pi)^3}\frac{2}{e^{\hbar\omega(\mathbf{k})/T}-1}\,, \tag{18}$$

where T is the temperature in energy units. The factor 2 in Eq. (18) is present because of the two polarizations of photons. It may need to be changed for other types of waves. Equation (17) is an expression of Kirchoff's law. For $\hbar\omega\ll T$, Eq. (18) becomes

$$f_0(\mathbf{k})\simeq\frac{1}{(2\pi)^3}\frac{2T}{\hbar\omega}. \tag{19}$$

This is the limiting form of interest in radio astronomy.

In order to compare Eq. (12) with the transfer equation found in the literature, we must write it in terms of the specific intensity I. In doing this we will assume that the medium is isotropic, so that ω depends on the magnitude of $\mathbf{k}$ but not on its direction. Then

$$\mathbf{v}_g=\frac{\partial\omega}{\partial\mathbf{k}}=\frac{\partial\omega}{\partial k}\frac{\partial k}{\partial\mathbf{k}}=v_g\mathbf{n}\,, \tag{20}$$

where

$$\mathbf{n}=\mathbf{k}/k \tag{21}$$

is a unit vector in the $\mathbf{k}$ direction. We shall also assume that all quantities are time-independent. It follows that

$$\dot{\omega}=\frac{\partial\omega}{\partial t}+\frac{\partial\omega}{\partial\mathbf{x}}\cdot\dot{\mathbf{x}}+\frac{\partial\omega}{\partial\mathbf{k}}\cdot\dot{\mathbf{k}}=\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\omega}{\partial\mathbf{k}}-\frac{\partial\omega}{\partial\mathbf{k}}\cdot\frac{\partial\omega}{\partial\mathbf{x}}=0\,, \tag{22}$$

so ω is a constant of the motion. We write

$$d^3k=k^2dkd\Omega=k^2(\partial k/\partial\omega)d\omega d\Omega=(k^2/v_g)d\omega d\Omega\,. \tag{23}$$

Then

$$\frac{k^2(\omega,\mathbf{x})}{v_g(\omega,\mathbf{x})}f(\mathbf{x},\omega,\mathbf{n},t)d^3xd\omega d\Omega$$

equals the number of photons in d^3x with ω in $d\omega$ and $\mathbf{n}$ in the solid angle $d\Omega$.

We now calculate the energy dE which crosses an element of area $d\sigma$ whose normal makes an angle θ with $\mathbf{n}$ in an interval dt with ω in $d\omega$ and $\mathbf{n}$ in $d\Omega$. This is clearly

$$dE=(k^2/v_g)\hbar\omega v_g f(\mathbf{x},\omega,\mathbf{n},t)d\omega d\Omega dtd\sigma\cos\theta\,, \tag{24}$$

since each photon carries an energy $\hbar\omega$ and moves with speed v_g in the direction of $\mathbf{n}$. One customarily writes

$$dE=I(\mathbf{x},\omega,\mathbf{n},t)d\omega d\Omega dtd\sigma\cos\theta\,, \tag{25}$$

so by comparison of Eqs. (24) and (25) one finds

$$I=\hbar\omega k^2 f. \tag{26}$$

Defining the index of refraction by

$$\mu=kc/\omega\,, \tag{27}$$

[5] T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), Secs. 3-3 and 3-6.

[6] S. Weinberg, Phys. Rev. **126**, 1899 (1962).

and using it to replace k, one finds

$$f=(c^2/\hbar\omega^3)(I/\mu^2). \tag{28}$$

This is used in

$$df/dt=\mathcal{E}-af, \tag{29}$$

which becomes

$$\frac{\partial}{\partial t}\left(\frac{I}{\mu^2}\right)+v_g\mathbf{n}\cdot\frac{\partial}{\partial \mathbf{x}}\left(\frac{I}{\mu^2}\right)+\dot{\mathbf{n}}\cdot\frac{\partial}{\partial \mathbf{n}}\left(\frac{I}{\mu^2}\right)=\frac{\hbar\omega^3}{c^2}\mathcal{E}-a\left(\frac{I}{\mu^2}\right). \tag{30}$$

We wish to compare Eq. (30) with the equation

$$(d/ds)(I/\mu^2)=K_\nu B(\nu,T)-K_\nu(I/\mu^2) \tag{31}$$

given by Wooley and Stibbs.[3] In this equation $\nu=\omega/2\pi$,

$$B(\nu,T)=\frac{2h\nu^3/c^2}{e^{h\nu/T}-1}=\frac{1}{(2\pi)^3}\frac{2\hbar\omega^3/c^2}{e^{\hbar\omega/T}-1}, \tag{32}$$

and K_ν is the absorption coefficient defined so that the intensity of a beam through the medium falls off as

$$e^{-K_\nu x}.$$

In terms of quantities in this paper it is

$$K_\nu=+2k_i=-2\omega_i/v_g=a/v_g, \tag{33}$$

where we have used Eqs. (6) and (16). Equation (31) may be written

$$v_g(d/ds)(I/\mu^2)=aB(\nu,T)-a(I/\mu^2). \tag{34}$$

If Eqs. (17) and (18) are used in Eq. (30), then Eqs. (30) and (34) will agree only if

$$\frac{d}{ds}=\mathbf{n}\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{1}{v_g}\dot{\mathbf{n}}\cdot\frac{\partial}{\partial \mathbf{n}}. \tag{35}$$

This interpretation of d/ds does not seem to have been explicitly stated in derivations of the transfer equation with which the writer is familiar. However, the solutions of the equation found in the literature are equivalent to the solution obtained in the following section. The solution is probably more obvious intuitively than the equation is.

III. SOLUTION OF THE EQUATION OF TRANSFER

A solution of Eq. (12) may be found in the following way. First, one solves the equations

$$\dot{\mathbf{x}}=\partial\omega_r/\partial\mathbf{k}, \tag{36}$$

$$\dot{\mathbf{k}}=-\partial\omega_r/\partial\mathbf{x}, \tag{37}$$

to obtain

$$\mathbf{x}=\mathbf{x}(\mathbf{x}_0,\mathbf{k}_0,t), \tag{38}$$

$$\mathbf{k}=\mathbf{k}(\mathbf{x}_0,\mathbf{k}_0,t), \tag{39}$$

where $\mathbf{x}_0$ and $\mathbf{k}_0$ are the values of $\mathbf{x}$ and $\mathbf{k}$ at the time $t=0$. Next Eqs. (38) and (39) are inverted to obtain

$$\mathbf{x}_0=\mathbf{x}_0(\mathbf{x},\mathbf{k},t), \tag{40}$$

$$\mathbf{k}_0=\mathbf{k}_0(\mathbf{x},\mathbf{k},t). \tag{41}$$

Note that

$$\frac{d\mathbf{x}_0}{dt}=\frac{\partial\mathbf{x}_0}{\partial t}+\dot{\mathbf{x}}\cdot\frac{\partial\mathbf{x}_0}{\partial\mathbf{x}}+\dot{\mathbf{k}}\cdot\frac{\partial\mathbf{x}_0}{\partial\mathbf{k}}=0 \tag{42}$$

and

$$\mathbf{x}_0(\mathbf{x},\mathbf{k},0)=\mathbf{x} \tag{43}$$

and that similar equations hold for $\mathbf{k}_0$. The solution of Eq. (12) is

$$f(\mathbf{x},\mathbf{k},t)=\int_0^t dt'\,\mathcal{E}(\mathbf{x}_0,(\mathbf{x},\mathbf{k},t-t'),\mathbf{k}_0(\mathbf{x},\mathbf{k},t-t'),t')\exp\left[-\int_{t'}^t dt''a(\mathbf{x}_0(\mathbf{x},\mathbf{k},t-t''),\mathbf{k}_0(\mathbf{x},\mathbf{k},t-t''),t'')\right]$$
$$+g(\mathbf{x}_0(\mathbf{x},\mathbf{k},t),\mathbf{k}_0(\mathbf{x},\mathbf{k},t))\exp\left[-\int_0^t dt'a(\mathbf{x}_0(\mathbf{x},\mathbf{k},t-t'),\mathbf{k}_0(\mathbf{x},\mathbf{k},t-t'),t')\right]. \tag{44}$$

That Eq. (44) is indeed a solution is readily verified by substituting it into Eq. (12). At $t=0$, Eq. (44) gives

$$f(\mathbf{x},\mathbf{k},0)=g(\mathbf{x},\mathbf{k}). \tag{45}$$

Generally we are interested only in the inhomogeneous part of Eq. (44) and we discard the term depending on $g(\mathbf{x}_0,\mathbf{k}_0)$ and change the lower limit of the first integral from zero to $-\infty$. If $\mathcal{E}$, a, and f are independent of t, then the solution may be written as

$$f(\mathbf{x},\mathbf{k})=\int_0^\infty dt\;\mathcal{E}(\mathbf{x}_0(\mathbf{x},\mathbf{k},t),\mathbf{k}_0(\mathbf{x},\mathbf{k},t))$$
$$\times\exp\left[-\int_0^t dt'a(\mathbf{x}_0(\mathbf{x},\mathbf{k},t'),\mathbf{k}_0(\mathbf{x},\mathbf{k},t'))\right]. \tag{46}$$

The interpretation of Eq. (46) is fairly obvious. The number of photons at the point $\mathbf{x}$, $\mathbf{k}$ at some instant is the sum of those emitted from points $\mathbf{x}_0$, $\mathbf{k}_0$ at times earlier by the time t, each contribution to the sum being weighted by the factor

$$\exp\left[-\int_0^t dt'a(t')\right] \tag{47}$$

because of the absorption undergone.

IV. A RIGOROUS DERIVATION OF THE EQUATION OF TRANSFER FOR A SPATIALLY UNIFORM MEDIUM

The definition of $f(\mathbf{x},\mathbf{k},t)$ which we have given clearly must represent some sort of an approximation since one cannot specify both $\mathbf{x}$ and $\mathbf{k}$ accurately. In fact

$$\Delta x_i \Delta k_i \sim 1.$$

We will try to make the nature of this approximation clear.

For concreteness we assume that the medium is a plasma in which a disturbance can be described by the linearized equations

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla\times\mathbf{B} - \sum 4\pi e n_0 \mathbf{v}, \tag{48}$$

$$\partial\mathbf{B}/\partial t = -c\nabla\times\mathbf{E} \tag{49}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\tau}\mathbf{v} + \frac{e}{m}\mathbf{E} + \frac{e}{mc}\mathbf{v}\times\mathbf{B}_0 - \frac{1}{n_0 m}\nabla p, \tag{50}$$

$$\partial p/\partial t = -\gamma p_0 \nabla\cdot\mathbf{v}. \tag{51}$$

In the above $\mathbf{E}$, $\mathbf{B}$, $\mathbf{v}$, and p are small perturbations in the medium. The quantities n_0, $\mathbf{B}_0$, and p_0 are unperturbed quantities. For the time being we assume that they are constants. There will be equations like Eqs. (50) and (51) for each species of particle in the plasma with appropriate values of e, m, n_0, γ, τ, and p_0. A summation sign without indices denotes summation over species.

We have assumed that the plasma is a medium described by Eqs. (48), (49), (50), and (51) in order to show explicitly how wave packets are to be interpreted. These equations are probably sufficient for problems of interest in radio astronomy, but we believe that the radiative transfer equation which we derive is applicable to a wider class of problems.

If Eq. (48) is multiplied by $\mathbf{E}/4\pi$, Eq. (49) multiplied by $\mathbf{B}/4\pi$, Eq. (50) multiplied by $n_0 m v$, and Eq. (51) multiplied by $p/\gamma p_0$ and the equations added, then one obtains

$$\frac{\partial}{\partial t}\left[\frac{E^2+B^2}{8\pi} + \sum \tfrac{1}{2}n_0 m v^2 + \sum \frac{p^2}{2\gamma p_0}\right] = -\sum \frac{n_0 m}{\tau}v^2 - \nabla\cdot\left[\frac{c}{4\pi}(\mathbf{E}\times\mathbf{B}) + \sum p\mathbf{v}\right]. \tag{52}$$

This is the equation of conservation of energy for the system. The energy density is

$$U = \frac{1}{8\pi}(E^2+B^2) + \sum \tfrac{1}{2}n_0 m v^2 + \sum \frac{p^2}{2\gamma p_0}.$$

Now we define the column vector

$$\psi(\mathbf{x},t) = \begin{pmatrix} \mathbf{E}/(8\pi)^{1/2} \\ \mathbf{B}/(8\pi)^{1/2} \\ (n_0 m/2)^{1/2}\mathbf{v} \\ p/(2\gamma p_0)^{1/2} \end{pmatrix} \tag{53}$$

and its adjoint

$$\psi^\dagger(\mathbf{x},t) = \left[\frac{\mathbf{E}^*}{(8\pi)^{1/2}}, \frac{\mathbf{B}^*}{(8\pi)^{1/2}}, \left(\frac{n_0 m}{2}\right)^{1/2}\mathbf{v}^*, \frac{p^*}{(2\gamma p_0)^{1/2}}\right], \tag{54}$$

where the asterisk indicates a complex conjugate. The scalar product

$$\psi^\dagger\psi = \frac{1}{8\pi}(|\mathbf{E}|^2+|\mathbf{B}|^2) + \sum \frac{n_0 m}{2}|\mathbf{v}|^2 + \sum \frac{1}{2\gamma p_0}|p|^2 \tag{55}$$

is just the energy density when ψ is complex. It obeys the equation

$$\frac{\partial}{\partial t}\psi^\dagger\psi = -\sum \frac{n_0 m}{\tau}|\mathbf{v}|^2 - \nabla\cdot \mathrm{Re}\left[\frac{c}{4\pi}(\mathbf{E}^*\times\mathbf{B}) + \sum p^*\mathbf{v}\right]. \tag{56}$$

The vector ψ satisfies the equation

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t} = -iH\left(\frac{1}{i}\frac{\partial}{\partial\mathbf{x}}\right)\psi(\mathbf{x},t), \tag{57}$$

where the operator H is a matrix operator which can be found by inspection from Eqs. (48), (49), (50), and (51). Similarly $\psi^\dagger$ satisfies

$$\frac{\partial\psi^\dagger}{\partial t} = +i\psi^\dagger H^\dagger\left(\frac{1}{i}\frac{\overleftarrow{\partial}}{\partial\mathbf{x}}\right). \tag{58}$$

(We indicate by an arrow over a differential operator the direction in which it operates when this adds clarity.)

We now look for a solution of Eq. (56) with spatial dependence given by a factor

$$e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Equation (56) becomes

$$\partial\psi/\partial t = -iH(\mathbf{k})\psi. \tag{59}$$

We assume a solution of the form

$$\psi = \phi(\mathbf{k})e^{-i\omega t}. \tag{60}$$

Then

$$[H(\mathbf{k}) - \omega 1]\phi(\mathbf{k}) = 0, \tag{61}$$

where 1 is the unit matrix. For a given $\mathbf{k}$, Eq. (61) will have a number of solutions representing electromagnetic waves, sound waves, plasma oscillations, etc. We will distinguish them by a subscript α; thus $\omega_\alpha(\mathbf{k})$ is an eigenvalue and $\phi_\alpha(\mathbf{k})$ is the corresponding eigenvector. We write

$$\psi_\alpha(\mathbf{k},t) = \phi_\alpha(\mathbf{k})e^{-i\omega_\alpha(\mathbf{k})t} \tag{62}$$

and

$$\psi_\alpha^\dagger(\mathbf{k},t) = \phi_\alpha^\dagger(\mathbf{k})e^{+i\omega_\alpha^*(\mathbf{k})t}. \tag{63}$$

[Note that we do not assume that $\omega_\alpha(\mathbf{k})$ is real.] From $\psi_\alpha(\mathbf{k},t)$ we can construct wave packet solutions. Thus

$$\psi_\alpha(\mathbf{x},t)=\int \frac{d^3k}{(2\pi)^3}e^{+i\mathbf{k}\cdot\mathbf{x}}\psi_\alpha(\mathbf{k},t) \tag{64}$$

is a packet of waves of type α.

It is convenient at this point to derive an expression for the group velocity. Let

$$\langle \mathbf{x}(t)\rangle=\frac{1}{N(t)}\int d^3x\, \psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\mathbf{x}\,, \tag{65}$$

where

$$N(t)=\int d^3x\, \psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\,. \tag{66}$$

Clearly $\langle \mathbf{x}\rangle$ is the centroid of the energy carried by the wave packet of type α. (For simplicity of notation we have suppressed the subscript α.) We now use Eq. (64) in Eqs. (65) and (66) together with

$$\int d^3x\, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}=(2\pi)^3\delta(\mathbf{k}'-\mathbf{k}) \tag{67}$$

and

$$\int d^3x\, \mathbf{x}e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}=\frac{(2\pi)^3}{i}\frac{\partial}{\partial\mathbf{k}'}\delta(\mathbf{k}'-\mathbf{k}) \tag{68}$$

to obtain

$$\langle \mathbf{x}(t)\rangle=\mathbf{x}_0(t)+t\mathbf{v}_g(t)\,, \tag{69}$$

where

$$\mathbf{x}_0(t)=\frac{i/2\int d^3k(\phi^\dagger(\partial\phi/\partial\mathbf{k})-(\partial\phi^\dagger/\partial\mathbf{k})\phi)e^{2\omega_i(\mathbf{k})t}}{\int d^3k\phi^\dagger(\mathbf{k})\phi(\mathbf{k})e^{2\omega_i(\mathbf{k})t}} \tag{70}$$

and

$$\mathbf{v}_g(t)=\frac{\int d^3k[\partial\omega_r(\mathbf{k})/\partial\mathbf{k}]\phi^\dagger(\mathbf{k})\phi(\mathbf{k})e^{2\omega_i(\mathbf{k})t}}{\int d^3k\phi^\dagger(k)\phi(\mathbf{k})e^{2\omega_i(\mathbf{k})t}} \tag{71}$$

$$=\frac{\int d^3k(\partial\omega_r/\partial\mathbf{k})\psi^\dagger(\mathbf{k},t)\psi(\mathbf{k},t)}{\int d^3k\psi^\dagger(\mathbf{k},t)\psi(\mathbf{k},t)}\,. \tag{72}$$

If ω is real, then $\mathbf{x}_0$ and $\mathbf{v}_g$ are constants and the centroid moves with constant velocity. If ω is complex, then $\mathbf{x}_0$ and $\mathbf{v}_g$ will change with time because the spectral composition of the packet is changing. However, if $\phi(\mathbf{k})$ is very sharply peaked about some value $\mathbf{k}_0$, then the factor

$$e^{2\omega_i(\mathbf{k})t}$$

can be replaced by

$$e^{2\omega_i(\mathbf{k}_0)t}$$

and removed from the integrand of Eqs. (70) and (71). It then cancels from numerator and denominator and $\mathbf{x}_0$ and $\mathbf{v}_g$ are again constants.

We now look for a function $f(\mathbf{x},\mathbf{k},t)$ which in the lowest order of some approximation satisfies an equation like Eq. (12). We will follow a method due to Wigner[4] which has been successful in a similar quantum mechanical problem. Define

$$f(\mathbf{x},\mathbf{k},t)=\int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\psi^\dagger(\mathbf{k}+\tfrac{1}{2}\mathbf{q},\, t)\psi(\mathbf{k}-\tfrac{1}{2}\mathbf{q},\, t)\,. \tag{73}$$

Note that

$$\int f(\mathbf{x},\mathbf{k},t)d^3x=\psi^\dagger(\mathbf{k},t)\psi(\mathbf{k},t)\,, \tag{74}$$

which is the energy density in $\mathbf{k}$ space. Also

$$\int f(\mathbf{x},\mathbf{k},t)d^3k=\int d^3k\int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\psi^\dagger(\mathbf{k}+\tfrac{1}{2}\mathbf{q},\, t)$$
$$\times\psi(\mathbf{k}-\tfrac{1}{2}\mathbf{q},\, t)=\frac{1}{(2\pi)^3}\int d^3u\int d^3v e^{i(\mathbf{u}-\mathbf{v})\cdot\mathbf{x}}$$
$$\times\psi^\dagger(\mathbf{u},t)\psi(\mathbf{v},t)=\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\,, \tag{75}$$

which is the energy density in configuration space. These are the properties one expects from $f(\mathbf{x},\mathbf{k},t)$. This definition of $f(\mathbf{x},\mathbf{k},t)$ differs from that of the last section where f was defined as the photon density in phase space rather than the energy density. For time-independent processes, one can be obtained from the other by multiplying by $\hbar\omega$, since ω is a constant of the motion. We note that

$$(\partial/\partial t)\psi^\dagger(\mathbf{k}',t)\psi(\mathbf{k}'',t)$$
$$=+i[\omega^*(\mathbf{k}')-\omega(k'')]\psi^\dagger(\mathbf{k}',t)\psi(\mathbf{k}'',t)\,. \tag{76}$$

We take the time derivative of Eq. (73) and use Eq. (76) with $\mathbf{k}'=\mathbf{k}+\frac{1}{2}\mathbf{q}$ and $k''=\mathbf{k}-\frac{1}{2}\mathbf{q}$. Note that

$$[\omega^*(\mathbf{k}')-\omega(\mathbf{k}'')]=[\omega_r(\mathbf{k}')-\omega_r(\mathbf{k}'')]$$
$$-i[\omega_i(\mathbf{k}')+\omega_i(\mathbf{k}'')]\,. \tag{77}$$

Next we expand Eq. (77) about $\mathbf{k}$ and obtain

$$[\omega^*(\mathbf{k}+\tfrac{1}{2}\mathbf{q})-\omega(\mathbf{k}-\tfrac{1}{2}\mathbf{q})]=2\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{n!}\omega_r(\mathbf{k})\left(\frac{\overleftarrow{\partial}}{\partial\mathbf{k}}\cdot\frac{\mathbf{q}}{2}\right)^n$$
$$-2i\sum_{\substack{n=0\\ \text{even}}}^{\infty}\frac{1}{n!}\omega_i(\mathbf{k})\left(\frac{\overleftarrow{\partial}}{\partial\mathbf{k}}\cdot\frac{\mathbf{q}}{2}\right)^n\,. \tag{78}$$

Now when it occurs in the integrand of Eq. (73), $\mathbf{q}$ is equivalent to the differential operator

$$\frac{1}{i}\frac{\partial}{\partial\mathbf{x}}\,.$$

Using Eqs. (76) and (78) in Eq. (73) we obtain

$$\frac{\partial f(\mathbf{x},\mathbf{k},t)}{\partial t}=-\omega_r(\mathbf{k})2\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\mathbf{k}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{x}}\right)f(\mathbf{x},k,t)$$

$$+\omega_i(\mathbf{k})2\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\mathbf{k}}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{x}}\right)f(\mathbf{x},\mathbf{k},t).\quad(79)$$

The sine and cosine functions arise from summing the series in Eq. (73).

If one assumes that f and ω are very slowly varying functions of their arguments so that all derivatives higher than the first can be neglected, one obtains

$$\frac{\partial f}{\partial t}+\frac{\partial\omega_r}{\partial\mathbf{k}}\cdot\frac{\partial f}{\partial\mathbf{x}}=2\omega_i f.\quad(80)$$

This is just the equation we should expect. The third term on the left of Eq. (12) does not occur because of our assumption that ω_r was independent of $\mathbf{x}$. No emission term occurs on the right-hand side because spontaneous emission cannot be obtained from the equations from which we started.

V. EXTENSION TO SPATIALLY DEPENDENT MEDIA

We now assume that $\mathbf{B}$, n_0, τ, and p_0 in Eqs. (48), (49), (50), and (51) are functions of position. Then Eq. (57) may be written

$$\frac{\partial\psi(\mathbf{x},t)}{\partial t}=-iH\left(\mathbf{x},\frac{1}{i}\frac{\overrightarrow{\partial}}{\partial\mathbf{x}}\right)\psi(\mathbf{x},t),\quad(81)$$

with a similar equation corresponding to Eq. (58). The equation of motion for $\psi(\mathbf{k},t)$ may be found by multiplying Eq. (76) by $e^{-i\mathbf{k}\cdot\mathbf{x}}$ and integrating over all space to obtain

$$\int e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\partial\psi(\mathbf{x},t)}{\partial t}d^3x=\frac{\partial}{\partial t}\psi(\mathbf{k},t)$$

$$=-i\int d^3x\,e^{-i\mathbf{k}\cdot\mathbf{x}}H\left(\mathbf{x},\frac{1}{i}\frac{\partial}{\partial\mathbf{x}}\right)\int\frac{d^3k'}{(2\pi)^3}e^{+i\mathbf{k}'\cdot\mathbf{x}}\psi(\mathbf{k},t)$$

$$=-i\int d^3k'H(\mathbf{k},\mathbf{k}')\psi(\mathbf{k}',t),\quad(82)$$

where

$$H(\mathbf{k},\mathbf{k}')=\int\frac{d^3x}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{x}}H\left(\mathbf{x},\frac{1}{i}\frac{\overrightarrow{\partial}}{\partial\mathbf{x}}\right)e^{+i\mathbf{k}'\cdot\mathbf{x}},\quad(83)$$

which can also be written as

$$H(\mathbf{k},\mathbf{k}')=H\left(\frac{1}{i}\frac{\overrightarrow{\partial}}{\partial\mathbf{k}'},\mathbf{k}'\right)\delta(\mathbf{k}-\mathbf{k}')$$

$$=H\left(-\frac{1}{i}\frac{\overrightarrow{\partial}}{\partial\mathbf{k}},\mathbf{k}\right)\delta(\mathbf{k}-\mathbf{k}').\quad(84)$$

The equation of motion then becomes

$$\frac{\partial\psi(\mathbf{k},t)}{\partial t}=-iH\left(-\frac{1}{i}\frac{\overrightarrow{\partial}}{\partial\mathbf{k}},\mathbf{k}\right)\psi(\mathbf{k},t).\quad(85)$$

The adjoint equation is

$$\frac{\partial\psi^\dagger(\mathbf{k},t)}{\partial t}=+i\psi^\dagger(\mathbf{k},t)H^\dagger\left(-\frac{1}{i}\frac{\overleftarrow{\partial}}{\partial\mathbf{k}},\mathbf{k}\right).\quad(86)$$

Next we divide H into Hermitian and anti-Hermitian parts. Thus

$$H=H_H+H_A,\quad(87)$$

where

$$H_H{}^\dagger=H_H\quad(88)$$

and

$$H_A{}^\dagger=-H_A.\quad(89)$$

We again define $f(\mathbf{x},\mathbf{k},t)$ by Eq. (73). In calculating the partial derivative of f with respect to t we need

$$\frac{\partial}{\partial t}\psi^\dagger(\mathbf{k}',t)\psi(\mathbf{k}'',t)=i\psi^\dagger(k',t)\left[H^\dagger\left(-\frac{1}{i}\frac{\overleftarrow{\partial}}{\partial\mathbf{k}'},\mathbf{k}'\right)\right.$$

$$\left.-H\left(-\frac{1}{i}\frac{\overrightarrow{\partial}}{\partial\mathbf{k}''},\mathbf{k}''\right)\right]\psi(\mathbf{k}'',t),\quad(90)$$

where

$$\mathbf{k}'=\mathbf{k}+\tfrac{1}{2}\mathbf{q},\quad(91)$$

$$\mathbf{k}''=\mathbf{k}-\tfrac{1}{2}\mathbf{q},\quad(92)$$

we expand with respect to $\mathbf{k}$ and $\mathbf{x}$, where $\mathbf{x}$ is the operator $-(1/i)\partial/\partial\mathbf{k}$, and keep only the leading terms of the expansion. Thus

$$[H^\dagger(\mathbf{x}',\mathbf{k}')-H(\mathbf{x}'',\mathbf{k}'')]$$

$$=2H_A+\frac{\partial H_H}{\partial\mathbf{k}}\cdot\mathbf{q}+\frac{1}{i}\frac{\partial H_H}{\partial\mathbf{x}}\cdot\left(\frac{\partial}{\partial\mathbf{k}'}+\frac{\partial}{\partial\mathbf{k}''}\right)\quad(93)$$

and

$$\frac{\partial f(\mathbf{x},\mathbf{k},t)}{\partial t}=\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\psi^\dagger(\mathbf{k}+\tfrac{1}{2}\mathbf{q},t)$$

$$\times\left[2iH_A+\frac{\overleftarrow{\partial}}{\partial\mathbf{x}}\cdot\frac{\partial H_H}{\partial\mathbf{k}}+\frac{\overleftarrow{\partial}}{\partial\mathbf{k}}\cdot\frac{\partial H_H}{\partial\mathbf{x}}+\frac{\partial H_H}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial\mathbf{k}}\right]$$

$$\times\psi(k-\tfrac{1}{2}\mathbf{q},t).\quad(94)$$

For the case of no spatial dependence we had

$$H(\mathbf{k})\psi(\mathbf{k})=\omega(\mathbf{k})\psi(\mathbf{k}).\quad(95)$$

We will assume that this is still approximately true so that

$$H(\mathbf{x},\mathbf{k})=1\omega(\mathbf{x},\mathbf{k}); \tag{96}$$

then Eq. (90) gives

$$\frac{\partial f}{\partial t}=2i\omega_A f+\frac{\partial\omega_H}{\partial\mathbf{k}}\cdot\frac{\partial f}{\partial\mathbf{x}}+\frac{\partial\omega_H}{\partial\mathbf{x}}\frac{\partial f}{\partial\mathbf{k}}. \tag{97}$$

If we identify ω_A with $-i\omega_i$ and ω_H with ω_r we have the desired result. The assumption contained in Eq. (96) seems to be difficult to justify in general.

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Statistical Descriptions of Free Boson Fields

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Several serious mathematical deficiencies in Sudarshan's probability-functional approach to the statistical description of light beams are demonstrated. In particular, it is shown that all the correlation functions of the beam do not necessarily determine its density matrix.

I. INTRODUCTION

RECENTLY, Sudarshan[1,2] has developed a probability-functional approach for describing all free boson fields. He concludes that "the description of statistical states of a quantum-mechanical system with an arbitrary (countably infinite) number of degrees of freedom is completely equivalent to the description in terms of classical probability distributions in the same (countably infinite) number of degrees of freedom." This conclusion and the methods introduced by Sudarshan have been used in several discussions of the statistical properties of light beams including that of an optical maser.[3–5]

The purpose of this note is to demonstrate several serious mathematical deficiencies in Sudarshan's probability functional approach. In particular, we will show that all the correlation functions of the beam do not necessarily determine its density matrix.

II. SUDARSHAN'S PROBABILITY FUNCTIONAL

The most general form taken by the density matrix of a free boson field is

$$\rho=\sum_{\{n_\mathbf{k}\},\{n_\mathbf{k}'\}}|\{n_\mathbf{k}\}\rangle\rho(\{n_\mathbf{k}\},\{n_\mathbf{k}'\})\langle\{n_\mathbf{k}'\}|\,, \tag{1}$$

where

$$\rho=\rho^\dagger,\quad \mathrm{Tr}\rho=1\,,$$

and[6]

$$|\{n_\mathbf{k}\}\rangle=\prod_\mathbf{k}|n_\mathbf{k}\rangle.$$

Sudarshan[1,2] has argued that all density matrices of the form given by Eq. (1), i.e., *every* free field boson density matrix, can be put into a special form in a unique way which allows the conclusion that "there is a one-to-one correspondence between density matrices of a quantized (free boson) field and classical probability functions." We shall now review for a single mode the demonstration which precedes this conclusion.

The most general density matrix for an isolated oscillator (field mode) is

$$\rho=\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}|n\rangle\rho(n,n')\langle n'|\,, \tag{2}$$

and the expectation value of the normal ordered product $(b^\dagger)^\lambda(b)^\mu$ for this statistical state is[7]

$$\begin{aligned}\xi_{\lambda;\mu}&=\mathrm{Tr}\{\rho(b^\dagger)^\lambda(b)^\mu\}\\&=\sum_{l=0}^{\infty}\rho(l+\mu,l+\lambda)(1/l!)[(l+\lambda)!(l+\mu)!]^{1/2}.\end{aligned} \tag{3}$$

[1] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).

[2] E. C. G. Sudarshan, *Proceedings of the Symposium on Optical Masers* (Brooklyn Polytechnic Press, New York and John Wiley & Sons, Inc., New York, 1963), pp. 45–50.

[3] L. Mandel, Phys. Letters **7**, 117 (1963).

[4] L. Mandel, Phys. Rev. **134**, A10 (1964).

[5] L. Mandel, Phys. Letters **10**, 166 (1964).

[6] $|n_\mathbf{k}\rangle$ is the occupation number state describing n bosons in the **k**th mode.

[7] b and b^+ are the annihilation and creation operators, respectively, for the bosons of the oscillator: $b|n\rangle=(n)^{\frac{1}{2}}|n-1\rangle$, $b^+|n\rangle=(n+1)^{\frac{1}{2}}|n+1\rangle$.