Structure of the Proton and the Hyperfine Shift in Hydrogen*

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(Received 26 October 1964)

The proton-size correction to the hyperfine structure in the ground state of atomic hydrogen is re-examined. It is shown by means of dispersion relations that this correction can be expressed as an integral over experimentally measurable cross sections for electron-proton scattering. This clarifies the physical nature of the correction, puts it on a rigorous basis and lends support to previous analyses. In the absence of experimental data, we give some theoretical estimates for the correction. They agree with previous estimates, and therefore we cannot explain the present experimental value for the hyperfine splitting. We discuss some possible implications of this disagreement and suggest some experiments which would clarify the situation.

I. INTRODUCTION

RECENTLY, the hyperfine splitting in the lowest lying s state of atomic hydrogen has been measured[1] with the remarkable precision of one part in 10^{11}. The purpose of this paper is to review the discrepancy between the calculated and measured values of the hyperfine structure (hfs), to discuss the implications of this disagreement concerning the structure of the proton, and the propose certain experimental measurements which would reduce the theoretical uncertainty in the calculated value of the hfs.

We start with the expression[2] for the energy difference between the lowest lying s-wave singlet and triplet states:

$$\delta E=\frac{32\pi\alpha^2}{3}R_\infty\frac{\mu_p\mu_e}{\mu_0{}^2}\left(1+\frac{m}{M}\right)^{-3}(1+\tfrac{3}{2}\alpha^2)\mathcal{E}\mathcal{R}\mathcal{P}. \quad (1.1)$$

Here α is the fine-structure constant, R_∞ the Rydberg for an electron in a fixed Coulomb field, μ_e/μ_0 the magnetic moment of the electron in Bohr magnetons and μ_p is the magnetic moment of the proton. The factor $(1+m/M)^{-3}$, involving the proton and electron masses is the reduced mass correction which may be obtained from the nonrelativistic Schrödinger equation.[3] The term $(1+\frac{3}{2}\alpha^2)$ is a relativistic correction which follows from the use of the single-particle Dirac equation for an electron in a fixed Coulomb field.[4] The electron shape factor $\mathcal{E}$ represents corrections to electron structure[5] arising from virtual photons surrounding the electron [graphs such as Fig. 1(A)], and the proton recoil terms $\mathcal{R}$ are additional relativistic corrections coming from the use of the field-theoretic bound-state equation.[6,7] The proton shape corrections $\mathcal{P}$ represents meson corrections to the point structure of the proton [graphs such as Fig. 1(B)]. As we shall see, the main difficulty in obtaining a theoretical value for the hfs is that the proton-shape correction cannot be expressed in terms of a simple, directly measurable, property of the proton (such as the static anomalous moment) but involves an unknown function of two variables which must be measured. Our reasons for identifying the various terms as separate entities in this conventional way is to gain some insight. Of course, this entire expression for the hfs is just a consistant expansion of the bound-state equation in powers of α and m/M up to including order $\alpha m/\pi(\mu+1)M$ where the lowest order hfs,

$$(\text{hfs})_0\equiv\Delta\equiv(32\pi\alpha^2m/3M)R_\infty(\mu+1)$$
$$=(8\pi\alpha/3mM)(\mu+1)|\varphi(0)|^2, \quad (1.2)$$

is taken as of order 1. ($\mu=1.79$.) It is essential to realize that although the different terms in the expression for the hfs, Eq. (1.1) do have physical meaning, the only

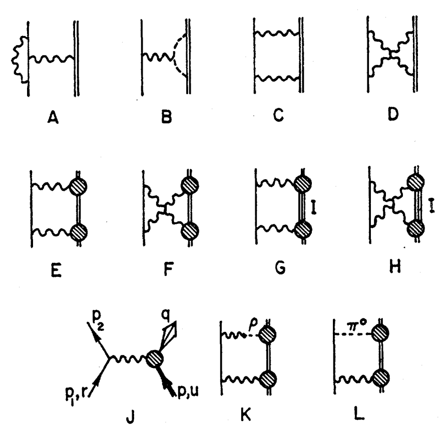

FIG. 1. Feynman graphs affecting the hfs. The electron is represented by a single solid line, the proton by a double solid line and photons by a wavy line.

* This study was supported in part by the U. S. Air Force through Air Force Office of Scientific Research Contract AF 49(638)-1389.

[1] S. B. Crampton, D. Kleppner, and N. F. Ramsey, Phys. Rev. Letters **11**, 338 (1963).

[2] R. Cohen, K. M. Crowe, and J. W. M. Dumond, *The Fundamental Constants of Physics* (Interscience Publishers, Inc., New York, 1957), p. 204.

[3] G. Breit and E. R. Meyerott, Phys. Rev. **72**, 1023 (1947).

[4] G. Breit, Phys. Rev. **35**, 1447 (1930).

[5] R. Karplus and A. Klein, Phys. Rev. **85**, 972 (1952); N. M. Kroll and F. Pollock, *ibid.* **86**, 876 (1952); A. Layzer, Nuovo Cimento **33**, 1538 (1964); D. E. Zwanziger, Bull. Am. Phys. Soc. **6**, 514 (1961); Brodsky, thesis, University of Minnesota (to be published).

[6] R. Arnowitt, Phys. Rev. **92**, 1002 (1953).

[7] W. A. Newcomb and E. E. Salpeter, Phys. Rev. **97**, 1146 (1955).

method we have to be sure that we did not double-count an effect or omit it is to make just this expansion of the bound-state equation. Of course this has already been done[6,7] and the corrections due to proton structure can be derived[8] simply by modifying the previous analysis. In particular, we always have in mind a system of bookkeeping based on the bound-state equation. Our later assertions that certain terms are negligible mean simply that a careful approximation to the corresponding terms in the expansion of the bound-state equation gives an effect which is too small to count.[9]

The difficulty with the proton-shape correction appears as a logarithmic infinity in the calculations of Arnowitt[6] and Newcomb and Salpeter.[7] Although this divergence appears in a rather complicated perturbation expansion of the bound-state equation, we can give intuitive reasons for its existence as follows: These authors took the coupling of the photon to protons as[10]

$$\Lambda_\mu(\kappa)=e[\gamma_\mu-\tfrac{1}{4}\mu(\gamma_\mu\boldsymbol{\kappa}-\boldsymbol{\kappa}\gamma_\mu)]. \tag{1.3}$$

The corrections to the hfs come from Fig. 1 graphs (C) and (D) where two photons are exchanged. (This point will be elaborated shortly.) Now it is known that the electrodynamics of fermions with anomalous moments is not renormalizable and therefore the appearance of the logarithmic infinity in the two photon graphs is not surprising. The above authors introduced an arbitrary cutoff in momentum space to obtain a finite answer. Although for a cutoff calculation the experimental and theoretical values for the hfs can be made to agree, this arbitrary cutoff is clearly not satisfactory.

An attempt has been made[11,8] to remove this arbitrariness by taking the interactions of the proton with photons as $F(\kappa^2)\Lambda_\mu$ where $F(\kappa^2)$ is a form factor which has been chosen to fit the electron-proton (elastic) scattering experiments.[12] This calculation may be understood very simply[13]: In atomic hydrogen the electron and proton have very small average velocities (order of α and $\alpha m/M$). A high-momentum photon which will probe the structure of the proton and give it enough momentum so that it has appreciable recoil velocity will leave the electron-proton system in a highly virtual state. This virtual state must soon go back to the initial state by the exchange of another photon of almost equal and opposite momentum. It is thus two-photon exchange [Fig. 1, graphs (C) and (D)] which involves states with large enough momenta so that the finite mass and structure of the proton are felt. We ask for the difference in the hfs for a proton with structure and a point proton. That is, we do not calculate Fig. 1, graphs (E) and (F) [with an interaction $F(\kappa^2)\Lambda_\mu$] but the difference of Fig. 1, graphs (E)+(F) and Figs. 1(C)+(D). This difference vanishes very rapidly for photons having momenta of order αm because the structure of the proton is about a pion Compton wavelength in "size." Any momentum small compared to a pion mass will "see" only a point proton. For this reason the binding effect of proton and electron can be neglected in the initial and final states for the difference of these graphs although it cannot be neglected for either pair, Figs. 1(C)+(D) or Figs. (E)+(F), alone. In fact, the mass of the electron in the intermediate state can also be neglected. These conclusions also follow from a detailed consideration of the bound-state equation, provided that one assumes all proton-photon interactions carry a form factor $F(\kappa^2)$. Although, as we shall see, the situation is not simple enough to be described by the one-photon form factor, the order of magnitude estimated for the various graphs by using such a form factor is certainly correct. Therefore the conclusion that the nuclear-structure effects in the hfs are due to two-photon exchange is on firm ground. Explicitly, we find[14] that the energy shift, hfs (extended proton) −hfs (point proton), is given by Fig. 1 graphs [(E)+(F)]−[(C)+(D)] as

$$\delta E=\left(\frac{\alpha m}{\pi(\mu+1)M}\right)\Delta\delta\mathcal{E}, \tag{1.4}$$

$$\delta\mathcal{E}=\frac{+3}{i\pi^2}\int\frac{d^4\kappa}{\kappa^4}\mathfrak{N}_{\nu\mu}\mathfrak{M}_{\mu\nu}, \tag{1.5}$$

$$\mathfrak{N}_{\nu\mu}=\tfrac{1}{4}\,\mathrm{Spur}\{C_{\nu\mu}(e)(\tfrac{1}{2}(1+\gamma_t))\gamma_z\gamma_5\} \tag{1.6}$$

$$\mathfrak{M}_{\mu\nu}=\tfrac{1}{4}\,\mathrm{Spur}\{[C_{\mu\nu}(p,\text{ extended}) -C_{\mu\nu}(p,\text{ point})](\tfrac{1}{2}(1+\gamma_t))\gamma_z\gamma_5\}. \tag{1.7}$$

$\delta\mathcal{E}$ is the energy shift in dimensionless units, momenta, and energies appearing in it being expressed in terms of proton mass. $C_{\mu\nu}(e)$ and $C_{\mu\nu}(p)$ are the Compton scattering operators for the electron and proton. We have normalized them so that

$$C_{\nu\mu}(e)=\left\{\gamma_\nu\frac{1}{\boldsymbol{p}_e+\boldsymbol{\kappa}-m}\gamma_\mu+\gamma_\mu\frac{1}{\boldsymbol{p}_e-\boldsymbol{\kappa}-m}\gamma_\nu\right\}, \tag{1.8}$$

$$e^2C_{\mu\nu}(p,\text{ extended})=\left\{\Lambda_\mu(-\kappa)\frac{1}{\boldsymbol{p}+\boldsymbol{\kappa}-1}\Lambda_\nu(\kappa) +\Lambda_\nu(\kappa)\frac{1}{\boldsymbol{p}-\boldsymbol{\kappa}-1}\Lambda_\mu(-\kappa)\right\}F^2(\kappa^2). \tag{1.9}$$

[8] C. K. Iddings and P. M. Platzman, Phys. Rev. **113**, 192 (1959). We shall refer to this article as I.

[9] Of course there are various forms of perturbation theory which are equivalent to the use of the bound-state equation; see for example, Ref. 7. The point to be made here is simply that starting with the exact theory (quantum electrodynamics) a consistent perturbation expansion generates *all* corrections to the hfs.

[10] From now on we shall use units where $\hbar=c=1$, $\mu=1.79$, $e^2=\alpha\cong 1/137$, the proton mass M is 1, and κ is the photon momentum.

[11] A. C. Zemach, Phys. Rev. **104**, 1771 (1956).

[12] For a thorough discussion of the form factors, see L. N. Hand, D. G. Miller, and Richard Wilson, Rev. Mod. Phys. **35**, 335 (1963).

[13] The ideas underlying this discussion are due to R. P. Feynman (unpublished).

[14] See Ref. 8 for the details.

$C_{\mu\nu}(p,\text{ point})$ simply replaces $F(\kappa^2)$ by unity. As noted above, we replace the electron propagators by $(\boldsymbol{\kappa})^{-1}$. There is no trouble with an infrared divergence because we are calculating the difference of an extended and a point proton. There is a logarithmic divergence in Eq. (1.5) coming from the $C_{\mu\nu}(p,\text{ point})$ term. This cancels with a similar divergence in the work of Newcomb and Salpeter[7] and yields a finite answer, the hfs for a proton with structure.[14]

Inspection of $C_{\mu\nu}(p)$ shows that what is needed is the forward Compton scattering of a virtual photon of initial polarization μ and final polarization ν and that this is being approximated by two interactions, each with the experimental one-photon form factor, and an intermediate state of one virtual proton calculated according to the usual Feynman rules [Fig. 1, graphs (E) and (F)]. The correct answer for the hfs, at least to this order in $[\alpha m/\pi(\mu+1)M]$, would be given by using the exact Compton amplitude in which intermediate states I, such as the $N^*(\frac{3}{2},\frac{3}{2})$, can occur as well as protons [see Fig. 1, graphs (G) and (H)]. In the exact amplitude, the intermediate state of one proton would presumably carry more complicated form factors than those of Eq. (1.9) since the proton as well as the photons are off the mass shell. The shift coming from the use of (1.6) and (1.7) in (1.5) disagrees with experiment but this is not impossible because important processes like isobar formation have been neglected.[15]

Rough estimates for Fig. 1, graphs (G) and (H) have been given[16]; however, it is not very clear in what way all the different intermediate states are to be included and whether or not these estimates can be improved. In other words, the inclusion of form factors and other graphs with isobar intermediate states does not necessarily follow from any self-consistent perturbation theory; it is an *ad hoc* hypothesis.

In this paper we first propose (Sec. II) a consistent dispersion-theoretic method of attack for obtaining the proton-structure corrections to the hfs. We show that the introduction of form factors [Fig. 1, graphs (E) and (F)] arises naturally and that there is also a contribution from isobar intermediate states [Fig. 1, graphs (G) and (H)]. [Contributions from Fig. 1, graphs (E) and (F) are discussed in detail in Appendix A.] Using the analyticity properties of the amplitudes involved, we are able to express the structure correction as an integral over *physically measurable* cross sections and an integral over a subtraction constant. In Sec. III we discuss the methods by which these cross sections may be obtained from electron-scattering experiments. Since these data are not available, at present, we give some theoretical estimates in Appendices B and C. Our conclusions are given in Sec. IV and summarized here: First, there is a discrepancy between the measured and theoretical values for the hfs. The previously given estimates for the nuclear-size corrections to the hfs are essentially unchanged. The theory is forced to rely on some rough estimates for processes which concern strongly interacting particles and this will be so until the electron scattering experiments (and possibly other experiments which fix subtraction functions) are done. Thus this disagreement is not indicative of any failure of quantum electrodynamics. Second, the disagreement would be reduced if there were many higher proton isobars which were copiously produced by γ rays, and if a certain subtraction function were large. "Reasonable" estimates suggest that even if both these effects contribute in the same direction a discrepancy will remain. Third, there is a close parallel in the calculation of the electromagnetic mass difference between neutron and proton and there also appears to be a similar discrepancy there. Progress in understanding either one of these problems may help with the other.

II. ROTATION OF THE CONTOUR OF INTEGRATION

We shall study the contribution of the extended proton to the term $\mathfrak{M}_{\mu\nu}$ of Eq. (1.7) by a method due to Cottingham and used by him to discuss the neutron-proton electromagnetic mass difference.[17] $\mathfrak{M}_{\mu\nu}$ (extended) is simply the spin-dependent part of the forward scattering amplitude for a proton and a photon of mass κ^2, polarization μ to a final state of polarization ν (mass κ^2). Using standard reductions, $\mathfrak{M}_{\mu\nu}$ (extended) can be brought into a form analogous to Eq. (1.7) of C.

$$\mathfrak{M}_{\mu\nu}(\text{extended}, \boldsymbol{\kappa},\kappa_0)=\frac{i}{16}\sum_Q\langle n,\alpha|j_\mu(0)|Q\rangle \times\langle Q|j_\nu(0)|n,\alpha\rangle\left\{\frac{\delta^3(\boldsymbol{\kappa}-\mathbf{q})}{q^0-1-\kappa^0-i\epsilon}+\frac{\delta^3(\boldsymbol{\kappa}+\mathbf{q})}{q^0-1+\kappa^0-i\epsilon}\right\} -(|n,\alpha\rangle\to|n,\beta\rangle), \quad (2.1)$$

where $\sum|Q\rangle\langle Q|$ is a sum over a complete set of outgoing states $|Q\rangle$ of momentum $(q^0,\mathbf{q})$, $|n,\alpha\rangle$ is the initial single-nucleon state with spin $\alpha,\sigma_z=+1$ (β is the spin state $\sigma_z=-1$); $j_\mu(x)$ is the Heisenberg current opera-

[15] In performing this cutoff calculation, Newcomb and Salpeter (Ref. 7) noticed a cancellation between various finite terms as well as the logarithmic divergence referred to above. The logarithmic divergence from Fig. 1, graphs (C) and (D) occurs only in terms proportional to the square of the anomalous moment μ and its dependence on the cutoff is quite weak. When the proton structure is included as in Ref. 8 (Fig. 1, graphs [(E)+(F)]−[(C)+(D)]) only about 7% of the correction comes from the μ^2 terms. The largest correction (∼93%) arises because the cancellations for a point proton no longer occur when the proton has structure. This explains why one does not obtain a realistic estimate of the size effects merely by replacing the cutoff in the logarithmic infinity by some length characterizing the proton structure. The structure effects are most important for terms which are perfectly finite (but tending to cancel) for a point proton.

[16] C. K. Iddings and P. M. Platzman, Phys. Rev. **115**, 919 (1959). We shall refer to this article as II.

[17] W. N. Cottingham, Ann. Phys. (N. Y.), **25**, 424 (1963). We shall refer to this article as C.

tor. Equation (2.1) shows that, as a function of κ^0 for fixed $\boldsymbol{\kappa}$, $\mathfrak{M}_{\mu\nu}$ is analytic in the entire κ^0 complex plane except for poles and cuts just below the positive real axis and just above the negative real axis. The integrand in Eq. (1.5) has exactly the same analytic structure as (2.1) for fixed $\boldsymbol{\kappa}$ and complex κ^0. [Of course there are additional poles in (2.2) because of the photon and electron propagators; these do not change the analytic structure since they are evaluated for the usual Feynman contour.] Therefore it is possible to rotate the contour of integration anticlockwise from the real (κ_0) to the imaginary ($i\kappa_0$) axis. Inspection of Eq. (1.9) shows that this rotation is also possible for the point-proton part of $\mathfrak{M}_{\mu\nu}$. We shall change variables, $d^4\kappa \rightarrow i d^4K$ after performing this rotation of the contour so that the integration is over a Euclidian four-dimensional space K. Equation (1.5) now becomes

$$\delta\mathcal{E}=\frac{+3}{\pi^2}\int\frac{d^4K}{K^4}\mathfrak{N}_{\nu\mu}(-K^2,iK_0)\mathfrak{M}_{\mu\nu}(-K^2,iK_0)\,. \quad (2.2)$$

In order to proceed further we must employ a dispersion representation for the function $\mathfrak{M}_{\mu\nu}$. The most general possible forward scattering amplitude $C_{\mu\nu}$ can be written

$$\begin{aligned}C_{\mu\nu}=&-[g_{\mu\nu}\kappa^2-\kappa_\mu\kappa_\nu]A(\kappa^2,\omega)\\&+[g_{\mu\nu}(p\cdot\kappa)^2+p_\mu p_\nu\kappa^2-(p_\mu\kappa_\nu+\kappa_\mu p_\nu)p\cdot\kappa]B(\kappa^2,\omega]\\&+[\epsilon(\mu\nu\kappa\gamma)\gamma_5]D(\kappa^2,\omega)+[\epsilon(\mu\nu\kappa p)\boldsymbol{\kappa}\gamma_5]G(\kappa^2,\omega)\,, \quad (2.3)\end{aligned}$$

where A, B, D, and G are functions of the two scalars κ^2 and $\kappa^0=p\cdot\kappa\equiv\omega$ and p is the laboratory momentum of the proton, (1,0,0,0).[18] Invariance under charge conjugation requires A, B, and D to be even in ω and G to be odd.

Dispersion relations for the functions A, B, D, and G can be proved for fixed, negative κ^2 by using the method of Bogoliubov.[17,19] For the functions D and G we have

$$D(\kappa^2,\omega)=\frac{d(\kappa^2)2\kappa^2}{\kappa^4-4\omega^2}+\frac{1}{\pi}\int_C^\infty\frac{2\omega'd\omega'\,\mathrm{Im}[D(\kappa^2,\omega')]}{\omega'^2-\omega^2}, \quad (2.4)$$

$$G(\kappa^2,\omega)=\frac{g(\kappa^2)4\omega}{\kappa^4-4\omega^2}+\frac{1}{\pi}\int_C^\infty\frac{2\omega d\omega'\,\mathrm{Im}[G(\kappa^2,\omega')]}{\omega'^2-\omega^2}, \quad (2.5)$$

where the cut starts with the inelastic pion-nucleon channel.

$$C=\tfrac{1}{2}(2m_\pi+m_\pi{}^2-\kappa^2)\,. \quad (2.6)$$

The pole terms are the one-nucleon contribution:

$$d(\kappa^2)=-\tfrac{1}{4}\mu(\kappa^2+4)f_1f_2-f_1{}^2-\tfrac{1}{4}\mu^2\kappa^2f_2{}^2 \quad (2.7)$$

$$g(\kappa^2)=+\tfrac{1}{2}\mu f_1f_2+\tfrac{1}{2}\mu^2f_2{}^2. \quad (2.8)$$

The functions $f_1(\kappa^2)$ and $f_2(\kappa^2)$ are the Dirac and Pauli form factors determined experimentally from elastic electron-proton scattering[12] and normalized to unity at $\kappa^2=0$. In terms of f_1 and f_2 the interaction of a real proton with a single virtual gamma ray of mass κ^2 and polarization μ is given by the vertex operator:

$$\Gamma_\mu(\kappa)=e\{\gamma_\mu f_1(\kappa^2)-\tfrac{1}{4}\mu(\gamma_\mu\boldsymbol{\kappa}-\boldsymbol{\kappa}\gamma_\mu)f_2(\kappa^2)\}\,. \quad (2.9)$$

We have so far ignored the question of subtraction "constants" (functions of κ^2) appearing in (2.4) or (2.5). We shall return to this point later and discuss whether or not subtractions are actually *required* for the convergence of the integrals (2.4) or (2.5). For the moment we shall make one subtraction in the dispersion relation for D. This is done only to facilitate the evaluation of Eq. (2.2) and not for any fundamental reason.

$$D(\kappa^2,\omega)=\frac{d(\kappa^2)2\kappa^2}{\kappa^4-4\omega^2}+d_0(\kappa^2)+\frac{1}{\pi}\int_C^\infty\frac{\omega^2 2\omega'd\omega'\,\mathrm{Im}[D(\kappa^2,\omega')]}{(\omega'^2-\omega^2)\omega'^2}. \quad (2.10)$$

If we compute D and G in the Born approximation using the interaction (2.9) we find D and G as given by (2.5), (2.7), (2.8), (2.10) with $\mathrm{Im}[D(\kappa^2,\omega)]=0=\mathrm{Im}[G(\kappa^2,\omega)]$ if $\omega\geq C$ and

$$d_0(\kappa^2)=+\tfrac{1}{2}\mu f_1f_2. \quad (2.11)$$

Writing (2.10) in the form

$$D=\left\{\frac{d(\kappa^2)2\kappa^2}{\kappa^4-4\omega^2}+\tfrac{1}{2}\mu f_1f_2\right\}+[d_0(\kappa^2)-\tfrac{1}{2}\mu f_1f_2]+\frac{1}{\pi}\int_C^\infty\frac{\omega^2 2\omega'd\omega'\,\mathrm{Im}[D(\kappa^2,\omega')]}{(\omega'^2-\omega^2)\omega'^2} \quad (2.12)$$

gives an expression, the first term of which is the Born approximation.

We now substitute (2.5), (2.12), into (2.2) and use the following relationships:

$$3\kappa^2\mathfrak{N}_{\nu\mu}\mathfrak{M}_{\mu\nu}(\text{extended})=[2\kappa^2+(p\cdot\kappa)^2]D+[(p\cdot\kappa)^2-\kappa^2](p\cdot\kappa)G, \quad (2.13)$$

$$\begin{aligned}ip\cdot\kappa=K_0=K\cos\psi,&\quad d^4K=d\Omega_{\mathbf{K}}K^3dK\sin^2\psi d\psi,\\ \theta\equiv\omega'^2/K^2,&\quad d^4K=4\pi K^3dK\sin^2\psi d\psi\end{aligned} \quad (2.14)$$

$$D_0(\kappa^2)\equiv[d_0(\kappa^2)-\tfrac{1}{2}\mu f_1f_2]. \quad (2.15)$$

We obtain

$$\delta\mathcal{E}=\delta\mathcal{E}(\text{Born})+\delta\mathcal{E}(\text{cut}), \quad (2.16)$$

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[18] We use the abbreviations $\epsilon(\mu\nu\kappa\gamma)\equiv\epsilon_{\mu\nu\rho\sigma}\kappa_\rho\gamma_\sigma$. Our normalization for the current j_μ of Eq. (2.1) is not the usual; this is implicit in our defining Eqs. (1.7), (1.8), (2.3).

[19] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), p. 610.

$$\delta\mathcal{E}(\text{cut})=\int_0^\infty\frac{dK}{K}\left[-\tfrac{9}{2}D_0(-K^2)+\int_C^\infty\frac{d\omega'}{\pi\omega'}\,\text{Im}[D(-K^2,\omega')]\{9+12\theta-8\theta^2-8(2-\theta)(\theta+\theta^2)^{1/2}\}\right.$$
$$\left.+\int_C^\infty\frac{d\omega'}{\pi}\,\text{Im}[G(-K^2,\omega')]\{-3-12\theta-8\theta^2+8\theta^{1/2}(1+\theta)^{3/2}\}\right].\quad(2.17)$$

The Born terms we want are those for the difference of the extended and point proton:

$$\delta\mathcal{E}(\text{Born})=\frac{1}{\pi^2}\int\frac{d^4K}{K^6}\left\{(K_0^2+2K^2)\left[\frac{K^2\{\mu(4-K^2)(f_1f_2-1)+4(f_1^2-1)-\mu^2K^2(f_2^2-1)\}}{2(4K_0^2+K^4)}+\tfrac{1}{2}\mu(f_1f_2-1)\right]\right.$$
$$\left.+(K^2-K_0^2)\left[\frac{4K_0^2\{\tfrac{1}{2}\mu^2(f_2^2-1)+\tfrac{1}{2}\mu(f_1f_2-1)\}}{4K_0^2+K^4}\right]\right\}.\quad(2.18)$$

All form factors are evaluated at $-K^2$. We have not done the angular integration in (2.18) because we shall give, in Appendix A, a more convenient formula for $\delta\mathcal{E}$ (Born), suitable for use when the form factors are expressed as a sum over poles. Evaluations of $\delta\mathcal{E}$ (Born) are also given for a number of form factors which fit the more recent electron-scattering data. The results are almost completely insensitive to the choice of form factor, $\delta\mathcal{E}$ (Born)≈-70 in all cases, and are in agreement with a previous estimate.[8] This is not surprising because the Born contribution is exactly the term which was calculated before except that f_1 and f_2 were taken as equal. It was also noticed[16] that this rotation of the contour was possible for the Born terms and that after this rotation, $\delta\mathcal{E}$ (Born) depended only on the form factors in a part of the physical region where they are well known. Thus the more recent data for the form factors makes little change in $\delta\mathcal{E}$ (Born). The point of this paper is that contour rotation is also possible for the non-Born terms, $\delta\mathcal{E}$ (cut), and that the Born terms arise naturally in a dispersion analysis.

In (2.17) and (2.18) we have achieved our goal; the hfs nuclear size correction is expressed as an integral over quantities which are, in principle, physically measurable.

III. EXPERIMENTAL DETERMINATION OF THE AMPLITUDES

We now turn to the experimental determination of the form factors and imaginary parts appearing in Eq. (2.16). Determination of the form factors f_1 and f_2 by elastic electron-proton scattering has been thoroughly discussed elsewhere and we shall simply assume that they are known.[12] The experiment which determines the imaginary parts of D and G is the inelastic scattering of a polarized electron by an initially polarized proton to a final state with known electron energy and momentum and unknown electron polarization. The final state of the proton system is arbitrary. Alternatively, the final electron polarization could be measured rather than the initial polarization.

We shall describe the two possible spin states, in direction s, of a fermion of momentum p by the covariant projection operators

$$P(\pm s)=\tfrac{1}{2}(1\pm is\gamma_5),\quad(3.1)$$

$$p\cdot s=0,\quad s^2=-1.\quad(3.2)$$

We consider electroproduction process shown in graph J whereby an electron of spin direction $+r$, momentum p_1 scatters from a proton of spin $+u$, momentum p into a final state of momentum p_2. The cross section for this process is given by

$$\frac{d^2\sigma}{d\Omega_2dp_2^0}=\frac{\alpha^2|\mathbf{p}_2|}{k^4|\mathbf{p}_1|}\,\text{Spur}\left\{\gamma_\nu\frac{(p_1+m)}{2}\frac{(1+ir\gamma_5)}{2}\gamma_\mu\frac{(p_2+m)}{2}\right\}\sum_Q\langle p,u|j_\mu(0)|Q\rangle\langle Q|j_\nu(0)|p,u\rangle\delta^4(qp-p-{}_1+p_2).\quad(3.3)$$

Using the relation implied by Eq. (2.1),

$$\text{Im}[\mathfrak{M}_{\mu\nu}]=(i\pi/16)\textstyle\sum_Q\langle p,\alpha|j_\mu(0)|Q\rangle\langle Q|j_\nu(0)|p,\alpha\rangle\delta^4(\kappa+p-q)-\{\alpha\to\beta\},\quad(3.4)$$

where $\kappa=(p_1-p_2)$ and $\kappa_0\geq0$, we find:

$$\frac{d^2\sigma}{d\Omega dp_2^0}=\frac{2\alpha^2|\mathbf{p}_2|}{\pi|\mathbf{p}_1|\kappa^4}\{(\kappa^4+2m^2\kappa^2)\,\text{Im}[A(\kappa^2,p\cdot\kappa)]-[((p\cdot p_1)^2+(p\cdot p_2)^2)\kappa^2+\tfrac{1}{2}\kappa^4+2m^2(p\cdot\kappa)^2]\,\text{Im}B(\kappa^2,p\cdot\kappa)$$
$$+2(\kappa\cdot u(\kappa\cdot r)-\kappa^2(u\cdot r))m\,\text{Im}[D(\kappa^2,p\cdot\kappa)]+2(\kappa\cdot r(\kappa\cdot p)-\kappa^2(p\cdot r))(\kappa\cdot u)m\,\text{Im}[G(\kappa^2,p\cdot\kappa)]\}.\quad(3.5)$$

If the final spin s rather than the initial spin r of the electron is measured, the replacement $r\to -s$ is to be made in Eq. (3.5). Clearly this formula enables us to find the imaginary parts of A, B, D, G as functions of κ^2, ω for $\kappa^2\leq 0$, $\omega\geq 0$. The kinematical problems involved in obtaining A and B are discussed in Ref. 17. Here we have the added complication of the spin. It is clear that if we take the difference of the cross section for $+u$ and $-u$ and hold everything else fixed, the A and B terms will drop out and we will be left with the terms involving D and G.

As an example we shall work out specifically what to expect for various spin directions in the laboratory frame. We introduce coordinates as follows:

$$\begin{aligned} &p_1=(\epsilon_1,\mathbf{p}_1,0,0)\,, && p_2=(\epsilon_2,\,\mathbf{p}_2\cos\theta,\,\mathbf{p}_2\sin\theta,\,0)\,, && p=(1,0,0,0)\,,\\ &r_1=(\mathbf{p}_1,\epsilon_1,0,0)/m\,, && && u_1=(0,1,0,0)\,,\\ &r_2=(0,0,1,0)\,, && && u_2=(0,0,1,0)\,,\\ &r_3=(0,0,0,1)\,, && && u_3=(0,0,0,1)\,. \end{aligned}\tag{3.6}$$

An arbitrary polarization can always be described as a linear combination of these vectors. Thus $r=(\alpha r_1+\beta r_2+\gamma r_3)$. (With real α, β, γ, $\alpha^2+\beta^2+\gamma^2\leq 1$.) Expressions for the dot products occurring in $d\sigma$ can now be obtained in terms of scattering angle and initial and final energies. For example,

$$\kappa\cdot r_1=-p_2\cdot r_1=-(\mathbf{p}_1\epsilon_2-\epsilon_1\mathbf{p}_2\cos\theta)/m\,.\tag{3.7}$$

The results are summarized in Tables I and II. Clearly there are enough independent functions of angle and polarization direction to enable us to find ImD and ImG as functions of κ^2 and $\omega\equiv p\cdot\kappa$. The particular choice made by the experimenter will depend upon the details of the apparatus and the techniques used. For example, if we confine ourselves to electron momenta, $\mathbf{p}_1$, $\mathbf{p}_2$, very large compared to an electron mass, $\epsilon_1/\mathbf{p}_1\approx 1\approx\epsilon_2/\mathbf{p}_2$, we find

$$d\sigma(u_1,r_1)\cong\mathrm{Im}[D+p\cdot\kappa G]I_1(u_1,r_1)-(\kappa^4/2m)\,\mathrm{Im}G\tag{3.8}$$

$$d\sigma(u_2,r_1)\cong\mathrm{Im}[D+p\cdot\kappa G]I_1(u_2,r_1)-(\kappa^4/2m)\cot\theta/2\,\mathrm{Im}G\,.\tag{3.9}$$

Our point is only that both ImD and ImG are independent contributions to $d\sigma$ and can therefore be determined separately. Table I and II may be used to give information about the boundedness of the amplitudes. We observe that in the lab system as $|\mathbf{p}_1|\to\infty$ holding κ^2 fixed that $\theta\sim 1/\sqrt{|\mathbf{p}_1|}\to 0$, $\omega=p\cdot\kappa\sim|\mathbf{p}_1|\to\infty$. If $d\sigma$ is bounded by a constant for any u, r we have

$$\begin{aligned}\mathrm{Im}D+|\mathbf{p}_1|\,\mathrm{Im}G&\lesssim\text{constant}\,,\\ \mathrm{Im}D&\lesssim|\mathbf{p}_1|\,\text{constant}\,,\end{aligned}\tag{3.10}$$

from u_1, r_1, and u_3r_3, respectively. These can both hold only if

$$\begin{aligned}|\mathrm{Im}D|&\lesssim\omega\,,\\ |\mathrm{Im}G|&\lesssim\text{constant}\,,\end{aligned}\tag{3.11}$$

for fixed κ^2. Thus the dispersion relation (2.10) requires at most one subtraction and (2.5) requires none.

Now let us look at the limit $\kappa^2\to 0$. The amplitude must pass over into the forward Compton scattering for real photons. For $\kappa^2=0$ as $\omega\to 0$ we have the low-energy theorem[20] which states that the first two powers of ω in the amplitude $C_{\mu\nu}$ are correctly given by the Born approximation. This means that [cf., Eq. (2.15)]

$$D_0(0)=0\,.\tag{3.12}$$

As can be seen from Eq. (2.17), if this did not hold there would be a logarithmic singularity in $\delta\mathcal{E}$ (cut).[21] If the dispersion relation for D [Eq. (2.4)] converges with no subtraction then D_0 can be determined by evaluating the integral over the cut at $\omega=0$. If the subtraction is required for convergence then the hfs depends upon a

TABLE I. The invariant $[\kappa\cdot u\kappa\cdot r-\kappa^2 u\cdot r]$ as a function of energy and angle for various choices of the polarizations u and r. Quantities evaluated in the laboratory frame.

Polarizations	I_1
(r_1,u_1)	$\{(\mathbf{p}_2\cos\theta-\mathbf{p}_1)[\epsilon_1\mathbf{p}_2\cos\theta-\epsilon_2\mathbf{p}_1]+\kappa^2\epsilon_1\}(1/m)$
(r_1,u_2)	$\mathbf{p}_2\sin\theta(\epsilon_1\mathbf{p}_2\cos\theta-\mathbf{p}_1\epsilon_2)(1/m)$
(r_2,u_1)	$\mathbf{p}_2\sin\theta(\mathbf{p}_2\cos\theta-\mathbf{p}_1)$
(r_2,u_2)	$(\mathbf{p}_2\sin\theta)^2+\kappa^2$
(r_3,u_3)	κ^2
$(r_1,u_3)\,(r_2,u_3)$, $(r_3,u_1)\,(r_3,u_2)$	0

TABLE II. The invariant $[\kappa\cdot r\kappa\cdot p-\kappa^2 p\cdot r]\kappa\cdot u$ as a function or energies and angle for various choices of the polarizations u and r. Quantities evaluated in the laboratory frame.

Polarizations	I_2
(r_1,u_1)	$(\mathbf{p}_2\cos\theta-\mathbf{p}_1)\{(\epsilon_1-\epsilon_2)[\epsilon_1\mathbf{p}_2\cos\theta-\epsilon_2\mathbf{p}_1]-\kappa^2\mathbf{p}_1\}(1/m)$
(r_1,u_2)	$(\mathbf{p}_2\sin\theta)\{(\epsilon_1-\epsilon_2)(\epsilon_1\mathbf{p}_2\cos\theta-\mathbf{p}_1\epsilon_2)-\kappa^2\mathbf{p}_1\}(1/m)$
(r_2,u_1)	$\mathbf{p}_2\sin\theta(\mathbf{p}_2\cos\theta-\mathbf{p}_1)(\epsilon_1-\epsilon_2)$
(r_2,u_2)	$(\mathbf{p}_2\sin\theta)^2(\epsilon_1-\epsilon_2)$
(r_3,u_3)	0
$(r_1,u_3)\,(r_2,u_3)$, $(r_3,u_1)\,(r_3,u_2)$	0

[20] F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* **96**, 1433 (1954).

[21] Of course bound-state effects and the finite electron mass would prevent a real divergence at low K; however, the calculation would be more involved. This was the motivation for making the subtraction in D.

function D_0 which cannot be determined from electron scattering alone. It is also possible that the dispersion integrals themselves converge but that there are functions D_0 or G_0 present. This means only that Re(D) or Re(G) has a different asymptotic behavior than the imaginary part. In either case, we have no proposal for determining these functions. What is needed is an analogue of the low-energy theorem but for $\kappa^2<0$. We do not know if such a theorem exists. In the absence of any other information, we shall assume that $D_0=0$. For the models considered in Appendices B and C, this is certainly true. We can see from Eq. (2.17) that the hfs is quite sensitive to D_0.[22]

IV. CONCLUSIONS

The results of Secs. II and III show what physical processes contribute to the proton-structure correction to the hfs; there is no further question of double counting or omitting terms. For example, a graph such as K is included in the form factor [Fig. 1, graph (E)].[23] There is one remaining uncertainty, the value of the subtraction function D_0. We do not know how large it is; the low-energy theorem tells us only that it is zero at $\kappa^2=0$. All other contributions to the hfs can be determined experimentally by means of electron-proton scattering experiments. Since these experiments are undoubtedly a long way off, we have given two estimates (Appendices B and C) of the contribution of a single resonance, $N^*(3,3)$ electroproduction, to the hfs. Both estimates are of the same order of magnitude: $\delta\mathcal{E}(N^*)\lesssim 0.5$. The single photon form factors are well known and we show in Appendix A that $\delta\mathcal{E}$ (Born)≈ -70. Taken at face value, this is exactly the same result obtained previously (Refs. 8 and 16). The previous conclusion that the value of α from fine structure measurements is in error, is not substantiated by a recent determination of α from the hfs of the μe system.[24] It therefore appears that either there are many, possibly overlapping, higher resonances which are strongly excited by photons or that our assumptions about subtractions are unjustified or that both of these possibilities hold. Neglecting the question of subtractions, even if there are, say, five resonances and each contributes about 2 to $\delta\mathcal{E}$, all in the same direction, there is still an over-all theoretical $\delta\mathcal{E}\simeq -60$ so $\mathcal{P}\cong(1-30\times10^{-6})$.[25] There appear to be no special interference effects which would give large enhancement with more than one resonance. On the other hand, if we allow a fairly wild behavior for D_0 then (cf. Ref. 22) $\delta\mathcal{E}\approx 10$ from D_0 and the total theoretical $\delta\mathcal{E}$ (Born, D_0 and resonance terms) is ≈ -50. So $\mathcal{P}\approx(1-23\times10^{-6})$. Thus even an extreme behavior of D_0 will not account for the discrepancy. We can give no reason to expect such behavior. At present, therefore, we are not able to account for the discrepancy between the measurements and our theoretical estimates. It would be very interesting to have the results of the electroproduction experiments to compare with our theoretical estimates for D and G since functions of two variables are then being compared rather than integrals over them (as is the case for $\delta\mathcal{E}$).

A similar situation exists for the electromagnetic neutron-proton mass difference.[17] A cutoff calculation of perturbation theory agrees with experiment.[26] The Born terms, taken with form factors, give an answer of the wrong sign and the contribution of a single isobar of reasonable widths and mass does not account for more than about 10% of the discrepancy.[27] This problem can also be investigated by means of inelastic electron-scattering experiments and it is possible that any unusually large isobar production cross sections will contribute to both the self-energy and hfs calculation and thus could be detected in either. Thus it would also be interesting to have the non-spin-dependent electroproduction data (for both proton and neutron).

ACKNOWLEDGMENTS

The author acknowledges helpful conversations and comments from his colleagues, in particular, Dr. W. N. Cottingham, Dr. S. Drell, Dr. N. Kroll, Dr. A. C. Hearn, Dr. T. A. Griffy, Dr. J. Bjorken, Dr. R. Socolow, and Dr. V. W. Hughes. He is indebted to Professor Vernon Hughes for repeatedly calling his attention to the widening gap between theory and experiment. He thanks Professor S. Drell for many encouraging and informative discussions.

APPENDIX A

Recent experimental results have shown that the form factors have a more complicated structure than was used in original estimates of $\delta\mathcal{E}$ (Born).[8] Accordingly, we have assumed that the form factors are a "sum of poles"[28]:

$$f_i(\kappa^2)=\sum_{n=1}^{N}\rho_i^{\,n}/(\kappa^2-\lambda_n^{\,2}). \tag{A1}$$

[22] As an estimate, if $D_0\sim 1$ for $0.1\leq -\kappa^2\leq 1$ and zero elsewhere then the contribution to $\delta\mathcal{E}$(cut) is of the order of $4.5\ \ln_e(10)\approx 10$.

[23] We have deliberately omitted terms like Fig. 1, graph (L) although they are, in principle, present in the kernel for the bound-state equation. A reasonable estimate of this graph gives the uninteresting result $\delta\mathcal{E}$[Fig. 1, graph (L)]≈ 0.002.

[24] W. E. Cleland, J. M. Bailey, M. Eckhause, V. W. Hughes, R. M. Mobley, R. Prepost, and J. E. Rothberg, Phys. Rev. Letters **13**, 202 (1964).

[25] In these units $\mathcal{P}-1\cong\delta\mathcal{E}(0.453\times10^{-6})$ experimentally (see Refs. 8 and 24) $\delta\mathcal{E}\approx +10\pm 20$.

[26] R. P. Feynman and G. Speisman, Phys. Rev. **94**, 500 (1954).

[27] W. N. Cottingham, private communication and Ref. 17.

[28] See for example S. D. Drell and F. Zachariasen, *Electromagnetic Structure of Nucleons* (Oxford University Press, London, 1961). This is a reasonable form of approximation to the integral representation. We are following the normalization conventions that $f_i(0)=1$. The derivative $(\partial f_i/\partial\kappa^2)_{\kappa^2=0}\equiv A^2$ is related to the rms radius R by $R=A\sqrt{6}$. We are using the convention that the proton mass is 1 so that R is measured to be about 3.81 in these units. The subscript $i=1$ refers to the Dirac part of the interaction and $i=2$ to the Pauli term. Cf. Eq. (2.9).

TABLE III. Hyperfine energy shifts, $\delta\mathcal{E}$ (Born), for various proton-single form factors. Logarithmic terms $[9\mu^2 \ln(2\kappa_m)/4]$ omitted; see text. Units of proton mass; all form factors except cases XII and XIII fit the experimental rms radius.

Case	λ_1^2	λ_2^2	λ_3^2	$\rho_1(D)$	$\rho_1(P)$	$\rho_2(D)$	$\rho_2(P)$	$\rho_3(D)$	$\rho_3(P)$	$\delta\mathcal{E}(0)$	$\delta\mathcal{E}(1)$	$\delta\mathcal{E}(2)$	$\delta\mathcal{E}_{\text{TOT}}$
I	0.52	2.06	...	−0.696	−0.696	+0.696	+0.696	...	...	−22.4	−44.8	−4.8	−72.0
II	0.75	0.924	...	−3.98	−3.98	+3.98	+3.98	...	...	−22.5	−45.2	−4.7	−72.6
III	0.4	0.511	1.11	0.184	0.475	0.298	+0.00859	−0.05	−0.225	−20.9	−45.3	−4.80	−71.1
IV	0.4	0.511	2.22	0.184	0.475	0.298	+0.00859	−0.10	−0.450	−21.1	−46.1	−4.74	−72.0
V	0.4	0.511	22.2	0.184	+0.475	0.298	+0.00859	−1.0	−4.50	−21.3	−47.0	−4.74	−73.0
VI	0.333	0.400	1.11	0.255	0	0	0.475	0.261	−0.255	−22.1	−44.7	−4.7	−71.5
VII	0.333	0.400	22.2	0.255	0	0	0.475	5.22	−4.50	−20.3	−45.8	−4.7	−70.8
VIII	0.333	1.11	...	0.255	0.333	0.261	0	...	...	−22.1	−47.0	−4.8	−73.9
IX	0.333	22.2	...	0.255	0.333	5.22	0	...	...	−20.3	−45.4	−4.8	−70.5
X	0.637	5.73	...	0.924	0.924	−2.58	−2.58	...	...	−21.5	−43.2	−4.8	−69.5
XI	0.637	15.8	...	0.905	0.905	−7.16	−7.16	...	...	−20.7	−41.4	−4.9	−67.0
XII	0.4	...	...	0.4	0.4	...	...	...	...	−22.4	−44.7	−4.9	−72.0
XIII	1.0	...	...	1.0	1.0	...	...	...	...	−12.7	−26.0	−5.6	−44.3
XIV	1.0	10.0	15.0			655.0	826.0	−968.0	−1224.	15.4	67.7	−501.	−417.0

When we introduce (A1) into (2.18), we obtain

$$\delta\mathcal{E}(\text{Born})=\sum_{\alpha=0,1,2}\sum_{i,j=1}^{N}R_{ij}(\alpha)\times\left\{\frac{J^\alpha(\lambda_i^2)-J^\alpha(\lambda_j^2)}{\lambda_i^2-\lambda_j^2}\right\}-\mathcal{G} \quad \text{(A2)}$$

$$\mathcal{G}=\mu^2\left\{\frac{17}{16}-\frac{9}{4}\ln(2\kappa_{\max})\right\}, \quad \text{(A3)}$$

where we put:

$$R_{ij}(0)=\rho_1{}^i\rho_1{}^j,\quad R_{ij}(2)=\rho_2{}^i\rho_2{}^j, \quad \text{(A4)}$$
$$R_{ij}(1)=\rho_1{}^i\rho_2{}^j,\qquad x=\tfrac{1}{2}(\lambda_i),$$

$$J^0(\lambda_i^2)=(1/\lambda_i^2)(\tfrac{9}{2}I_Q+4I_1-\tfrac{1}{2}I_2), \quad \text{(A5)}$$

$$J^1(\lambda_i^2)=(\mu/\lambda_i^2)(6I_Q+4I_1-2I_2), \quad \text{(A6)}$$

$$J^2(\lambda_i^2)=(\mu^2/\lambda_i^2)(\tfrac{3}{2}I_Q-\tfrac{3}{2}I_2+9/8\ln(\lambda_i^2)), \quad \text{(A7)}$$

$$I_Q=(\cos^{-1}x)/2x(1-x^2)^{1/2}, \quad \text{(A8)}$$

$$I_1=\tfrac{1}{2}\ln(4x^2)+(1-2x^2)I_Q, \quad \text{(A9)}$$

$$I_2=1+(1-2x^2)\ln(4x^2)+(1-8x^2+8x^4)I_Q. \quad \text{(A10)}$$

If $i=j$ then $\partial J^\alpha(\lambda_i^2)/\partial\lambda_i^2$ is to be used in place of the bracketed term in (A2). The logarithmic divergence coming from $C_{\mu\nu}(p,$ point) has been separated explicitly in the term $\mathcal{G}$; when the expression for $\delta\mathcal{E}$ (Born) is added to the previously obtained expression for the hfs,[7] it will cancel a similar divergence there. For this reason, we omit the logarithmic part of $\delta\mathcal{E}$ (Born), $-(9/4)\mu^2\ln 2\kappa_m$, and state only the finite part of $\delta\mathcal{E}$ (Born). The results for a number of form factors are given in Table III.

Case I was done by hand computation also[29] and case II simulates the form used in Ref. 7, Eq. (37) by two poles, close together. The results agree, within the accuracy of the previous calculations (1 part in 70) and thus give an additional check of both the algebra and computer program used to obtain Table I. Cases III, IV, V are approximations to the form factors which fit the more recent data.[30] The "hard core" (constant term) which the experimenters used to fit their data has been replaced by a pole, located farther from the physical region than the other two poles. As was explained in Sec. I, a constant (in κ^2) Pauli moment leads to a logarithmic divergence in the expression for $\delta\mathcal{E}$ (Born). There are other theoretical reasons for believing that as $\kappa^2\to\infty$, the Pauli form factor f_2 may vanish. At any rate, we have assumed that this is the case and therefore what the experimenters regard as a constant in f_2, we have replaced by a pole at very large, positive, κ^2. We have similarly eliminated constant terms in f_1 although there is no particularly compelling reason for doing so. As the results of Table I show, the hfs is not at all sensitive to the location of this pole. Cases VI to IX show by example that as long as the derivative of the form factor is correctly chosen, its structure can change radically with little effect on the hfs. Cases X and XI are form factors chosen to fit the preliminary results of Zichichi *et al.*[31] on $p+\bar{p}\to e+\bar{e}$. Here the form factors are measured at large timelike κ^2 and as is apparent from the table, even a form factor which remains large at much higher κ^2 than those of cases I and II makes practically no difference in the hfs. All the form factors in the table except XII and XIII have been fitted to the correct slope and value at $\kappa^2=0$. Thus the value of $\delta\mathcal{E}$ (Born) seems dependent only on the slope of the

[29] See footnote 10 of Ref. 8.

[30] C. de Vries, R. Hofstadter, and R. Herman, Phys. Rev. Letters, 8, 381 (1962).

[31] M. Conversi, T. Massam, Th. Muller, and A. Zichichi, Phys. Letters 5, 195 (1963).

form factor at $\kappa^2=0$, a region which has been thoroughly investigated by experiment. This conclusion agrees with those of articles I and II. It is *possible* to fit the experimental slope at $\kappa^2=0$ and obtain a different value of $\delta\mathcal{E}$ (Born); however, brief consideration of the form factor required, shows that it is definitely *not* in agreement with experiment. Case XIV is included only as an illustration of what would be required to change the results appreciably and is not intended as a realistic example. Thus it seems quite unlikely that any better form factors will make much change in $\delta\mathcal{E}$ (Born) from its value of -65 to -75.

APPENDIX B

We now consider an approximate evaluation of $\delta\mathcal{E}$ (cut). First, we shall ignore the term involving G and put $D(\kappa^2, p\cdot\kappa)=D(0, p\cdot\kappa)$. If we consider only $D(0,\omega)$ we are asking for the forward spin-dependent, Compton scattering of real photons, the contribution to the imaginary part is then given by the usual unitary condition in terms of photon-production cross sections. In view of the current state of knowledge, we think that the most that can be done is to estimate ImD in an extremely simple, one pole approximation. We assume that the only inelastic channel to which photon and proton are coupled is the $M1\ \frac{3}{2}\to N^*(3,3)$ channel and that this resonance has zero width for πN decay. Then to lowest order the imaginary part of D is just the imaginary part of the isobar graph (calculated in Appendix C)

$$\mathrm{Im}[D(0,\omega)]=\left\{\frac{8\Gamma_\gamma N^4[(2N^2-1)\omega_0+\omega_0^2]}{\alpha N^2(N^2-1)^3(3N^2+1)}\right\}\times\pi\{\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\},\quad \text{(B1)}$$

where $\omega_0=\frac{1}{2}(N^2-1)$, N is the isobar mass and Γ_γ the partial width for decay of the isobar into $p+\gamma$. (All units are those of a proton mass.) It should be realized that although we use the bookkeeping system of the spin-$\frac{3}{2}$ theory to keep track of kinematical factors in obtaining Eq. (B1) this is only for the sake of convenience; the imaginary part of D, in a zero-width approximation is quite independent of the method used to obtain it. Employing a dispersion relation gives the real part of D on the cut.

$$\mathrm{Re}[D]=\left[\frac{\Gamma_\gamma}{\alpha}\frac{(3N^2-1)}{(3N^2+1)}\frac{2N^2}{(N^2-1)^2}\right]\times\left[\frac{-1}{\omega_0+\omega}-\frac{1}{\omega_0-\omega}\right].\quad \text{(B2)}$$

Following the usual line of attack, we shall depart from the zero width approximation at this point and replace ω_0 in (B2) by $\omega_0-\frac{1}{2}iN\Gamma$ where Γ is the total width.

$$\mathrm{Re}[D]=\left[\frac{\Gamma_\gamma(3N^2-1)2N^2}{\alpha(3N^2+1)(N^2-1)^2}\right]\times\left\{\frac{\omega-\omega_0}{(\omega-\omega_0)^2+N^2\frac{1}{4}\Gamma^2}-\frac{\omega+\omega_0}{(\omega+\omega_0)^2+N^2\frac{1}{4}\Gamma^2}\right\},\quad \text{(B3)}$$

$$\Gamma=\frac{\gamma(qa)^3}{1+(qa)^3},\quad \Gamma_\gamma=\frac{\gamma_\gamma(ka)_{\mathrm{c.m.}}{}^3}{1+(ka)_{\mathrm{c.m.}}{}^3},\quad \text{(B4)}$$

$$s=2|\omega|+1,\quad q^2=\left[\frac{(s-(\mu+1)^2)(s-(\mu-1)^2)}{4s}\right],$$
$$(ka)_{\mathrm{c.m.}}=a(s-1)/2\sqrt{s},\quad \omega_0=\tfrac{1}{2}(N^2-1),\quad \text{(B5)}$$

where γ is the reduced width, γ_γ the reduced photon width, a is the channel radius. In making the widths functions of the momenta in the c.m. system, we get a better fit to the experimental data[32] and to the threshold behavior in ω for D.[33]

The dispersion relation for D is subtracted at $\omega=0$. Since the form (B3) vanishes here, we see that it satisfies Eq. (2.12) with $D_0=0$, at $\kappa^2=0$. Substituting (B3) into (2.13) and (1.5), performing the integral over $d^3\boldsymbol{\kappa}$ we obtain

$$\delta\mathcal{E}(N^*)=+\frac{9}{2}\int_0^\infty\frac{\mathrm{Re}[D(0,\omega)]d\omega}{\omega}.\quad \text{(B6)}$$

We have used the evenness of D in ω so that we need D only for positive frequencies—as given by (B3). $\delta\mathcal{E}(N^*)$ from Eq. (B6) is to be added to $\delta\mathcal{E}$ (Born) as required by Eq. (2.16). There seems to be little to be learned by doing these integrals (B6) analytically so we have made the change of variable $\omega=x/(1-x)$ and done them numerically. The results are given in Table IV for various values of the parameters.

A previous estimate of the N^* contribution to the hfs was given in Ref. 16 and for the sake of completeness we remark here that it corresponds to the choice of D and G such that

$$\delta\mathcal{E}(N^*,II)=\left[\frac{8N^3\Gamma_\gamma i}{\alpha\pi^2(N^2-1)^3}\right]\int\frac{d^4\kappa}{\kappa^6}\frac{(p\cdot\kappa)^3F^2(\kappa^2)}{(p+\kappa)^2-N^2},$$
$$F(\kappa^2)=\Lambda^4/(\kappa^2-\Lambda^2)^2,\quad \Lambda=0.91.\quad \text{(B7)}$$

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[32] M. Gell-Mann and K. M. Watson, Ann. Rev. Nucl. Sci. 4, 219 (1954).

[33] We have not used the usual Breit-Wigner one-level form for the width. See, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 410 and also Ref. 32. This is because the asymptotic behavior of the usual form ($\Gamma\to\infty$ as $\omega\to\infty$) leads to convergence troubles in the expression for the hfs [see Eq. (B6)]. This lack of convergence is not physically significant since we do not expect our expression for the amplitude (B3) has much validity outside the resonance region. Accordingly, we have chosen a form for widths which approaches a constant at large frequencies, is similar to the usual form near the peak of the resonance and has the correct threshold behavior.

TABLE IV. $\delta\mathcal{E}(N^*)$ all units in proton masses, energy shift to be added to that of Sec. II.

Case	π Meson mass	N Isobar mass	Γ	Γ_γ	a	Λ	$\delta\mathcal{E}$(III)	$\lvert\delta\mathcal{E}(\text{II})\rvert$	$\lvert\delta\mathcal{E}(\text{I})\rvert$
I	0.15	1.32	0.1	0.0003	5.86	0.91	−0.34	0.10	2.28
II	0.15	1.32	0.1	0.003	5.86	0.91	−3.41	1.02	22.8
III	0.15	1.32	1.00	0.0003	5.86	0.91	−0.34	0.10	2.42
IV	0.15	2.64	0.10	0.0003	5.86	0.91	−0.0013	0.00005	0.104
V	0.15	2.04	0.20	0.001	5.86	0.91	−0.027	0.002	1.02

The kinematical factors were treated in a slightly different manner and the dependence on the mass of the photon was taken as a multiplicative form factor $F(\kappa^2)$. (Cf. the discussion of Ref. 16.) We have also evaluated $\delta\mathcal{E}$ (N^*,II) in Table IV.

Finally we give, $\delta\mathcal{E}$(III), the energy shift obtained when the nucleon resonance is treated as an elementary particle. The details of this "isobar" analysis are to be found in Appendix C. Our reason for including $\delta\mathcal{E}$(III) here is to emphasize that although there are several *ad hoc* assumptions in each of these expressions for $\delta\mathcal{E}(N^*)$ they all represent quite different approaches to the inclusion of nucleon excited states and the agreement between the results is probably an indication that the answers are reasonable. In particular the isobar approach does not assume that G is zero.

Case I is that with parameters appropriate to the $N^*(3,3)$. All three $\delta\mathcal{E}$ are of the same order of magnitude and not significant as far as the hfs is concerned. [$\delta\mathcal{E}$(II) is a more accurate evaluation of the energy shift than the upper bound given in Ref. 16.] Case II shows that the shift is proportional to the partial width Γ_γ. Unless this reaches very large values, there is little effect of a higher resonance on the hfs. Case III shows that a larger width slightly raises the value of $|\delta\mathcal{E}(\text{I})|$. [$\delta\mathcal{E}$(II) and $\delta\mathcal{E}$(III) do not depend upon this width Γ.] Case IV shows that doubling the isobar mass makes the shift much smaller. Case V is the mass and width appropriate to the second $\frac{3}{2}$ π-N resonance. We see that this higher resonance contributes even less than the $N^*(3,3)$ by any one of our methods of calculation. Our conclusion is that a single isobar contributes very little to hfs size corrections. It is clear from Eqs. (B6) and (B3) that, since the sign of the real part of D changes in crossing the resonance, different regions in ω give contributions to $\delta\mathcal{E}$ which cancel. This is also the case for the N^* contribution to two-photon exchange in e-p elastic scattering.[34]

APPENDIX C

We shall now calculate $\delta\mathcal{E}(N^*)$ under the assumption that the N^* is an elementary particle described by a Rarita-Schwinger spin-$\frac{3}{2}$ field, φ_μ or v_μ. The propagator for such a particle is[35]

$$S_{\mu\nu}(p)=\frac{1}{p-N}\left\{\delta_{\mu\nu}-\frac{\gamma_\mu\gamma_\nu}{3}-\frac{1}{3N}(\gamma_\mu p_\nu-p_\mu\gamma_\nu)-\frac{2}{3N^2}p_\mu p_\nu\right\}. \quad (C1)$$

The N^* can be photoproduced and we take the interaction for this process as[36]

$$L=(eC/m_\pi)(\bar{v}_\nu\gamma_\mu\gamma_5 u)(e_\nu\kappa_\mu-\kappa_\mu e_\nu)+\text{H.c.} \quad (C2)$$

Experimentally, the magnetic dipole channel is found to be most important in photoproduction of N^* and (C2) has been chosen for this reason. Its limit for low N, N^* velocities is pure $M1$ photoproduction.[37] We have also included a form factor, $F(\kappa^2)=\Lambda^4/(\Lambda^2-\kappa^2)^2$, at the $NN^*\gamma$ vertex ($\Lambda/M\approx 0.91$). This is seen experimentally[38] to be about the same as the form factor which is measured in e-p elastic scattering experiments and in certain approximations, it is the same[39] but there is no general agreement which requires the identity of the form factors in the two cases.

With this interaction, we find for the N^* contribution:

$$(\tfrac{1}{2})C_{\mu\nu}(p)=F^2(\kappa^2)(C/m_\pi)^2(\delta_{\mu\rho}\kappa_\lambda-\kappa_\rho\delta_{\mu\lambda})\times\gamma_5\gamma_\lambda S_{\rho\sigma}(p+\kappa)(\delta_{\nu\sigma}\kappa_\eta-\delta_{\nu\eta}\kappa_\sigma)\gamma_5\gamma_\eta. \quad (C3)$$

When the traces are taken and the contractions of μ

[34] See p. 17 of Ref. 28.

[35] N is the isobar mass which we take as real and equal to 1.32 proton masses. We compute diagrams according to the conventions of R. P. Feynman, Phys. Rev. **84**, 108 (1951). Thus $\gamma_5\equiv\gamma_0\gamma_1\gamma_2\gamma_3$; $\gamma_5^2=-1$.

[36] M. Gourdin and Ph. Salin, Nuovo Cimento, **27**, 193 and 309 (1963).

[37] The two other possible interactions correspond in the static limit to linearly independent combinations of $E2$ and longitudinal-quadrupole photoproduction of N^*. Thus when the experiments are fitted, these interactions are expected to have very small coupling constants. Gourdin and Salin (Ref. 36) have used this spin-$\frac{3}{2}$ formalism in an "isobar model" to describe photoproduction and they find that this is the case; the interaction (C5) is by far the most important.

[38] W. K. H. Panofsky and E. A. Allton, Phys. Rev. **110**, 1155 (1958); L. N. Hand, *ibid.* **129**, 1834 (1963).

[39] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958); S. Gartenhaus and C. N. Lindner, *ibid.* **113**, 917 (1959); P. Dennery, *ibid.* **124**, 2000 (1961); M. Gourdin, *Proceedings of the International Conference on Nucleon Structure, Stanford, June 1963* (Stanford University Press, Stanford, 1964), p. 325.

and ν performed there results

$$\left(\frac{3m_\pi}{C}\right)^2 \mathfrak{N}_{\nu\mu}\mathfrak{N}_{\mu\nu}=\{[(p\cdot\kappa)^2+3p\cdot\kappa+2\kappa^2][4(\kappa^2+p\cdot\kappa)^2-6\kappa^2N^2]$$
$$+8[\kappa^2-(p\cdot\kappa)^2][(N^2+1)p\cdot\kappa+\kappa^2]+2N\kappa^2[3N^2(p\cdot\kappa)+4\kappa^2-4(p\cdot\kappa)^2]\}\frac{F^2(\kappa^2)}{\kappa^2[(p+\kappa)^2-N^2]}. \quad \text{(C4)}$$

In obtaining (C4) we have used the fact that we are going to integrate over solid angles $d\Omega_\kappa$ so κ_z^2 may be replaced by its spherical averages $3\kappa_z^2=(p\cdot\kappa)^2-\kappa^2$. We have not included Fig. 2, graph (H), only twice Fig. 1, graph (G) since an integration $d^4\kappa$ with a crossing symmetric $\mathfrak{N}_{\mu\nu}$ will give the same contribution for either Fig. 1 graph (G) or (H). In the usual way this integral can be parametrized and finally expressed as:

$$\delta\mathcal{E}(N^*,\text{III})=\left(\frac{C}{m_\pi}\right)^2\left(\frac{-6}{\pi^2}\right)\int_0^1 dx(P_6Q_6+P_4Q_4+P_2Q_2+P_0Q_0+QL), \quad \text{(C5)}$$

$$P_6=-2\pi^2/N^2\Lambda^4,$$
$$P_4=\frac{\pi^2}{2N^2\Lambda^4}\left\{\frac{1}{3}\left[(N^2-1)\left(\frac{-27}{2}\right)+1+64x(x-1)\right]+2N\right\},$$
$$P_2=\frac{2\pi^2}{9N^2\Lambda^4\epsilon}\left\{\left(\frac{99x}{2}-21\right)N^2+3(1-x)[5-23x(1-x)]+3N(3N^2-2x)\right\}, \quad \text{(C6)}$$
$$P_0=\frac{5\pi^2}{3N^2\Lambda^4\epsilon^3}\{2(1-x)[3N^2-2(1-x)^2]-2N^3\},$$

$$Q_6=\frac{\alpha}{3}(G2)+\frac{\alpha}{2}(F1)(G1)+\alpha(F2)-(F3)(GL)+\alpha\gamma(F4)-\frac{\alpha\gamma}{2}(G1),$$
$$Q_4=\frac{\alpha}{2}(G1)-\alpha(F1)+(10\gamma+8\alpha)(GL)-\frac{\alpha(F3)}{(F)}+\frac{\alpha(F4)(G1)}{2(F)}-\frac{\alpha(G2)}{3(F)},$$
$$Q_2=\alpha\gamma-(F1)(GL)-\frac{\alpha(F2)}{(F)}-\frac{\alpha(F3)(G1)}{2(F)^2}+\frac{\alpha(F4)(G2)}{3(F)^2}-\frac{(G3)}{4(F)^2}, \quad \text{(C7)}$$
$$Q_0=\gamma^2(GL)-\frac{\alpha\gamma^2(F1)}{(F)}+\frac{\alpha\gamma(F2)(G1)}{2(F)^2}-\frac{\alpha\gamma(F3)(G2)}{3(F)^3}+\frac{\gamma(F4)(G3)}{4(F)^3}-\frac{\gamma(G4)}{5(F)^3},$$
$$QL=\frac{\pi^2}{N^2\Lambda^4}\left\{\frac{-32\alpha^3(GL)}{3\Lambda^2\epsilon}+\frac{\Lambda^4}{18}\left[\frac{-27}{2}(N^2-1)+1+6N\right]\left(\frac{x^2-3x+3}{\epsilon(F)}\right)[\alpha(x+\epsilon)+\Lambda^2\gamma]\right\},$$

$$F1=5\gamma+2\alpha, \quad F3=10\gamma^2+12\alpha\gamma+3\alpha^2,$$
$$F2=10\gamma^2+8\alpha\gamma+\alpha^2, \quad F4=5\gamma+3\alpha, \quad \text{(C8)}$$
$$F=\alpha+\gamma,$$

$$G1=2\gamma+\alpha, \quad G3=(\gamma+\alpha)^4-\gamma^4,$$
$$G2=3\gamma^2+3\gamma\alpha+\alpha^2, \quad G4=(\gamma+\alpha)^5-\gamma^5, \quad \text{(C9)}$$
$$GL=\gamma\ln\left(\frac{\alpha+\gamma}{\gamma}\right),$$

$$\alpha=\Lambda^2(1-x), \quad \epsilon=(N^2-1+x), \quad \gamma=\epsilon x. \quad \text{(C10)}$$

The analytic form of this definite integral is not expected to give any particular insight into the proton structure and therefore we have evaluated it numerically with the result [parameters chosen for the $N^*(3,3)$]

$$\delta\mathcal{E}=(C/m_\pi)^2(6/\pi^2)(-0.471). \quad \text{(C11)}$$

The value for C is taken from the photoproduction width. If the cross section for $\gamma+p\to N^*$ is written as:

$$\sigma_{\text{total}}=\frac{\pi}{k^2}\left(\frac{\Gamma_\gamma\Gamma}{(E-N)^2+\frac{1}{4}\Gamma^2}\right). \quad \text{(C12)}$$

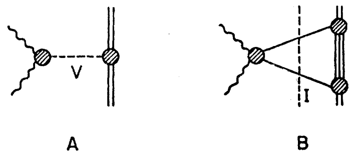

FIG. 2. Processes which contribute to subtraction functions in D or G.

The Γ_γ can be calculated by using the interaction (C2):

$$\frac{\Gamma_\gamma}{m_\pi}=\frac{2\alpha}{3}\left(\frac{C}{m_\pi}\right)^2 k^3\left[\frac{(E-N)(E+1)+(N+1)^2}{N(E+1)m_\pi}\right], \quad \text{(C13)}$$

where k is the photon (three-) momentum in the c.m. system and E is the total nucleon energy in this system. Putting $\Gamma_\gamma \cong 0.3$ MeV,[32] we find

$$C^2 \cong 0.03 \quad \text{(C14)}$$

and $\delta\mathcal{E}(N^*)=-0.34$ the first entry in Table IV.

Since the interaction amplitude involves the field strength, $F_{\mu\nu}=(e_\mu\kappa_\nu-\kappa_\mu e_\nu)$, it vanishes at zero frequency if $\kappa^2=0$ and thus obeys a one-subtraction dispersion relation with subtraction constant D_0 zero. If we had employed the dispersion approach to the calculation of $\delta\mathcal{E}$, we would have found no contribution from D_0.

APPENDIX D (*added in proof*)

This appendix contains a simple, physically motivated, discussion of the subtraction functions and kinematic singularities which may (possibly) appear in the amplitudes D and G.

Figure 2(A) shows a typical graph which would contribute to the subtractions. The imaginary part of this graph is nonzero when V is on the mass shell. It is clear that there is no way of obtaining such a term from the inelastic eN cross sections. Of course, *parts* of such graphs may be included in these cross sections. [For example, if V is taken as two ρ's, each coupled separately to one γ then there is a partial contribution to Fig. 1, graph (E) or (G).] Since terms which affect the hfs involve a flip of the proton spin but no change of its momentum, one unit of angular momentum is transferred by V to the photon; hence, V must have the quantum number 1^+. Because the two γ and V are virtual, they can couple although if the particles were physical this would be forbidden. This is also clear from the phenomenological terms which describe the $V\gamma\gamma$ vertex. They are

$$\partial_\mu V_\eta-\partial_\eta V_\mu)F_{\mu\nu}F_{\lambda\sigma}\epsilon(\nu\lambda\sigma\eta)\,,$$
$$V_\rho\partial_\mu F_{\mu\nu}F_{\lambda\sigma}\epsilon(\nu\lambda\sigma\rho)\,,$$

where $F_{\mu\nu}$ is the field tensor for the photon and V_η is the (four-vector) potential for the V particle. If the photon (V) polarizations are described by the three-vectors $\mathbf{e}$, $\mathbf{e}'(\mathbf{v})$ and if the four-momentum of the V is $(\Lambda,0)$ and of the photons is $(\omega,\pm\mathbf{K})\equiv k$ then these interactions have the noncovariant forms:

$$[(\mathbf{e}\times\mathbf{e}'\cdot\mathbf{v})\omega^2-(\mathbf{e}\times\mathbf{e}'\cdot\mathbf{K})(\mathbf{K}\cdot\mathbf{v})]\Lambda$$
$$[(\mathbf{e}\times\mathbf{e}'\cdot\mathbf{v})\omega k^2].$$

For the scattering in the forward direction, $\Lambda=0$ and the first term vanishes. The interaction of V and nucleon is of the form

$$\bar{N}[aV+b\partial_\mu V_\nu\sigma_{\mu\nu}]\gamma_5 N\,.$$

We conclude that the graph of Fig. 2(A) will contribute to a subtraction in the amplitude D. Since the interaction of V with 2γ is not required by gauge invariance, it must involve the photon field tensor $F_{\mu\nu}$. Only if there were some similar gauge principle which required that the field tensor for the V particle had to appear in the $V\gamma\gamma$ coupling, could we assert that there is no subtraction coming from Fig. 2, graph (A). (For forward scattering, the V field tensor, $[\partial_\mu V_\nu-\partial_\nu V_\mu]$, vanishes.) We see then, that if such a V meson were sufficiently strongly coupled to both N and γ, the hfs could perhaps be understood. Since this would imply an effective ep interaction other than that mediated by photons, its influence in other processes might be measurable. Recently, attempts have been made to understand the electromagnetic mass differences in the terms of "tadpoles" (Ref. 40). That is, the electromagnetic mass difference of neutron and proton is due to an isotopic spin 1, 0^+ particle. The contributing graph is the same as Fig. 2(A). In the hfs we see that the corresponding particle needed as a 1^+ particle, possibly of zero isotopic spin. It is interesting to speculate about the existence of such particles and the relative symmetry under the interchange of spin and isotopic spin. As far as the author is aware; there is no experimental evidence for these mesons. One final remark should be made. Even if no new particles are found, nonresonant graphs such as Fig. 2(B) with an intermediate state I of quantum numbers 1^+ could also give rise to subtraction functions in D or G and thus affect the hfs. We have not attempted estimates of these processes since a dispersion treatment would apparently have to consider nonzero momentum transfers as well as forward scatterings. In this case, a much more general form for $C_{\mu\nu}$ is required than that of Eq. (2.3). There are, in fact, twelve amplitudes rather than just four.

The absence of kinematical singularities in the amplitudes ImG and ImD can be shown as follows. We assume that the differential cross section given in Eq. (3.5) is finite for fixed, arbitrary k^2, r, s and all ω. Then, as is evident from this equation, any singularities in ImG and ImD could produce singularities in the cross section, in contradiction to our assumption.

A second proof is based on perturbation theory. We suppose we have written an arbitrary graph X contributing to $C_{\mu\nu}$ of Eq. (2.3). After integrating over any

[40] S. Coleman and S. L. Glashow, Phys. Rev. **134**, B671 (1964); S. Coleman and H. J. Schnitzer, *ibid.* **136**, B223 (1964).

virtual quanta which appear, we are left with a term which has the form (spinor invariant)$\times$(amplitude). We assume that the "(amplitude)" obeys a dispersion relation with no kinematical singularities. In order to cast X into the form given in Eq. (2.3) it is (possibly) necessary to combine this graph with others so as to obtain a gauge-invariant expression and (possibly) divide the resulting (amplitude) by a function of k^2 or ω, thus introducing a kinematical singularity, but obtaining the invariants I_1, I_2. Here we have put

$$I_1=\epsilon(\mu\nu k\gamma)\gamma_5\,,\quad I_2=\epsilon(\mu\nu kp)k\gamma_5\,.$$

Now we shall show that no division by functions of k^2 or ω is necessary and hence that in an arbitrary order of perturbation theory, there are no kinematical singularities in the amplitudes D and G. This proof is more general than the first one given above which held only for the imaginary parts of the amplitudes.

One simply writes down all possible terms which could appear in the (spinor invariant). Since only forward scattering is being considered, the only four-vectors which can appear are k, p, and the photon polarization which we shall temporarily denote by the four-vectors e, e'. For reasons which should be obvious by now, the following must hold: e and e' must appear linearly and antisymmetrically; $\boldsymbol{k}$ can appear at most once; $\boldsymbol{p}$ can be replaced by 1 and an invariant involving a single boldbace italic quantity; $2\boldsymbol{e}$, $2\boldsymbol{e}'$, $2\boldsymbol{k}$ can be replaced by a scalar quantity, $(e\cdot p)$, $(e'\cdot p)$, $(k\cdot p)$. Using these rules, it is quite easy to reduce the number of possible invariants to five:

$$\begin{aligned}
F_1&=(e\cdot ke'\cdot p-e\cdot pe'\cdot k)\,,\\
F_2&=[\boldsymbol{e},\boldsymbol{e}']\,,\\
F_3&=[\boldsymbol{e},\boldsymbol{e}']\boldsymbol{k}-[\boldsymbol{e},\boldsymbol{k}]\boldsymbol{e}'+[\boldsymbol{e}',\boldsymbol{k}]\boldsymbol{e}\,,\\
F_4&=e\cdot k[\boldsymbol{e}',\boldsymbol{k}]-e'\cdot k[\boldsymbol{e},\boldsymbol{k}]\,,\\
F_5&=e\cdot p[\boldsymbol{e}',\boldsymbol{k}]-e'\cdot p[\boldsymbol{e},\boldsymbol{k}]\,,
\end{aligned}$$

where $[a,b]\equiv ab-ba$. Using the fact that the matrix element must vanish if e or e' is replaced by k, one can show that F_1 cannot appear and that $F_2-k^2F_4-p\cdot kF_5=0$. Hence F_2 may be eliminated without introducing any kinematical singularities. It is a simple matter to obtain explicit expressions for F_3, F_4, and F_5 in terms of I_1 and I_2 in the laboratory frame and to verify that replacement of the F's by the I's introduces no kinematical singularities.

PHYSICAL REVIEW VOLUME 138, NUMBER 2B 26 APRIL 1965

Model for the First Nucleon Recurrence

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(Received 23 November 1964)

A calculation of the position and width of the $\frac{5}{2}^+$ (1688) pion-nucleon resonance is made. Inelastic unitarity is used to generate the dynamics where single-pion exchange coupling $N\pi$ to Nf states provides an approximation for the force. Good agreement with experiment is obtained. A similar mechanism for generating higher $N\pi$ resonances is suggested.

A FAMILY principle has been proposed[1] to hold among the resonant states in the πN system. In particular the $\frac{5}{2}^+$(1688) resonance, being a $T=\frac{1}{2}$ state, appears to be the first Regge recurrence of the nucleon. Carruthers[2] and Freedman[3] have considered models of nucleon recurrence based on single-channel unitarity. This note describes a dynamical model, based on inelastic unitarity,[4] which yields such a resonance and which suggests a mechanism yielding all recurrences of both the nucleon and the (3,3) isobar.

Our mechanism is a modification of the Cook-Lee model.[5] We invoke two-channel unitarity in which the coupled channels are $N\pi$ and Nf, where f denotes the 2^+ $(T=0)$ $\pi\pi$ resonance with mass $\sqrt{\sigma_f}=1250$ MeV.[6] We assume that the force is purely inelastic and is described by the π exchange diagram of Fig. 1(a). For πN states with positive parity, s waves can occur in the Nf system so that absorption is maximal, and the subsequent attraction can lead to elastic scattering resonances below the inelastic threshold. In the following we shall assume that s waves dominate in the Nf state and treat both $\frac{3}{2}^+$ and $\frac{5}{2}^+$ possibilities for the $N\pi$ system.

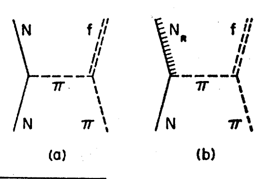

Fig. 1. π exchange diagrams coupling elastic and inelastic channels.

[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1962); R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).
[2] P. Carruthers, Phys. Rev. Letters **10**, 540 (1963).
[3] D. Z. Freedman, Phys. Rev. **134**, B652 (1964).
[4] R. Blankenbecler, Phys. Rev. **122**, 983 (1961).
[5] L. F. Cook and B. W. Lee, Phys. Rev. **127**, 283 (1962); **127**, 297 (1962). We denote the latter of these by CL hereafter.
[6] A. H. Rosenfeld, A. Barbaro-Galtieri, W. Barkas, P. Bastien, J. Kirz, and M. Roos, Rev. Mod. Phys. **36**, 977 (1964). References to the individual experimental papers may be found here.