Three-Nucleon Problem with Separable Potentials*

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The low-energy three-nucleon problem with two-particle separable potentials is solved. The formalism of Faddeev in recasting the three-body equations into Fredholm form and subsequent angular-momentum reductions are carried out. Upon searching for a bound state, two poles of the amplitude are found. Upon further examination, one of the poles is found to have a residue whose sign is opposite to those encountered in scattering with local field theory or with local potentials.

INTRODUCTION

RECENTLY there has been extensive work on recasting the equations for the three-body problem into a form that is tractable by numerical computation. Faddeev[1] has obtained a set of equations for this problem that is analogous to the Lippmann-Schwinger equations for two-body scattering, and whose kernel is such that standard Fredholm theory of integral equations may be applied. Weinberg[2] has extended these results to the n-body problem.

For the three-body case a further simplification has been obtained by Omnès[3] in performing an angular-momentum separation which is symmetric in the three particles involved.

If one were to take as the interaction between the particles a local potential, one would be faced with three coupled integral equations in two unknowns, which would have to be solved for a range of parameters. Although this problem is not insurmountable on present-day computers, a further simplification may be obtained by replacing the two-particle local potentials by simple separable ones. For two-body scattering separable potentials have been extensively discussed.[4] They have likewise been applied to the three-body problem.[5,6]

The specific problem treated in this article is the low-energy three-nucleon problem. Two-body separable potentials may be constructed which reproduce low-energy scattering parameters, such as scattering length and effective range. They may involve spin-spin, spin-orbit, or tensor forces. The physical transition matrix is the same as would be obtained from a Bargmann[7] potential; however, off the energy-momentum shell the amplitudes in the separable and local case are different. Nevertheless, it was felt that one would obtain a solution to the three-body problem not too different from one obtained through the use of local potentials.

Even further simplification may be obtained by restructing the spin dependence of the interaction to spin-spin forces, but not to include spin-orbit or tensor forces. In the latter case, the z component of the spin is not a good quantum number, and one has to keep track of the helicities of all the particles, making the problem quite complicated. If one treats the spin dependence by a spin-spin interaction, both the total spin and its z component are good quantum numbers and one may decouple the equations further. A formalism for projecting the Faddeev equations into states of total angular momentum, in the case of a generally spin-dependent potential, is presented in the Appendix, although no further use is made of it.

Section II is devoted to a discussion of the two-body potentials and Sec. III treats the integral equations for the three-body problem.

The results may be stated briefly as the problem presented an unphysical situation when a bound-state search was made. The three-body scattering amplitude was found to have two poles. One of these poles had a residue of the wrong sign and may be classified under the broad category of ghosts. It is felt that this situation is not due to the three-body nature of the problem but to the nonlocality of the potentials employed. These states may even occur in two-body scattering with separable potentials if the forces are strong enough to pull a bound state beyond the beginning of the left-hand cut. They have been extensively discussed in connection with the N/D equations. To remove the ghost, the forces have to be made sufficiently weak so as to produce the bound state with an energy of the order of 4 MeV as opposed to the experimental value of approximately 8.5 MeV. As this is a poor approximation to the triton, the calculation was not pursued further.

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TWO-BODY POTENTIALS

One of the purposes of this calculation was its possible future comparison to a treatment of the three-body problem using local interactions, which give the same two-particle physical scattering. A convenient set of local potentials whose S-wave part has an analytic solution are those of the Bargmann type.[7] The simplest Bargmann potential yields a phase shift which may be expressed by an effective-range formula valid at all energies,

$$k\cot\delta=-(1/a)+\tfrac{1}{2}rk^2. \tag{1}$$

Such a behavior of the phase shift may likewise be obtained from a separable potential of the form

$$\langle\mathbf{k}|V|\mathbf{q}\rangle=\frac{\lambda}{(k^2+\beta^2)^{1/2}(q^2+\beta^2)^{1/2}}. \tag{2}$$

The resultant two-body amplitude T (which is a solution of the Lippmann-Schwinger equations), keeping both initial and final momenta and the extended energy Z as independent variables, is

$$\langle\mathbf{k}|T(Z)|\mathbf{q}\rangle=\frac{\lambda}{(k^2+\beta^2)^{1/2}(q^2+\beta^2)^{1/2}}\times\frac{1}{1+2\pi^2\lambda[\beta-(-Z)^{1/2}]/(\beta^2+Z)}. \tag{3}$$

The physical T matrix is obtained by setting $|k|=|q|=(Z)^{1/2}$,[8]

$$T(k)=\frac{\lambda}{k^2+\beta^2+2\pi^2\lambda(\beta+ik)}. \tag{4}$$

The unitary amplitude $A(k)$, obtained from $T(k)$ by multiplying it by $-2\pi^2$, may be written as

$$A(k)=\left[-\left(\beta+\frac{\beta^2}{2\pi^2\lambda}\right)-\frac{k^2}{2\pi^2\lambda}-ik\right]^{-1}, \tag{5}$$

from which we note that the scattering length $a=(\beta+\beta^2/2\pi^2\lambda)^{-1}$ and $r=-1/\pi^2\lambda$.

The analytic structure of $A(k)$ in the complex k plane consists of two poles. One is at $k=i\beta$ and is the fixed pole representing the potential. The other pole moves as we vary λ; it is located at $k=-i(\beta+2\pi^2\lambda)$. As λ is made more and more negative, i.e., as the force becomes more attractive, the pole moves into the upper half-plane and represents a bound state. For $\lambda<-\beta/\pi^2$, this pole crosses the fixed pole and its residue changes sign. In a formal sense it still represents a bound state as we may assign a normalizable wave function to it, and it has a positive norm in Hilbert space. However, if we try to interpret the scattering as an exchange of this "bound state" between the incident and outgoing particles, as would be the case if we drew an analogy with field theory, we are forced to assign an imaginary coupling constant to this interaction. These states are referred to as ghosts.[9] Although the potential parameters were kept below the limit for the appearance of a two-body ghost, such a state did show up in the three-body problem.

[8] Units are such that $\hbar=M=1$.

In the three-nucleon problem we know that the forces are not purely central, but that there are spin-spin, spin-orbit, and tensor forces. As discussed in the introduction the inclusion at spin-orbit or tensor forces would complicate the calculation considerably. As a start only spin-spin forces were considered. The low-energy two-body scattering parameters may be represented with sufficient accuracy by such a potential. Restricting ourselves to this case the formalism of Ref. 3 may be taken over with minor modifications. The complications that would be inherent in a treatment with general spin dependence are presented in the Appendix.

The neutron-proton and neutron-neutron potentials are taken to be

$$\langle\mathbf{k}|V_{np}|\mathbf{q}\rangle=(\lambda_t^{np}P_t+\lambda_s^{np}P_s)/[(k^2+\beta^2)^{1/2}(q^2+\beta^2)^{1/2}],$$
$$\langle\mathbf{k}|V_{nn}|\mathbf{q}\rangle=\lambda_s^{nn}P_s/[(k^2+\beta^2)^{1/2}(q^2+\beta^2)^{1/2}], \tag{6}$$

where

$$P_t=(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)/4;\quad\text{triplet projection operator},$$
$$P_s=(1-\sigma_1\cdot\sigma_2)/4;\quad\text{singlet projection operator}. \tag{7}$$

λ_t^{np} and β were adjusted to give a triplet scattering length of 5.397 F and a deuteron binding energy of 2.2245 MeV.[10] λ_s^{nn} was chosen to give a neutron-neutron scattering length of -17.5 F. To obtain considerable simplification in the three-body problem, the range of the interactions β was taken the same in all cases. λ_s^{np} was adjusted to reproduce the singlet n-p scattering length of -23.679 F, and as we had no more freedom in varying the singlet range the effective range turned out to be 2.257 F as opposed to the experimental 2.459 F. It was felt that these potentials were a reasonable approximation to the low-energy two-body problem.

THREE-BODY SCATTERING EQUATIONS

Using the potentials discussed in the previous section, the three-body Hamiltonian is

$$H=\sum_{i=1}^{3}P_i^2/2+\sum_{i\neq j}V_{ij}, \tag{8}$$

where the V_{ij} have the form of Eq. (6). The total spin $S^2=(\sum\sigma_i/2)^2$ and $S^Z=\sum\sigma_i^Z/2$ commute with H.

[9] It may be amusing to note that if one tries to construct a Bargmann potential producing a ghost, one is forced to introduce a singularity into the potential.

[10] The two-body scattering parameters are due to H. Pierre Noyes, Nucl. Phys. (to be published).

Three spin-$\frac{1}{2}$ particles add up to one state of spin $\frac{3}{2}$ and two states of spin $\frac{1}{2}$. From now on we shall only concern ourselves with the spin-$\frac{1}{2}$ state. Let us denote by the subscripts 1, 2 the two neutrons and by 3 the proton. If $\alpha(\beta)$ is the eigenstate of σ^Z with eigenvalue $+1(-1)$, the two spin-$\frac{1}{2}$ states with $S^Z=\frac{1}{2}$ are

$$\begin{aligned} u&=(2\alpha_1\alpha_2\beta_3-(\alpha_1\beta_2+\beta_2\alpha_1)\alpha_3)/\sqrt{6}\,,\\ v&=(\alpha_1\beta_2-\alpha_2\beta_1)\alpha_3/\sqrt{2}\,. \end{aligned} \tag{9}$$

The total wave function of H has the form

$$\psi=\varphi u+\chi v\,, \tag{10}$$

where φ and χ are functions of the momenta of the particles and, owing to the Pauli principle, φ is antisymmetric in the interchange of 1 and 2, while χ is symmetric. Using u and v as basis spin states, the transition amplitude restricted to $S=\frac{1}{2}$ is a 2×2 matrix.

Following Faddeev[1] we decouple the transition amplitude into a sum of terms,

$$T=T_1+T_2+T_3\,. \tag{11}$$

Let $\hat{T}_{ij}$ denote the two-body scattering amplitude between particles i and j. The T_i satisfies the following set of equations:

$$T_i=\hat{T}_{jk}[1+G_0(T_j+T_k)]\,. \tag{12}$$

Let us remember that the T_i's and $\hat{T}_{ij}$'s are matrices in both Hilbert space spanned, say by the free-particle three-body states, and likewise matrices in the two-dimensional spin space. They are also functions of the extended energy Z. G_0 is the free-particle Green's function.

Writing Eq. (12) explicitly for say T_1 it is,

$$\begin{aligned} &\langle \mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3|T_1^{\mu\nu}(Z)|\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\rangle\\ &\quad=\langle \mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3|T_{23}^{\mu\lambda}(Z)|\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3'\rangle\\ &\qquad\times\{\delta(\mathbf{k}_1'-\mathbf{q}_1)\delta(\mathbf{k}_2'-\mathbf{q}_2)\delta(\mathbf{k}_3'-\mathbf{q}_3)\delta^{\lambda\nu}\\ &\qquad+(Z-(k_1'^2+k_2'^2+k_3'^2)/2)^{-1}\\ &\qquad\times\langle \mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3'|(T_2^{\lambda\nu}(Z)+T_3^{\lambda\nu}(Z))|\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\rangle\}\,. \end{aligned} \tag{13}$$

The superscript indicates spin-space coordinates.

At this point the method of Ref. 3 may be applied. Specializing to states of orbital momentum zero, all operators leave invariant a subspace of the Hilbert space spanned by vectors which we may label by the energies of the three particles, $\omega_i=k_i^2/2$. Equation (13) reduced to this subspace and using the potentials defined in Eq. (6), is

$$T_1(\boldsymbol{\omega},\boldsymbol{\omega}',Z)=\pi\left(\frac{2}{\omega_1}\right)^{1/2}\frac{R_{23}(Z-\omega_1)}{(\omega_2+\omega_3-\frac{1}{2}\omega_1+\beta^2)^{1/2}}\left\{\frac{\delta(\omega_1-\omega_1')}{(\omega_2'+\omega_3'-\frac{1}{2}\omega_1+\beta^2)^{1/2}}+\int\frac{d\boldsymbol{\omega}''\delta(\omega_1-\omega_1'')}{(\omega_2''+\omega_3''-\frac{1}{2}\omega_1+\beta^2)^{1/2}(Z-\omega_1-\omega_2-\omega_3)}(T_2(\boldsymbol{\omega}'',\boldsymbol{\omega}',Z)+T_3(\boldsymbol{\omega}'',\boldsymbol{\omega}',Z))\right\}\,, \tag{14}$$

where the notation $\boldsymbol{\omega}$ is shorthand for ω_1, ω_2, ω_3. The integration over $\boldsymbol{\omega}''$ is limited by the triangle inequality $|P_i''+P_j''|\rangle P_k''\rangle|P_i''-P_k''|$, and

$$R_{23}(Z)=\tfrac{1}{4}[\lambda_s^{np}/(1+2\pi^2\lambda_s^{np}(\beta-\sqrt{-Z})(\beta^2+Z)^{-1})+3\lambda_t^{np}/(1+2\pi^2\lambda_t^{np}(\beta-\sqrt{-Z})(\beta+Z)^{-1})]\,. \tag{15}$$

Analogous equations may be obtained for T_2 and T_3. Owing to the factorizability of the two-body amplitude we may read off from Eq. (14) the functional dependence of T_1 on ω_2 and ω_3. If we let

$$F_1(\omega_1,\boldsymbol{\omega}',Z)=\int\frac{d\boldsymbol{\omega}''\delta(\omega_1-\omega_1'')(T_2(\omega_1'',\omega_1',Z)+T_3(\omega_1'',\omega_1',Z))}{(\omega_2''+\omega_3''-\frac{1}{2}\omega_1+\beta^2)^{1/2}(Z-\omega_1-\omega_2''-\omega_3'')}\,, \tag{16}$$

we obtain

$$T_1(\boldsymbol{\omega},\boldsymbol{\omega}',Z)=\frac{\pi(2/\omega_1)^{1/2}R_{23}(Z-\omega_1)}{(\omega_2+\omega_3-\frac{1}{2}\omega_1+\beta^2)^{1/2}}\left\{\frac{\delta(\omega_1-\omega_1')}{(\omega_2'+\omega_3'-\frac{1}{2}\omega_1+\beta^2)^{1/2}}+F_1(\omega_1,\omega_1',Z)\right\}\,. \tag{17}$$

Substituting expressions of the type of Eq. (17) for T_2 and T_3 in terms of F_2 and F_3 into Eq. (16) we obtain six coupled integral equations in one variable. In searching for bound states we may set $\omega_1'=\omega_2'=\omega_3'$, and owing to the symmetry between 1 and 2 we may further simplify the problem and we are faced with three coupled equations.

These equations were solved numerically on the Stanford 7090 computer by means of matrix inversion. Matrices up to orders of 100×100 were employed.

The extended energy Z was varied below the n-d breakup energy. In this region all quantities are real. Two poles were found for the T matrix. Upon setting all the ω_1 and ω_1' equal to $Z/3$ the residues were determined. The residue of the less tightly bound pole was found to be negative, and in accordance with the discussion in the Introduction we refer to it as a ghost. One further check was made. It was observed that most of the three-body state was in the space symmetric wave function [χ of Eq. (10)]. A problem where the antisymmetric wave function was ignored was solved and compared to the full solution. Again two poles appeared at approximately the same position residues as above.

The results of Ref. 6(a) likewise showed two poles. The nature of the residues was not further discussed. In Ref. 6(b), which is a treatment analogous to 6(a) with the inclusion of spin, only one pole appears. The difference may be due to the different two-body separable potentials employed. The interactions used in Ref. 6 go beyond the effective range, and in some sense represent less attraction than the potentials employed in the present calculation. As mentioned previously, one of the purposes of this calculation is to compare it with a future calculation employing local two-body potentials which give the same two-body scattering amplitudes. It is felt that states with negative residues will not occur in the local potential calculations. If this continues to be true, it will indicate that the manner of taking the scattering amplitude off the energy shell is critical.

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APPENDIX

The three-body problem, recast into the Faddeev form, lends itself to a partial-wave decomposition even in the presence of arbitrary spin particles and general spin-dependent potentials. In the spinless case, as is shown in Ref. 3, through the use of linear and angular momentum integrals of motion we may reduce the problem to one involving three continuous variables and one discrete one. In the spin problem involving particles of spin S_1, S_2, S_3, we have to adjoin $\prod(2S_i+1)$ discrete variables. The notation of this Appendix is that of Ref. 3.

As a start we label each particle by its three-momentum $\mathbf{P}_i$ and helicity λ_i. The normalization is such that

$$\langle \mathbf{P},\lambda | \mathbf{P}',\lambda' \rangle = \delta(\mathbf{P}-\mathbf{P}')\delta_{\lambda\lambda'}, \quad \text{(A1)}$$

a two-body scattering matrix will have the form

$$\langle \mathbf{P}_i,\lambda_i,\mathbf{P}_j,\lambda_j | T_{ij}(Z) | \mathbf{P}_i',\lambda_i',\mathbf{P}_j',\lambda_j' \rangle = \delta(\mathbf{P}_i+\mathbf{P}_j-\mathbf{P}_i'-\mathbf{P}_j') \mathcal{T}_{ij}^{\lambda_i,\lambda_j,\lambda_i',\lambda_j'} (\mathbf{P}_i-\mathbf{P}_j;\, \mathbf{P}_i'-\mathbf{P}_j';\, Z). \quad \text{(A2)}$$

Let us transform to the center-of-mass frame of the three particles; $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ form a triangle. To specify the system completely we introduce the three energies ω_i, corresponding to the $\mathbf{P}_i$, a space-fixed set of axes and a body-fixed set of axes, with the Z axis along some fixed direction in the plane of the triangle, conveniently along one of the momenta. As a complete set of variables we introduce the total momentum $\mathbf{P}$, ω_1, ω_2, ω_3 (which when no ambiguity arises we label by $\boldsymbol{\omega}$), the total angular momentum J, the projection of J along the space-fixed axis M, its projection along the body-fixed axis M_1, and the helicities λ_i (which we likewise label $\boldsymbol{\lambda}$). The normalization is chosen as

$$\delta(\mathbf{P})\langle \mathbf{P},\boldsymbol{\omega},J,M,M_1,\boldsymbol{\lambda} | \mathbf{P}',\boldsymbol{\omega}',J',M',M_1',\boldsymbol{\lambda}' \rangle = \delta(\mathbf{P})\delta(\mathbf{P}')\delta(\boldsymbol{\omega}-\boldsymbol{\omega}')\delta_{J,J'},\delta_{M,M'},\delta_{M_1,M_1'},\delta_{\lambda,\lambda'} \quad \text{(A3)}$$

with the following metric in Hilbert space:

$$\sum_{(J,M,M_1,\lambda)} \int d^3P d\boldsymbol{\omega}. \quad \text{(A4)}$$

The $\boldsymbol{\omega}$ integration is limited by the triangle inequalities satisfied by the momenta. The transformation law between the old and new states is

$$\delta(\mathbf{P})\langle \mathbf{P},\boldsymbol{\omega},J,M,M_1,\boldsymbol{\lambda} | \mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\boldsymbol{\lambda}' \rangle = \left(\frac{2J+1}{8\pi^2 m_1m_2m_3}\right)^{1/2} \delta(\mathbf{P})\delta(\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3) \times \prod_{i=1}^{3} [\delta(\omega_i - P_i^2/2m_i)\delta_{\lambda_i,\lambda_i'}] D_{M_1,M}{}^J(\psi,\theta,\varphi). \quad \text{(A5)}$$

ψ, θ, φ are the Euler angles of the triangle. The coefficient in front of Eq. (A5) insures the normalization condition with respect to the metric Eq. (A4). From now on the procedure follows that of Ref. 3.

A word should be said about Eq. (A2). In general a scattering amplitude is given in the two-particle center-of-mass system, and the helicities are taken along $\mathbf{P}_i-\mathbf{P}_j$ and $\mathbf{P}_i'-\mathbf{P}_j'$.[11] If we denote by μ_i the helicity along $\mathbf{P}_i-\mathbf{P}_j$ and by θ the angle between $\mathbf{P}_i-\mathbf{P}_j$ and $\mathbf{P}_i'-\mathbf{P}_j'$, the amplitude is usually expressed as

$$T_{\mu_i,\mu_j,\mu_i',\mu_j'} = \sum(2J+1)\langle \mu_i,\mu_j | T^J | \mu_i',\mu_j' \rangle \times d_{\mu_i-\mu_j,\mu_i'-\mu_j'}{}^J(\theta) e^{i(\mu_i-\mu_j-\mu_i'+\mu_j')\varphi}. \quad \text{(A6)}$$

Let α_i be the angle between $\mathbf{P}_i$ and $\mathbf{P}_i-\mathbf{P}_j$; then

$$\mathcal{T}_{\lambda_i,\lambda_j,\lambda_i',\lambda_j'} = d_{\lambda_i,\mu_i}{}^{S_i}(-\alpha_i) d_{\lambda_j,\mu_j}{}^{S_j}(-\alpha_j) \times T_{\mu_i,\mu_j,\mu_i',\mu_j'} d_{\mu_i',\lambda_i'}{}^{S_i}(\alpha_i') d_{\mu_j',\mu_j'}{}^{S_j}(\alpha_j').$$

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