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Treatment of Neutron-Proton Pairing Correlations

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The short-range correlation between neutrons and protons in the configuration $(j)^N$ is discussed in the BCS approximation. The interaction is assumed to exist only in the $J=0$ and $T=1$ state of two nucleons. The ground- and excited-state energies are calculated and compared with the exact values.

1. INTRODUCTION

In this paper we study the short-range correlations between neutrons and protons, making use of the approximate methods of Bardeen, Cooper, and Schrieffer (BCS), Bogoliubov, and Valatin.[1] For the sake of simplicity we have considered a configuration of the type $(j)^N$, where the N nucleons (both neutrons and protons) occupy the same shell-model orbital j. The interaction Hamiltonian[2] we have chosen is a generalization of the "pairing Hamiltonian" to include isotopic spin. Such a Hamiltonian has been discussed by one of the authors[2] and by Flowers and Szpikowski,[3] where methods of group theory are used to obtain the energy spectrum and the classification of the states.

The main purpose of this paper is to rederive the results of Ref. 2 using the approximate methods of Bogoliubov and others.[1] Similar methods have been used by various authors[4] to discuss the correlation of neutrons and protons.

2. THE GROUND-STATE WAVE FUNCTION AND THE QUASIPARTICLE TRANSFORMATION

We will consider a system with equal numbers of neutrons and protons so that the total $T_z=0$. The pairing Hamiltonian is

$$H=-\tfrac{1}{4}g\sum_{\substack{m,m'\\ p+q=T_z=p'+q'\\ T_z}} s_{jm}s_{jm'}\langle\tfrac{1}{2}\,\tfrac{1}{2}pq|1T_z\rangle \times\langle\tfrac{1}{2}\,\tfrac{1}{2}p'q'|1T_z\rangle a_{mp}{}^\dagger a_{-mq}{}^\dagger a_{m'p'}a_{-m'q'}\equiv\textstyle\sum_{T_z}H(T_z);\qquad T_z=-1,0,1, \tag{2.1}$$

where $s_{jm}=(-1)^{j-m}$ and the subscripts p and q refer to the third component of the isotopic spin. $p=\frac{1}{2}(-\frac{1}{2})$ refers to a proton (neutron). The operators a_{mp} satisfy the usual anticommutation relations

$$\{a_{m'p'}{}^\dagger,a_{mp}\}=\delta_{mm'}\delta_{pp'}, \quad \{a_{mp},a_{m'p'}\}=0.$$

We assume the ground-state wave function for such a system to be of the BCS form:

$$\psi_{\text{g.s.}}=\prod_m(u_m+s_{jm}v_m a_{m1/2}{}^\dagger a_{-m-1/2}{}^\dagger)|0\rangle, \tag{2.2}$$

where the u_m's and v_m's have their usual meanings and $u_m{}^2+v_m{}^2=1$. The wave function (2.2) has total $T_z=0$, but it does not have a definite value for the total isotopic spin T. It is in fact a linear combination of wave function with different values of T. We now perform a canonical transformation from the operators a_{mp} to the quasiparticle operators α_{mp}, such that $\psi_{\text{g.s.}}$ is the vacuum for the quasiparticles.

$$\begin{pmatrix}\alpha_{m1/2}{}^\dagger\\ s_{jm}\alpha_{-m-1/2}\end{pmatrix}=\begin{pmatrix}u_m & -v_m\\ v_m & u_m\end{pmatrix}\begin{pmatrix}a_{m1/2}{}^\dagger\\ s_{jm}a_{-m-1/2}\end{pmatrix} \tag{2.3}$$

and

$$\alpha_{mp}|\psi_{\text{g.s.}}\rangle=0.$$

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957); N. N. Bogoliubov, Nuovo Cimento **7**, 794 (1958); J. G. Valatin, *ibid.* **7**, 843 (1958); B. R. Mottelson, in *The Many Body Problem*, edited by C. DeWitt (Dunod Cie., Paris, 1959).

[2] J. C. Parikh, Nucl. Phys. (to be published).

[3] B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. (London) **84**, 193 (1964); see also A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A214**, 515 (1952).

[4] B. Brémond and J. G. Valatin, Nucl. Phys. **41**, 640 (1963). M. Baranger, Phys. Rev. **130**, 1244 (1963); M. K. Pal and M. K. Banerjee, Phys. Letters **13**, 155 (1964); see also B. H. Flowers and M. Vujičić, Nucl. Phys. **49**, 586 (1963).

The inverse transformation is

$$\begin{pmatrix} a_{m1/2}{}^{\dagger} \\ s_{jm}a_{-m-1/2} \end{pmatrix} = \begin{pmatrix} u_m & v_m \\ -v_m & u_m \end{pmatrix} \begin{pmatrix} \alpha_{m1/2}{}^{\dagger} \\ s_{jm}\alpha_{-m-1/2} \end{pmatrix}. \quad (2.4)$$

Note that the charge is a good quantum number for the quasiparticles. The transformation (2.3) does not leave the number operator n diagonal and so we must add to the Hamiltonian a term λn, where λ is a Lagrange multiplier. λ is determined by the condition that the expectation value of n is equal to the total number of particles, i.e.,

$$\langle\psi_{\mathrm{g.s.}}|n|\psi_{\mathrm{g.s.}}\rangle = N. \quad (2.5)$$

The total number operator is

$$n = \sum_{mp} a_{mp}{}^{\dagger} a_{mp}. \quad (2.6)$$

Introducing the quasiparticle transformation in (2.6) and evaluating the matrix element $\langle\psi_{\mathrm{g.s.}}|n|\psi_{\mathrm{g.s.}}\rangle$ we get from (2.5)

$$2\sum_m v_m{}^2 = N. \quad (2.7)$$

For our simple Hamiltonian all the v_m's are equal and therefore

$$v_m = (N/4\Omega)^{1/2}, \quad u_m = (1-N/4\Omega)^{1/2}, \quad (2.8)$$

where

$$2\Omega = 2j+1.$$

In terms of the quasiparticle operators the Hamiltonian $H = \sum_{T_z} H(T_z)$ becomes

$$H(T_z=1) = -\tfrac{1}{4}g \sum_{m,m'} s_{jm}s_{jm'}[(u_m\alpha_{m1/2}{}^{\dagger} + v_m s_{jm}\alpha_{-m-1/2})(u_m\alpha_{-m-1/2}{}^{\dagger} - v_m s_{jm}\alpha_{m-1/2}) \times (u_{m'}\alpha_{m'1/2} + v_{m'}s_{jm'}\alpha_{-m'-1/2}{}^{\dagger})(u_{m'}\alpha_{-m'1/2} - v_{m'}s_{jm'}\alpha_{m'-1/2}{}^{\dagger})]. \quad (2.9)$$

A similar expression for $H(T_z=-1)$ is obtained by replacing $\frac{1}{2}$ by $-\frac{1}{2}$ and vice versa. Finally,

$$H(T_z=0) = -\tfrac{1}{2}g \sum_{m,m'} s_{jm}s_{jm'}[(u_m\alpha_{m1/2}{}^{\dagger} + v_m s_{jm}\alpha_{-m-1/2})(u_m\alpha_{-m-1/2}{}^{\dagger} - v_m s_{jm}\alpha_{m1/2}) \times (u_{m'}\alpha_{m'1/2} + v_{m'}s_{jm'}\alpha_{-m'-1/2}{}^{\dagger})(u_{m'}\alpha_{-m'-1/2} - v_{m'}s_{jm'}\alpha_{m'1/2}{}^{\dagger})]. \quad (2.10)$$

3. REDUCTION OF THE HAMILTONIAN

Since we wish to evaluate the diagonal matrix elements of H, the only terms in (2.9) and (2.10) that will contribute to the expectation value are the ones that conserve the number of quasiparticles. Following the usual notation we will call these terms $H_{22}(T_z)$. From (2.9) and (2.10) we obtain

$$H_{22}(T_z=1) + H_{22}(T_z=-1) = -\tfrac{1}{4}g \sum_{m,m'} s_{jm}s_{jm'}[u_m{}^2u_{m'}{}^2(\alpha_{m1/2}{}^{\dagger}\alpha_{-m1/2}{}^{\dagger}\alpha_{m'1/2}\alpha_{-m'1/2} + \alpha_{m-1/2}{}^{\dagger}\alpha_{-m-1/2}{}^{\dagger}\alpha_{m'-1/2}\alpha_{-m'-1/2})$$
$$+ v_m{}^2v_{m'}{}^2(\alpha_{m'1/2}\alpha_{-m'1/2}\alpha_{m1/2}{}^{\dagger}\alpha_{-m1/2}{}^{\dagger} + \alpha_{m'-1/2}\alpha_{-m'-1/2}\alpha_{m-1/2}{}^{\dagger}\alpha_{-m-1/2}{}^{\dagger})$$
$$+ 4u_mu_{m'}v_mv_{m'}s_{jm}s_{jm'}(\alpha_{m1/2}{}^{\dagger}\alpha_{m-1/2}\alpha_{m'1/2}\alpha_{m'-1/2}{}^{\dagger} + \alpha_{m'1/2}\alpha_{m'-1/2}{}^{\dagger}\alpha_{m1/2}{}^{\dagger}\alpha_{m-1/2})] \quad (3.1)$$

and

$$H_{22}(T_z=0) = -\tfrac{1}{2}g \sum_{mm'} s_{jm}s_{jm'}[u_m{}^2u_{m'}{}^2\alpha_{m1/2}{}^{\dagger}\alpha_{-m-1/2}{}^{\dagger}\alpha_{m'1/2}\alpha_{-m'-1/2}$$
$$+ v_m{}^2v_{m'}{}^2\alpha_{-m-1/2}\alpha_{m1/2}\alpha_{-m'-1/2}{}^{\dagger}\alpha_{m'1/2}{}^{\dagger} + u_mu_{m'}v_mv_{m'}s_{jm}s_{jm'}(\alpha_{m1/2}{}^{\dagger}\alpha_{m1/2}\alpha_{m'1/2}\alpha_{m'1/2}{}^{\dagger} - \alpha_{m1/2}{}^{\dagger}\alpha_{m1/2}\alpha_{-m'-1/2}{}^{\dagger}\alpha_{-m'-1/2}$$
$$- \alpha_{-m-1/2}\alpha_{-m-1/2}{}^{\dagger}\alpha_{m'1/2}\alpha_{m'1/2}{}^{\dagger} + \alpha_{-m-1/2}\alpha_{-m-1/2}{}^{\dagger}\alpha_{-m'-1/2}{}^{\dagger}\alpha_{-m'-1/2})]. \quad (3.2)$$

We will now evaluate the expectation value of H, in the ground state and the excited states of the system, and compare with the exact expression[2]

$$E(N,T,s,t) = \tfrac{1}{2}g[\tfrac{1}{4}(N-s)(4\Omega+6-N-s) - T(T+1) + t(t+1)]. \quad (3.3)$$

Equation (3.3) is the energy of an N-particle state with total isotopic spin T, seniority s, and reduced isotopic spin t. We now rewrite the sum $H_{22}(T_z=+1)+H_{22}(T_z=-1)$ in a suitable form, in order to facilitate the calculations, and also to show explicitly the occurrence of $T(T+1)$ and $t(t+1)$ terms in the energy.

The various terms in (3.1) can be grouped and, after some manipulations, we get

$$H_{22}(T_z=1) + H_{22}(T_z=-1) = -\tfrac{1}{4}g \sum_{m,m'} s_{jm}s_{jm'}(u^4+v^4)(\alpha_{m1/2}{}^{\dagger}\alpha_{-m1/2}{}^{\dagger}\alpha_{m'1/2}\alpha_{-m'1/2} + \alpha_{m'-1/2}\alpha_{-m'-1/2}\alpha_{m-1/2}{}^{\dagger}\alpha_{-m-1/2}{}^{\dagger})$$
$$- \tfrac{1}{2}g \sum_{m,m'} (u^2-v^2)^2\alpha_{m1/2}{}^{\dagger}\alpha_{m-1/2}\alpha_{m'-1/2}{}^{\dagger}\alpha_{m'1/2} + \tfrac{1}{2}g \sum_{m,m'} \alpha_{m1/2}{}^{\dagger}\alpha_{m-1/2}\alpha_{m'-1/2}{}^{\dagger}\alpha_{m'1/2}$$
$$+ g\sum_m (u^2\alpha_{m-1/2}{}^{\dagger}\alpha_{m-1/2} - v^2\alpha_{m1/2}{}^{\dagger}\alpha_{m1/2}) + g\Omega(v^2-u^2). \quad (3.4)$$

We have dropped the subscripts in u_m and v_m since they are independent of m. Now the three components of the total isotopic spin vector $\mathbf{T}$ are[2]

$$T^+=\sum_m a_{m1/2}{}^\dagger a_{m-1/2}\,,$$
$$T^-=\sum_m a_{m-1/2}{}^\dagger a_{m1/2}\,,$$
$$T^0=\tfrac{1}{2}\sum_m(a_{m1/2}{}^\dagger a_{m1/2}-a_{m-1/2}{}^\dagger a_{m-1/2})\,.$$

Substituting the transformation (2.4) in these operators we get for the product

$$T^+T^-=\sum_{m,m'}(u^2-v^2)^2\alpha_{m1/2}{}^\dagger\alpha_{m-1/2}\alpha_{m'-1/2}{}^\dagger\alpha_{m'1/2}$$
$$-\sum_{m,m'}s_{jm}s_{jm'}u^2v^2(\alpha_{m1/2}{}^\dagger\alpha_{m-1/2}{}^\dagger\alpha_{m'1/2}\alpha_{-m'1/2}+\alpha_{m'-1/2}\alpha_{-m'-1/2}\alpha_{m-1/2}{}^\dagger\alpha_{-m-1/2}{}^\dagger)\,, \quad (3.5)$$

where again we have retained only those terms which conserve the number of quasiparticles. Now since the charge is a good quantum number for the quasiparticles we can, in analogy with the operators T^+ and T^- above, write,

$$\sum_{m,m'}\alpha_{m1/2}{}^\dagger\alpha_{m-1/2}\alpha_{m'-1/2}{}^\dagger\alpha_{m'1/2}=t^+t^- \quad (3.6)$$

for the quasiparticles. t^+ and t^- are the components of the reduced isotopic spin vector $\mathbf{t}$. Then from (3.4), (3.5), and (3.6) we obtain

$$H_{22}(T_z=1)+H_{22}(T_z=-1)=-(\tfrac{1}{2}g)T^+T^-+(\tfrac{1}{2}g)t^+t^-$$
$$-\tfrac{1}{4}g\sum_{m,m'}s_{jm}s_{jm'}(\alpha_{m1/2}{}^\dagger\alpha_{-m1/2}{}^\dagger\alpha_{m'1/2}\alpha_{-m'1/2}+\alpha_{m-1/2}\alpha_{-m-1/2}\alpha_{m'-1/2}{}^\dagger\alpha_{-m'-1/2}{}^\dagger)$$
$$+g\sum_m(u^2\alpha_{m-1/2}{}^\dagger\alpha_{m-1/2}-v^2\alpha_{m1/2}{}^\dagger\alpha_{m1/2})+g\Omega(v^2-u^2)\,. \quad (3.7)$$

It is clear now that the first two terms in (3.7), when acting on a state with total isotopic spin T and reduced isotopic spin t, will give the required $T(T+1)$ and $t(t+1)$ terms.

4. CALCULATION OF ENERGY

The ground state of the system is the quasiparticle vacuum and the excited states are those in which quasiparticles are present. The state

$$\psi(N,s)=(\alpha_{m_1 1/2}{}^\dagger\alpha_{-m_1-1/2}{}^\dagger\alpha_{m_2 1/2}{}^\dagger \times\alpha_{-m_2-1/2}{}^\dagger\cdots\alpha_{m_{s/2}1/2}{}^\dagger\alpha_{-m_{s/2}-1/2}{}^\dagger)\psi_{\mathrm{g.s.}} \quad (4.1)$$

with s quasiparticles present, is a state of seniority s and if the isotopic spin of quasiparticles is coupled to $\mathbf{t}$, then $\mathbf{t}$ is the reduced isotopic spin. The ground-state wave function does not have a definite value for the isotopic spin, but a state with definite isotopic spin (say $\mathbf{T}'$) can be projected out of $\psi_{\mathrm{g.s.}}$ and then $\mathbf{t}$ and $\mathbf{T}'$ can be coupled to give the total isotopic spin $\mathbf{T}$. Then $\psi(N,T,s,t)$ is a wave function of the form (4.1) with the proper couplings to give the desired values of $\mathbf{T}$ and $\mathbf{t}$. From (3.2), (3.7), and (4.1) we get after a straightforward calculation

$$(\psi(N,T,s,t),H_{22}\psi(N,T,s,t))$$
$$=(\tfrac{1}{2}g)[\tfrac{1}{4}\{(N-s)(4\Omega-N-s)+6s(1-N/2\Omega)+\tfrac{3}{2}N^2/\Omega+4\Omega s(1-N/2\Omega)^2-s^2(1-N/2\Omega)^2\}-T(T+1)+t(t+1)]\,. \quad (4.2)$$

This result agrees with the exact expression (3.3) for $E(N,T,s,t)$ to within terms of order $1/\Omega$, so long as $s<N$.

5. DISCUSSION

We have thus shown that the wave function $\psi_{\mathrm{g.s.}}$ is a good approximation to the exact wave function for the simple model we have considered. These considerations can be easily generalized to the case of pairing interaction between particles in nondegenerate orbitals. It is to be emphasized that this wave function does not contain any explicit four-particle correlation. The necessity of such correlations has been discussed in recent literature.[4] Although we have considered only a limited problem, our results indicate that four-particle correlations may not be necessary in the general case.

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