Finite-Temperature Transport Properties of Normal Fermi Systems. I. The General Transport Equation

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We express the response of a normal Fermi system to a weak external charge in terms of a one-particle Wigner distribution function which obeys a generalized transport equation valid at all temperatures. The derivation of this transport equation is closely related to the method developed by Prigogine and co-workers in nonequilibrium situations, both classical and quantum; it is based on a reclassification of the perturbation expansion of the autocorrelation formula expressing the response. The proof is greatly facilitated by the use of a suitable diagram technique which describes both the dynamical processes in the system as well as the effect of the interactions on equilibrium properties; full advantage is taken of the contraction theorem for averages over the unperturbed equilibrium ensemble.

1. INTRODUCTION

During the last few years, normal Fermi systems have provoked much interest from the points of view of both equilibrium and nonequilibrium properties. Work in this field was mostly initiated by Landau[1] when he formulated his phenomenological description of Fermi liquids at zero temperature in terms of quasiparticles. Since then, much progress has been realized in the justification of this model starting from first principles, mainly by Landau himself[2] and by Luttinger and Nozières.[3]

The physical interest of this model is well known: He^3 at low temperature and electrons in metals, are strongly coupled systems which should behave in many ways like normal Fermi liquids. Also, from a purely theoretical aspect, a normal Fermi system offers almost the only example where the full apparatus of modern perturbation calculus may be developed and tested. It is thus of great importance to have a good understanding of the behavior of these systems.

In particular, the program which we will attack here has a threefold objective: What is the general relationship between a quasiparticle description and the more usual particle description for studying transport properties in low-temperature systems?[4] What is the connection between the Green's function technique and the usual transport equation approach? What is the finite-temperature generalization of Landau's model, if there is any?

It is known indeed that, at zero temperature at least, all physical properties of normal Fermi systems may be analyzed in terms of a quasiparticle distribution function (d.f.), which roughly speaking gives the number of dressed particles in a state of given momentum; as a matter of fact, this remark was the main key in the success of Landau's theory. On the other hand, usual nonequilibrium statistical mechanics describes transport properties of any system, weakly as well as strongly coupled, in terms of Wigner distribution functions which always refer to average occupation number of bare particles in states of given momentum. As far as we know, the connection between these two approaches has only been clarified for equilibrium properties.[5] Also the existing theories of transport in normal Fermi systems have mostly been developed using the Green's function technique, which is specially useful in the zero-temperature limit; there is however a large amount of work in quantum transport theory which is based on the derivation of kinetic equations[4,6] for the density matrix or for the above-mentioned Wigner distribution functions. Here again, there is no clear understanding of the relationship between the two methods.

Finally, it has to be stressed that, except for very particular cases,[7] existing theories always refer to nondissipative properties: Indeed, at zero temperature (excluding the case of scattering by impurities) transport coefficients like electrical conductivity, viscosity, etc., are not defined because the system has no dissipative mechanism. It is then of great interest to develop a nonequilibrium theory of Fermi systems at finite temperature, in order to be able to compute such transport coefficients.

In this paper, we shall derive the transport equation for the one-particle Wigner distribution function of a Fermi system submitted to an external test charge. The line of thought we shall follow is very similar to the one used in the general theory of nonequilibrium processes of Prigogine and co-workers, both for classical and quantum systems.[8] However, the mathematical technique we shall utilize is somewhat different because we want to benefit from the fact that we are interested in the simple case of the linear response to the external

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[1] L. D. Landau, Zh. Eksperim. i Teor. Fiz. 32, 59 (1957) [English transl.: Soviet Phys.—JETP 5, 101 (1957)].
[2] L. D. Landau, Zh. Eksperim. i Teor. Fiz. 35, 97 (1958) [English transl.: Soviet Phys.—JETP 8, 70 (1959)].
[3] P. Nozières and J. M. Luttinger, Phys. Rev. 127, 1423, 1431 (1962).
[4] See for instance, P. Résibois, Physica 27, 541 (1961).
[5] J. M. Luttinger, Phys. Rev. 119, 1853 (1960).
[6] L. Van Hove, Physica 23, 144 (1957).
[7] See, for instance, J. S. Langer, Phys. Rev. 127, 5 (1962).
[8] See for instance Ref. 4 and I. Prigogine, *Non-Equilibrium Statistical Mechanics* (Interscience Publishers Inc., New York, 1962).

perturbation; it is well known that the response of the system may then be expressed in terms of the autocorrelation of the current operators, averaged over the equilibrium distribution. We shall thus be allowed to use the familiar contraction theorems[9] for the mean value of creation and destruction operators while in the general theory of approach to equilibrium, the initial ensemble describing the system was arbitrary and no such theorems could be used.

In the first section, we present the general formulation; we perform a perturbation analysis of the autocorrelation formula valid for our particular problem and we decompose the elementary processes describing the dynamics of the system according to their time ordering. In the next paragraph, the well-known contraction rules are briefly recalled and a diagrammatic representation is developed for the perturbation expansion derived in the previous section; the diagrams we shall use are essentially the same as the one introduced previously by Fujita and co-workers[10] and we shall thus be very brief.

There is however an essential difference with Fujita's method in the correspondence rule between a diagram and a contribution: A diagram will correspond to a term of the perturbation series for which *the time ordering between the various elementary processes is fixed.* This point is very important because it will allow us to define in a unique way a collision process.

After these preliminaries, the main part of the work is given in Sec. 4, where an exact transport equation is derived for the one-particle Wigner distribution function. This equation is discussed and briefly compared with previously derived general transport equations.[11] We also give the limiting form of this equation when the external field is slowly varying both in time and space.

In order to illustrate the method, we analyze in Sec. 5 two very simple situations: the weakly coupled, spatially homogeneous gas and the collision-free, spatially inhomogeneous gas; we recover, respectively, the well-known Pauli and Vlassov equations. Of course, these trivial examples should only be considered as tests for the method we outline here; more complicated situations will be analyzed in further publications. Also, in Appendix C, we briefly summarize a simple model calculation which clearly shows the connection between the present Wigner distribution function method and the quasiparticle formalism, in the zero-temperature limit.

2. GENERAL FORMULATION

We consider an N-particle Fermi system enclosed in a box of volume Ω. We decompose the Hamiltonian into a kinetic part H_0 and the perturbation λV

$$H=H_0+\lambda V \tag{2.1}$$

with

$$H_0=\sum_k \epsilon_k a_k^\dagger a_k\,, \tag{2.2}$$

$$\lambda V=(\lambda/2\Omega)\times\sum_{klpr} v(k-r)a_k^\dagger a_l^\dagger a_p a_r \delta^{\mathrm{Kr}}(k+l-p-r)\,, \tag{2.3}$$

$$v(k-r)=\int dx\; V(x)\exp-i(k-r)x\,, \tag{2.4}$$

where $V(x)$ is the pair interaction between the particles.

In these formulas, we have used the well-known creation and destruction operators $a_k^\dagger$ and a_k obeying the anticommutation rules:

$$[a_k,a_{k'}^\dagger]_+=\delta_{kk'}^{\mathrm{Kr}}\,, \tag{2.5}$$

$$[a_k,a_{k'}]_+=[a_{k'}^\dagger,a_{k'}^\dagger]_+=0\,. \tag{2.6}$$

We assume that, at time zero, the system is at equilibrium; its density matrix

$$\rho^{\mathrm{eq}}=\exp(-\beta H)/\mathrm{Tr}\exp(-\beta H)\,. \tag{2.7}$$

We further suppose that all the particles have the charge e and we introduce as usual a continuous background in order to ensure electroneutrality. We then submit the system to a small external test charge

$$q^e(r,t)=er_q\exp-i(\omega t-qr) \tag{2.8}$$

which creates an electric field

$$E^\alpha\equiv E_{q\omega}{}^\alpha\exp[-i(\omega t-qr)] = -(4\pi i q^\alpha/q^2)r_q\exp[-i(\omega t-qr)]\,. \tag{2.9}$$

Applying the well-known linear response function formalism,[12] it is then easy to show that a current $\langle j_{q\omega}{}^\beta\rangle_t$ is created in the system; it may be written, in the limit of long times

$$\langle j_{q\omega}{}^\beta\rangle_t=\langle j_{q\omega}{}^\beta\rangle e^{-i\omega t}\,, \tag{2.10}$$

$$\langle j_{q\omega}{}^\beta\rangle=\frac{E_{q\omega}{}^\beta}{\Omega\,\mathrm{Tr}\exp(-\beta H)}\int_0^\infty d\tau\int_0^\beta d\gamma\exp(i\omega\tau)\times\mathrm{Tr}\{\exp(-\beta H)\hat{J}_{-q}{}^\beta(-i\gamma)\hat{J}_q{}^\alpha(\tau)\}\,. \tag{2.11}$$

The derivation of (2.11) is very simple and is reported in the appendix. The remarkable feature of this formula is of course that the long-time response is given by the average over the equilibrium distribution (2.7) of the autocorrelation of the current operator

$$\hat{J}_q{}^\alpha(\tau)=\exp(iH\tau)J_q{}^\alpha\exp(-iH\tau) \tag{2.12}$$

with

$$J_q{}^\alpha=e\sum_k k^\alpha a_{k+q/2}{}^\dagger a_{k-q/2}\,. \tag{2.13}$$

[9] C. Bloch and C. De Dominicis, Nucl. Phys. **7**, 459 (1958).
[10] S. Fujita and R. Abe, J. Math. Phys. **3**, 350 (1962). S. Fujita and F. Mayné, Physica **29**, 1201 (1963).
[11] O. Konstantinov and V. Perel, Zh. Eksperim. i Teor. Fiz. **39**, 197 (1960) [English transl.: Soviet Phys.—JETP **12**, 142 (1961)].
[12] P. Nozières and D. Pines, Nuovo Cimento **9**, 470 (1958).

However, (2.11) is still a purely formal expression because if involves the unitary operator of motion $\exp(-iH\tau)$ describing the dynamics of the N-particle system.

In order to get an explicit expression for $\langle j_{q\omega}{}^{\alpha}\rangle$, we shall thus have to analyze further this formula (2.11) by some adequate technique. Before proceeding in this direction, we want to make the following remarks:

(1) The model considered here is rather unrealistic; indeed a direct consequence of the assumptions that all particles have the same charge e is that, by conservation of momentum, the system has no mechanism to dissipate the current and the real part of the conductivity tensor

$$\sigma_{q\omega}{}^{\alpha\beta}=\langle j_{q\omega}{}^{\alpha}\rangle/E_{q\omega}{}^{\beta} \tag{2.14}$$

is infinite. It would however introduce no new feature in the theory—except a heavier notation—to take a two-component system (with charge e and $-e$) in which case we would get a dissipative part in (2.14).

(2) We have supposed that the particles are charged: there is thus a Coulombic part in the interaction $V(x)$; this in turn leads to divergence difficulties in perturbation calculus. As the procedure to eliminate these divergences is well known,[13] we shall not consider this point any further here and we shall merely assume that the potential $V(x)$ is short ranged.

(3) The two aforementioned difficulties would have been avoided by discussing thermal flows (like momentum flow) for neutral particles. Formulas similar to (2.11) have indeed been proposed for thermal transport properties. However, as we have shown recently for the classical case,[14] a satisfactory justification of their validity is rather involved; we preferred thus to limit ourselves to the simpler case of external disturbances. There would, however, be no new difficulty in discussing a formula of the type (2.11), with the current operator (2.13) replaced by any thermal-flow operator.

Let us now consider the perturbation expansion of (2.11). We first extend (2.11) to a grand ensemble with given chemical potential $\mu\equiv-\alpha/\beta$; we have

$$\langle j_{q\omega}{}^{\alpha}\rangle=\frac{E_{q\omega}{}^{\beta}}{\Xi\Omega}\int_0^{\infty}d\tau\int_0^{\beta}d\gamma\,\exp(i\omega\tau)\times\mathrm{Tr}\{\exp[-\beta(H-\mu N)]\hat{J}_{-q}{}^{\beta}(-i\gamma)\hat{J}_{q}{}^{\alpha}(\tau)\}\,, \tag{2.15}$$

where now

$$\Xi=\mathrm{Tr}\exp[-\beta(H-\mu N)]\,, \tag{2.16}$$

N being the operator for the total number of particles:

$$N=\textstyle\sum_k a_k{}^{\dagger}a_k\,. \tag{2.17}$$

We then use the well-known perturbation expansion

$$\exp(-iH\tau)=\exp(-iH_0\tau)\;U(\tau;0) \tag{2.18}$$

with

$$U(\tau;0)=\left[\sum_{n=0}^{\infty}\left(\frac{\lambda}{i}\right)^n\int_0^{\tau}d\tau_n\int_0^{\tau_n}\cdots\times\int_0^{\tau_2}d\tau_1\,V(\tau_n)\cdots V(\tau_1)\right] \tag{2.19}$$

and

$$V(\tau)=\exp(iH_0\tau)V\exp(-iH_0\tau)\,. \tag{2.20}$$

Similarly, we have

$$\exp(+iH\tau)=U(0;\tau)\exp(iH_0\tau)=U^{\dagger}(\tau;0)\exp(iH_0\tau)\,, \tag{2.21}$$

$$\exp(-\gamma H)=\exp(-\gamma H_0)U(-i\gamma;0)\,. \tag{2.22}$$

Inserting (2.18), (2.21), and (2.22) into (2.15), we obtain

$$\langle j_{q\omega}{}^{\alpha}\rangle=\frac{E_{q\omega}{}^{\beta}}{\Omega}\int_0^{\infty}dt\,\exp(i\omega t)\int_0^{\beta}d\gamma\,K^{\alpha\beta}(t,\gamma;\beta)\,, \tag{2.23}$$

where

$$\begin{aligned}K^{\alpha\beta}(t,\gamma;\beta)&=\{\mathrm{Tr}\exp[-(\beta H_0+\alpha N)]U(-i\beta;0)U(0;-i\gamma)\\&\quad\times\exp(\gamma H_0)J_{-q}{}^{\beta}\exp(-\gamma H_0)U(-i\gamma;0)U^{\dagger}(t;0)\\&\quad\times\exp(iH_0t)J_q{}^{\alpha}\exp(-iH_0t)U(t;0)\}/\\&\quad\times\{\mathrm{Tr}\exp[-(\beta H_0+\alpha N)]U(-i\beta;0)\}\,.\end{aligned} \tag{2.24}$$

If we define a current operator in interaction representation

$$J_q{}^{\alpha}(\tau)=\exp(iH_0\tau)J_q{}^{\alpha}\exp(-iH_0\tau) \tag{2.25}$$

and set for an arbitrary operator A

$$\langle A\rangle_0=(1/\Xi_0)\,\mathrm{Tr}\exp[-(\beta H_0+\alpha N)]A \tag{2.26}$$

we may write for (2.24)

$$K^{\alpha\beta}(t,\gamma;\beta)=\langle U(-i\beta;-i\gamma)J_{-q}{}^{\beta}(-i\gamma)U(-i\gamma;0)U(0;t)\times J_q{}^{\alpha}(t)U(t;0)\rangle_0/\langle U(-i\beta;0)\rangle_0\,. \tag{2.27}$$

If we put the expansion (2.19) into the numerator of (2.27), we obtain

$$\begin{aligned}\int_0^{\beta}d\gamma\,K^{\alpha\beta}(t,\gamma;\beta)&=\sum_{n,m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(-\lambda)^{n+m}\int_0^{\beta}d\gamma_{n+m+1}\int_0^{\gamma_{n+m+1}}\cdots\int_0^{\gamma_2}d\gamma_1\left(-\frac{\lambda}{i}\right)^p\int_0^t dt_p'\cdots\int_0^{t_2'}dt_1'\left(\frac{\lambda}{i}\right)^r\int_0^t dt_r\cdots\int_0^{t_2}dt_1\\&\quad\times\langle V(-i\gamma_{n+m+1})\cdots V(-i\gamma_{n+2})J_{-q}{}^{\beta}(-i\gamma_{n+1})V(-i\gamma_n)\cdots V(-i\gamma_1)\\&\quad\times V(t_1')\cdots V(t_p')J_q{}^{\alpha}(t)V(t_r)\cdots V(t_1)\rangle_0/\langle U(-i\beta;0)\rangle\,.\end{aligned} \tag{2.28}$$

[13] See for instance Ref. 3.
[14] P. Résibois, J. Chem. Phys. 41, 2979 (1964).

The order of the terms appearing in (2.28) should be noticed: Reading from right to left, we first have a series of r interactions classified in order of *increasing* times, a current operator taken at time t, another series of p interactions written in order of *decreasing* times, and finally a sequence of $n+m$ interactions taken at increasing temperatures with a current operator at γ_{n+1}. We see thus that in (2.28) we have no relative order between the first set of dynamical interactions and the following sequence of dynamical processes: They both run independently on the whole term scale $0 \rightarrow t$.

As was discussed in detail elsewhere,[15] there is a great advantage in classifying these two classes of events relative to each other; this is in fact the whole key for the definition of a collision process, as will appear clearly in Sec. 4. In order to realize such a classification, we notice the following identity for arbitrary functions $f_j(t)$:

$$\int_0^t dt_{n'}\cdots\int_0^{t_{2'}} dt_{1'}\int_0^t dt_m\cdots\int_0^{t_2} dt_1 f_{1'}(t_{1'})\cdots f_{n'}(t_{n'})f_m(t_m)\cdots f_1(t_1)$$
$$=\int_0^t dt_{n'+m}\int_0^{t_{n'+m}} dt_{n'+m-1}\cdots\int_0^{t_2} dt_1 \sum_{\mathcal{P}} \mathcal{P}\{f_{1'}(t_\alpha)\cdots f_{n'}(t_\beta)f_m(t_\gamma)\cdots f_1(t_\delta)\}\,, \quad (2.29)$$

where $\mathcal{P}$ represents all possible permutations of the arguments such that

$$t_\alpha<\cdots<t_\beta\,,\quad t_\gamma>\cdots>t_\delta$$

and such that $\alpha\cdots\beta\gamma\cdots\delta$ exhausts the set of indices $1, 2\cdots n'+m$. The proof of (2.29) is readily obtained by recurrence, for instance, we have for $n'=1$, $m=2$:

$$\int_0^t dt_{1'}\int_0^t dt_2\int_0^{t_2} dt_1 f_{1'}(t_{1'})f_2(t_2)f_1(t_1)$$
$$\equiv\int_0^t dt_3\int_0^{t_3} dt_2\int_0^{t_2} dt_1 \sum_{\mathcal{P}} \mathcal{P}\{f_{1'}(t_1)f_2(t_2)f_1(t_3)\}$$
$$\equiv\int_0^t dt_3\int_0^{t_3} dt_2\int_0^{t_2} dt_1\{f_{1'}(t_3)f_2(t_2)f_1(t_1)+f_{1'}(t_2)f_2(t_3)f_1(t_1)+f_{1'}(t_1)f_2(t_3)f_1(t_2)\}\,. \quad (2.30)$$

Other examples are discussed in Ref. 15.

As may be seen from (2.30), the use of this identity amounts to expressing two series of integrals running both from 0 to t as a sum of integrals *ordered along a single time scale 0 to t.*

We obtain from (2.28) and (2.29):

$$\int_0^\beta d\gamma\, K^{\alpha\beta}(t,\gamma;\beta)=\sum_{n,m=0}^{\infty}\sum_{p,r=0}^{\infty}(-\lambda)^{n+m}\left(-\frac{\lambda}{i}\right)^p\left(\frac{\lambda}{i}\right)^r\int_0^\beta d\gamma_{n+m+1}\cdots\int_0^{\gamma_2} d\gamma_1\sum_{\mathcal{P}}\int_0^t dt_{p+r}\cdots\int_0^{t_2} dt_1$$
$$\times\langle V(-i\gamma_{n+m+1})\cdots V(-i\gamma_{n+2})J_{-q}{}^{\beta}(-i\gamma_{n+1})V(-i\gamma_n)\cdots V(-i\gamma_1)$$
$$\times\mathcal{P}\{V(t_\alpha)\cdots V(t_\beta)J_q{}^{\alpha}(t)V(t_\gamma)\cdots V(t_\delta)\}\rangle_0/\langle U(-i\beta;0)\rangle_0\,. \quad (2.31)$$

This formula is the basis for the diagram technique we shall discuss presently.

3. DIAGRAM TECHNIQUE

As may be seen from the definition of $V(t)$ and $J_q{}^\alpha(t)$, the evaluation of (2.31) involves the calculation of grand canonical averages of the type

$$\langle a_k{}^{\pm}(-i\gamma_{n+m+1})\cdots a_l{}^{\pm}(-i\gamma_1)\cdots a_n{}^{\pm}(t_\alpha)\cdots \times a_m{}^{\pm}(t)\cdots a_r{}^{\pm}(t_\delta)\rangle_0\,, \quad (3.1)$$

where we have set, for t real or imaginary

$$a_l{}^+(t)=a_l{}^\dagger\exp(i\epsilon_l t)\,, \quad (3.2)$$

$$a_l{}^-(t)=a_l\exp(-i\epsilon_l t)\,. \quad (3.3)$$

As was shown by Bloch and De Dominicis,[16] averages of that sort are obtained by taking all possible systems of contraction of creation and destruction operators, the only nonvanishing contraction being

$$\langle a_k{}^+(t)a_k{}^-(t')\rangle_0=\exp[i\epsilon_k(t-t')]F_k{}^0\,, \quad (3.4)$$

$$\langle a_k{}^-(t')a_k{}^+(t)\rangle_0=\exp[i\epsilon_k(t-t')](1-F_k{}^0)\,, \quad (3.5)$$

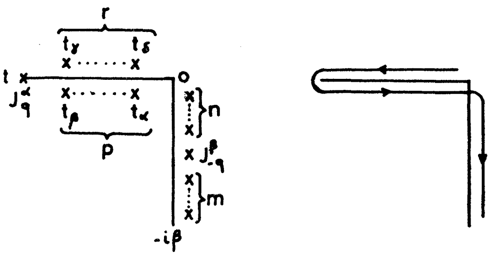
where we have introduced the unperturbed Fermi distribution

$$F_k{}^0=[\exp(\alpha+\beta\epsilon_k)+1]^{-1}\,. \quad (3.6)$$

Moreover, the proper sign has to be attached to each system of pairings, according to rules which will be made precise in the following.

[15] P. Résibois, Physica 29, 721 (1963).

[16] See Ref. 9.



(a) (b)

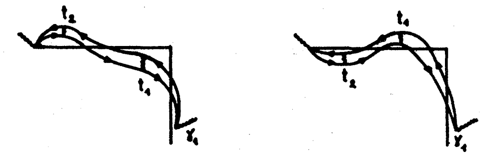
FIG. 1. Schematic description of a diagram. (a) Representation of the various vertices. (b) Ordering of the vertices in (2.31).

At this stage, it is most convenient to introduce a diagram technique in order to represent the various contributions to (2.31). To each interaction $V(\tau)$ (τ real or imaginary) as well as to the two current operators $J_q^\alpha(\tau)$ appearing in (2.31), we shall associate a *vertex* (provisionally represented by a cross). As may be seen from (3.4) and (3.5), the factor associated with a contraction depends upon the relative position of the creation and destruction operators which are averaged; let us use the following device:

(1) We draw r vertices, ordered from right to left and corresponding to $V(t_\delta)\cdots V(t_\gamma)$, above an horizontal axis 0–t; we then represent the operator $J_q^\alpha(t)$ by a vertex at time t [see Fig. 1(a)].

(2) We represent the p interactions $V(t_\beta)\cdots V(t_\alpha)$ by vertices below the horizontal axis and ordered from left to right.

(3) Finally, we draw the $(n+m)$ remaining interactions V along a vertical axis running from 0 to $-i\beta$, with a current operator inserted at $-i\gamma_{n+1}$.



(a) p=1, r=1, n=m=0, $t_2>t_1$ (b) p=1, r=1, n=m=0, $t_2>t_1$

FIG. 2. Examples of simple diagrams.

This procedure provides us with a rule for deciding whether a given vertex appears before or after another one in a given contribution of (2.31): it is easily checked that the various vertices of a given diagram are ordered in (2.31) according to their positions on the oriented line in Fig. 1(b).

With these definitions, we may now proceed according to the usual rules for writing down a diagram and its corresponding contribution: a vertex V will be represented from now on by two points joined by a wavy line expressing the transfer of momentum $k-r$ [see Eq. (2.3)]; a vertex J_q^α will be indicated by one point from which starts a wavy line with momentum q; a contraction is indicated by a line joining the two vertices where the operators a_k and $a_k^\dagger$ appear, with an arrow orienting the line from the creation operator to the destruction operator. A simple example is given in Fig. 2(a). Before proceeding further, we still have to stress one important point; in going from (2.29) to (2.31), we have decomposed each term of the initial series (2.28) in a sum of terms corresponding to different relative time orderings of the two sets of p and r dynamical interactions appearing in (2.28). For instance, we have written:

$$\int_0^t dt_1\int_0^t dt_1'\langle J_{-q}^\beta(-i\gamma_1)V(t_1')J_q^\alpha(t)V(t_1)\rangle_0$$
$$\equiv\int_0^t dt_1\int_0^{t_1} dt_2\langle J_{-q}^\beta(-i\gamma_1)V(t_1)J_q^\alpha(t)V(t_2)+J_{-q}^\beta(-i\gamma_1)V(t_2)J_q^\alpha(t)V(t_1)\rangle_0. \quad (3.7)$$

As we wish to associate a diagram to each contribution of (2.31), we shall be led, in this particular example, to consider the diagrams of Figs. 2(a) and 2(b) as different; they both correspond to a possible system of contractions, respectively, of the first and of the second term of the right-hand side of Eq. (3.7).

More generally, we shall consider as distinct the diagrams with different relative orderings between the vertices above and below the horizontal axis. This rule is the only one which differs from Fujita and co-workers' technique[17]; as we shall see later, it is, however, fundamental in order to derive a transport equation in a simple form.

Expression (2.31) may be considerably simplified if we use an appropriate "*linked cluster expansion theorem*[18]"; it is indeed a very simple matter—although rather long—to prove that the denominator $\langle U(-i\beta;0)\rangle_0$ of (2.31) is precisely cancelled by the sum of all disconnected diagrams in the numerator; we are then left with vacuum-vacuum connected diagrams, i.e.:

$$\int_0^\beta K^{\alpha\beta}(t,\gamma;\beta)d\gamma=\sum_{n,m=0}^\infty\sum_{p,r=0}^\infty(-\lambda)^{n+m}\left(-\frac{\lambda}{i}\right)^p\left(\frac{\lambda}{i}\right)^r\int_0^\beta d\gamma_{n+m+1}\cdots\int_0^{\gamma_2}d\gamma_1\sum_{\mathcal{P}}\int_0^t dt_{p+r}\cdots\int_0^{t_2}dt_1$$
$$\times\langle V(-i\gamma_{n+m+1})\cdots V(-i\gamma_{n+2})J_{-q}^\beta(-i\gamma_{n+1})V(-i\gamma_n)\cdots V(-i\gamma_1)\mathcal{P}\{V(t_\alpha)\cdots V(t_\beta)J_q^\alpha(t)V(t_\gamma)\cdots V(t_\delta)\}\rangle_{0,c}. \quad (3.8)$$

[17] See Ref. 10.
[18] See Ref. 9 as well as P. Nozières, "*Le Problème à N Corps*" (Dunod et Cie, Paris, 1963).

From (2.23) and (3.8), one obtains readily the rules which allow to calculate the current $\langle j_{q\omega}{}^{\alpha}\rangle$ to an arbitrary order in the coupling constant; we first draw *all distinct connected vacuum-vacuum diagrams*, involving an arbitrary number of interaction vertices and two current operators, one, $J_q{}^{\alpha}$, at time t and the second, $J_{-q}{}^{\beta}$, at time $-i\gamma_{n+1}$. The contribution to $\langle j_{q\omega}{}^{\alpha}\rangle$ of a given diagram is then obtained through the following *rules I*:

(1) To each interaction vertex, associate a factor

$$(1/2\Omega)v(k-r)\delta^{\mathrm{Kr}}(k+l-p-r).$$

(2) To the vertices $J_q{}^{\alpha}$ and $J_{-q}{}^{\beta}$ associate, respectively, a factor ek^{α} and ek^{β}.

(3) Label the vertices along the horizontal axis by $1, 2, \cdots p+r$ in the order where they appear starting from $t=0$; in this labeling, no account is taken from the fact that a vertex is above or below this horizontal axis; similarly, label the vertices along the temperature axis by $1, 2, \cdots n, n+2, \cdots n+m+1$ in order of increasing temperature; the current operator is labeled with an index $n+1$.

(4) To a line with momentum k, associate a factor

$$\exp[i\epsilon_k(t_\alpha-t_\beta)](1-F_k{}^0)\,, \tag{3.9}$$

if the line is parallel to the contour of Fig. 1(b), and

$$(-)\exp[i\epsilon_k(t_\alpha-t_\beta)]F_k{}^0\,, \tag{3.9'}$$

if the line is antiparallel to the contour of Fig. 1(b). In (3.9) and (3.9′), the times t_α and t_β (real or imaginary) refer to the label of the vertices where particle k is respectively created and absorbed.

(5) Integrate over

$$\int_0^\beta d\gamma_{n+m+1}\cdots\int_0^{\gamma_2}d\gamma_1\int_0^\infty e^{i\omega t}dt\int_0^t dt_{p+r}\cdots\int_0^{t_2}dt_1\,. \tag{3.10}$$

(6) Multiply by

$$g=(-\lambda)^{n+m}(-\lambda/i)^p(\lambda/i)^r(-1)^L s\,, \tag{3.11}$$

where L is the number of closed loops and s the symmetry factor of the diagram (see especially Ref. 18).

(7) Sum over all momenta.

(8) Multiply by $E_{q\omega}{}^{\beta}/\Omega$.

For instance, after expressing all conservation of moments, the diagram of Fig. 2(a) corresponds to

$$D_{2a}=\frac{E_{q\omega}{}^{\beta}}{\Omega}2^2(-1)^1\left(\frac{\lambda}{i}\right)\left(-\frac{\lambda}{i}\right)\sum_{k,l,l'}\int_0^\beta d\gamma_1\int_0^\infty dt\,e^{i\omega t}\int_0^t dt_2\int_0^{t_2}dt_1(k+l+l')^{\alpha}k^{\beta}\frac{v(l)}{2\Omega}\frac{v(l')}{2\Omega}$$
$$\times(1-F_{k+l+l'}{}^0)(-F_{k+l+l'+q}{}^0)(1-F_{k+l+q}{}^0)(1-F_{k+q}{}^0)(-F_k{}^0)(-F_{k+l}{}^0)\exp[i\epsilon_{k+l-l'}(t_2-t)]$$
$$\times\exp[i\epsilon_{k+l+l'-q}(t-t_2)]\exp[i\epsilon_{k+l+q}(t_2-t_1)]\exp[i\epsilon_{k+q}(t_1+i\gamma_1)]\exp[i\epsilon_k(-i\gamma_1-t_2)]\exp[i\epsilon_{k+l}(t_1-t_2)]\,. \tag{3.12}$$

As is seen from this example, the explicit expression associated with a given diagram becomes rapidly complicated and it is often much simpler to work with diagrams. However, the rules given above may be somewhat simplified if we reorganize properly the integrals over times and temperatures. Let us first consider the case of (3.12); we may write the time-temperature-dependent factors as

$$\int_0^\beta d\gamma_1\int_0^\infty dt\,\exp(i\omega t)\int_0^t dt_2\int_0^{t_2}dt_1\exp[-i(\epsilon_{k+l+l'}-\epsilon_{k+l+l'+q})(t-t_2)]$$
$$\times\exp[-i(\epsilon_{k+l}-\epsilon_{k+l+q})(t_2-t_1)]\exp[-i(\epsilon_k-\epsilon_{k+q})(t_1-0)]\exp[-(\epsilon_{k+q}-\epsilon_k)(\gamma_1-0)]\,. \tag{3.13}$$

In this way, we have expressed these integrals as convolution (both in time and in temperature) involving the "lifetimes" $(t-t_2)$, (t_2-t_1), (t_1-0), (γ_1-0) of each intermediate state; the energy factor associated with each of these intermediate states is just the difference in energy between the particles propagating in the forward direction (i.e., 0 to t or 0 to $-i\beta$) and in the backward direction.

More generally, it is easily verified that the time-temperature factor associated with an arbitrary diagram may always be expressed as

$$\int_0^\infty e^{i\omega t}dt\int_0^t dt_{p+r}\cdots\int_0^{t_2}dt_1\int_0^\beta d\gamma_{n+m+1}\cdots\int_0^{\gamma_2}d\gamma_1\exp[-i\Delta\epsilon_{p+r+1}(t-t_{p+r})]\exp[-i\Delta\epsilon_{p+r}(t_{p+r}-t_{p+r-1})]\cdots$$
$$\times\exp[-i\Delta\epsilon_1(t_1-0)]\exp[-i\Delta\epsilon_{n+m+1}{}'(\gamma_{n+m+1}-\gamma_{n+m})]\cdots\exp[-i\Delta\epsilon_1{}'(\gamma_1-0)]\,, \tag{3.14}$$

where we have introduced the notation

$$\Delta\epsilon_i=\sum_k \nu_k{}^i\epsilon_k\,, \tag{3.15}$$

which represents the energy difference in the ith intermediate state on the horizontal axis [i.e., (t_i-t_{i-1})]; we have $\nu_k{}^i=+1$ if particle k is propagating in the forward direction, $\nu_k{}^i=-1$ if particle k is propagating in the backward direction, and $\nu_k{}^i=0$ if state k is not excited (see Fig. 3). Similarly,

$$\Delta\epsilon'_j=\sum \nu'_k{}^j\epsilon_k \tag{3.16}$$

is the energy difference in the jth intermediate state on the temperature axis [i.e. $(\gamma_j-\gamma_{j-1})$] with: $\nu'_k{}^j=+1$ if particle k propagates from 0 to $-i\beta$, $\nu'_k{}^j=-1$ if particle k propagates from $-i\beta$ to 0, $\nu'_k{}^j=0$ if state k is not excited (see Fig. 3). In order to apply (3.15) and (3.16) correctly, one has to keep in mind that all the dynamical events (along the horizontal axis) take place at $-i\gamma=0$, while all temperature-dependent effects occur at time $t=0$. As is seen in the example of Fig. 3, a line may participate *both* to $\Delta\epsilon_i$ and to $\Delta\epsilon'_j$. It is now a simple matter to perform formally the time integrations in (3.14); introducing a convergence factor $\omega\to\omega+i\alpha$ in order to give a meaning to this expression we have indeed

$$\int_0^\infty \exp[i(\omega+i\alpha)t]dt\int_0^t dt_{p+r}\cdots\int_0^{t_2} dt_1 \exp[-i\Delta\epsilon_{p+r+1}(t-t_{p+r})]\cdots\exp[-i\Delta\epsilon_1(t_1-0)]$$

$$=\prod_{j=1}^{p+r+1}\int_0^\infty \exp[-(\Delta\epsilon_j-\omega-i\alpha)\tau_j]d\tau_j$$

$$=\prod_{j=1}^{p+r+1}\frac{1}{i(\Delta\epsilon_j-\omega-i\alpha)}. \tag{3.17}$$

It is of course also possible to calculate explicitly the temperature-dependent part, but we shall not need it here.

Using (3.15) and (3.17), we may reformulate the rules for calculating the contribution of a given diagram.

Rules I': (1') To each interaction, associate a factor $(1/2\Omega)v(k-r)\delta^{\mathrm{Kr}}(k+l-p-r)$.

(2') To the vertex $J_{-q}{}^\beta$, associate a factor ek^β [and a factor ek^α to $J_q{}^\alpha$].

(3') To a line with momentum k, associate a factor $(1-F_k{}^0)$, if the line is parallel to the contour of Fig. 2(b), or $(-F_k{}^0)$, if it is antiparallel to this contour.

(4') For each intermediate state j in the range 0 to t, write a factor

$$1/i(\Delta\epsilon_j-\omega-i\alpha)\,, \tag{3.18}$$

where $\Delta\epsilon_j$ is defined by (3.15).

(5') For each intermediate state j in the range 0 to $-i\beta$, write a factor $\exp[-i\Delta\epsilon_j'(\gamma_j-\gamma_{j-1})]$, where $\Delta\epsilon_j'$ is defined by (3.16), except for $\Delta\epsilon'=0$ when $\gamma_j=\beta$.

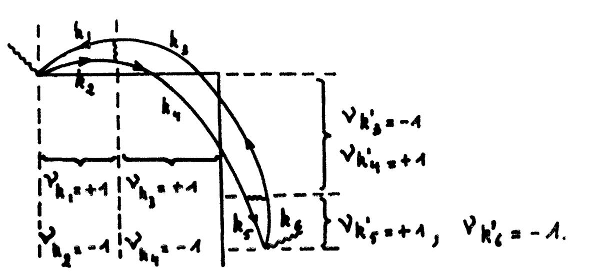


FIG. 3. A diagram with the corresponding factors ν_k and ν'_k.

(6') Multiply by $g=(-\lambda)^{n+m}(-\lambda/i)^p(\lambda/i)^r(-1)^Ls$.

(7') Integrate over the temperatures

$$\int_0^\beta d\gamma_{n+m+1}\cdots\int_0^{\gamma_2} d\gamma_1.$$

(8') Sum over all momenta [including k^α].

(9') Multiply by $E_{q\omega}{}^\beta$ [and by Ω^{-1}].

4. THE GENERALIZED TRANSPORT EQUATION

From (2.11) and (2.13), we may write

$$\langle j_{q\omega}{}^\alpha\rangle=\Omega^{-1}\sum_k ek^\alpha f_{q\omega}(k) \tag{4.1}$$

where

$$f_{q\omega}(k)=\frac{E_{q\omega}{}^\beta}{\Xi}\int_0^\infty dt\int_0^\beta d\gamma\,\exp(i\omega t)$$

$$\times\mathrm{Tr}\{[\exp-(\beta H+\alpha N)]\hat{J}_{-q}{}^\beta(-i\lambda)$$

$$\times[\exp(-iHt)a_{k+q/2}{}^\dagger a_{k-q/2}\exp(iHt)]\}\,. \tag{4.2}$$

As is physically clear, $f_{q\omega}(k)$ is nothing but the Fourier-Laplace transform of the Wigner distribution function of our Fermi system submitted to the external field (2.9). We thus expect this function $f_{q\omega}(k)$ to obey a transport equation which generalizes to normal Fermi systems transport equations derived previously for classical systems[19]: We shall try to obtain an integral

[19] R. Balescu, Physica **27**, 693 (1961). P. Résibois, in *Many Particle Physics*, edited by E. Meeron (Gordon and Breach, New York, to be published).

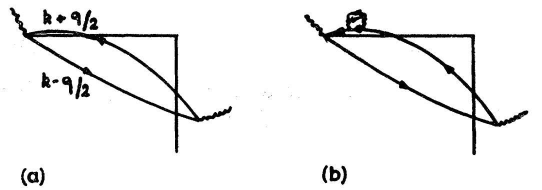


FIG. 4. Examples of normal contributions. (a) The λ^0 diagram. (b) A second-order collision process.

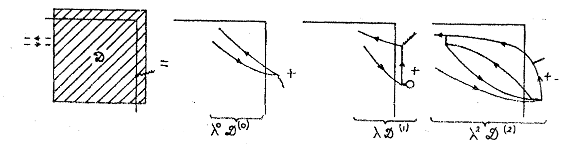


FIG. 6. The destruction region: examples.

equation for $f_{q\omega}(k)$ expressing the balancing between the collision processes on the one hand and the free flow plus acceleration due to the external field on the other hand. As usual, this equation will be derived through an adequate classification of the diagrams contributing to $f_{q\omega}(k)$. These diagrams are the same as the one discussed in the previous section for computing $\langle j_{q\omega}{}^{\alpha}\rangle$; as a matter of fact, it is immediately seen from (4.1) that the rules for obtaining $f_{q\omega}(k)$ are the ones given at the end of Sec. 3 (Rules I′), except that the bracketed prescriptions should be omitted.

Our classification of diagrams will be based on the generalization of the concept of collision operator, which was used extensively in our previous analysis of irreversible phenomena in quantum spatially homogeneous systems (see Refs. 4 and 15); in this latter case, the collision operator was defined as the most general transition bringing the system from a state $\Delta\epsilon_1 \equiv 0$ to a state $\Delta\epsilon_2 \equiv 0$, with no intermediate state having this property (irreducibility condition). In the present formulation, each intermediate states implies an energy denominator [see (3.18)]

$$\Delta\epsilon_j - \omega\,. \tag{4.3}$$

Of course, for finite frequency and wave number (ω, $q\neq 0$), this quantity will never vanish identically; for instance, the simplest possible diagram (λ^0) represented in Fig. 4(a), involves the factor:

$$\Delta\epsilon - \omega = \epsilon_{k+q/2} - \epsilon_{k-q/2} - \omega \neq 0\,. \tag{4.4}$$

However, in the limit ω, $q\rightarrow 0$, this energy difference becomes identically zero for all k; the general criterion we shall use for defining a collision process is that it is *a transition process such that in the limit* q, $\omega\rightarrow 0$, *we have an initial state and a final state with* $\Delta\epsilon \equiv 0$, *while all intermediate states are such that* $\Delta\epsilon_j \neq 0$. An example is given in Fig. 4(b). It is easy to check that

$$\lim_{q,\omega\to 0} \Delta\epsilon_1 = \lim_{q,\omega\to 0} \Delta\epsilon_3 = 0\,. \tag{4.5}$$

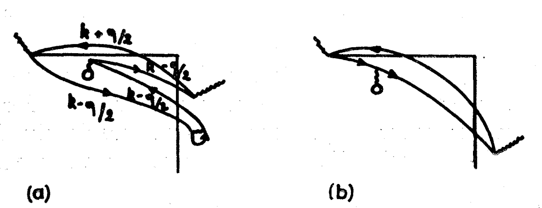


FIG. 5. Example of an anomalous contribution and its normal skeleton.

From (4.4), it is of course very tempting to say that the most general collision process involves a transition from a particle-hole state ($k+q/2$, $k-q/2$, with $V_{k+q/2} = +1$ and $V_{k-q/2} = -1$) to a similar state ($k'+q/2$, $k'-q/2$). These contributions, which we shall call *normal*, do not, however, exhaust all possibilities; indeed, one has also to take into account *anomalous contributions* where we have in addition an arbitrary number of pairs with the same momentum and opposite energy: these supplementary pairs do not alter the energy denominators (4.4). An example is given in Fig. 5(a). These anomalous diagrams pose a somewhat tricky problem because the number of anomalous pairs is in principle arbitrary and makes it difficult to obtain an integral equation for $f_{q\omega}(k)$. However, the following simplifies the matter considerably.

Theorem I

(a) the only anomalous graphs which contribute to $f_{q\omega}(k)$ are such that the interactions on these anomalous lines are purely temperature-dependent (i.e., along the vertical axis 0, $-i\beta$).

(b) Moreover the sum of these graphs will properly be accounted for if we retain only normal graphs and replace everywhere the unperturbed Fermi factor (3.6) by the *exact* momentum distribution:

$$\langle F_k\rangle = (1/\Xi)\,\mathrm{Tr}\hat{n}_k \exp[-(\beta H + \alpha N)]\,. \tag{4.6}$$

The proof of this theorem is rather long and has been reported in Appendix B. However, its physical content is quite clear. Let us consider a particle which is perturbed by the external field. As we are working in a linear theory, this particle only interacts with particles at equilibrium. These fermions at equilibrium are however not described correctly by (3.6) in the presence of interaction, but rather by the exact distribution (4.6) The role of the anomalous diagrams is precisely to "renormalize" the fermion distribution to its correct value (4.6). Using the second part of the theorem, we shall thus limit ourselves to normal diagrams, replacing everywhere $F_k{}^0 \rightarrow \langle F_k\rangle$; in Appendix B, we shall indi-

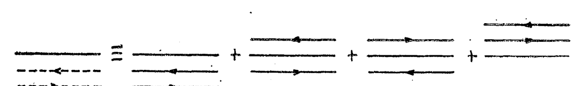


FIG. 7. The meaning of the dashed lines.

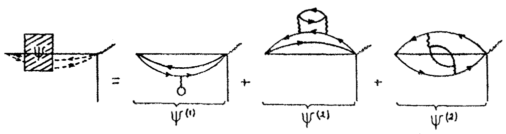


FIG. 8. Examples of diagonal fragments.

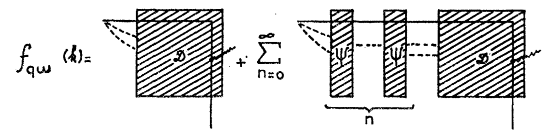


FIG. 9. Schematic expansion for $f_{q\omega}(k)$.

cate how a perturbation expansion for this $\langle F_k\rangle$ may be obtained.

The most general normal graph contributing to $f_{q\omega}(k)$ is then decomposed in a *destruction region* and a sequence of *diagonal fragments*.

The destruction region is the region of the diagram starting at $t=0$ and *including thus all temperature-dependent interactions* $0\rightarrow -i\beta$, and ending with a pair particle-hole state. A few simple destruction diagrams are indicated in Fig. 6. In general the destruction region of Fig. 6, we have represented the outgoing lines by dashed lines, which correspond to the four possibilities of Fig. 7. Indeed, it is readily verified from the rules given previously that the statistical factor associated with a line only depends on the position of the vertex from which it starts (i.e., in the "box "$\mathfrak{D}$) and not at all on the vertex where it ends. In this way, the box $\mathfrak{D}$ is a completely defined entity, independent of the remaining part of the diagram.

A diagonal fragment is the most general transition from a pair-hole state to another pair-hole state; it will be represented by a box ψ (see Fig. 8) and, for normal diagrams, which we consider here, it only involves dynamical interactions; indeed all temperature-dependent processes have been included in the destruction region. In order to be able to apply in a simple fashion the rules I′ of Sec. 3 for evaluating a diagonal fragment, we shall always close the corresponding diagram by a current vertex taken at temperature $-i\gamma=0$; the contribution associated with a diagonal fragment will then be obtained from the rules I′ *omitting points* (2′), (5′), (7′), *and* (9′) *altogether*. Examples of d.f. are given in Fig. 8 and the correspondence between a graph and its contribution is exemplified in Sec. 5. With these definitions, we represent thus $f_{q\omega}(k)$ by the schematic equation of Fig. 9. It is now a simple matter to obtain a transport equation for $f_{q\omega}(k)$; we have just to express our results in analytical form.

We define a "destruction" function $\mathfrak{D}_{q\omega}(k;\beta)$ and a collision operator $\psi_{q\omega}(k,k';\beta)$ through the following equations:

$$[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega-i\alpha)]^{-1}\mathfrak{D}_{q\omega}(k;\beta)=\sum\nolimits^{\mathfrak{D}}[\text{all destruction diagrams of } f_{q\omega}(k)] \tag{4.7}$$

$$\begin{aligned}&[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega-i\alpha)]^{-1}[i(\epsilon_{k'+q/2}-\epsilon_{k'-q/2}-\omega-i\alpha)]^{-1}\psi_{q\omega}(kk';\beta)\\&\quad\equiv-\sum\nolimits_{\psi}(\text{all diagonal fragments } k\rightarrow k',\ \textit{excluding}\text{ the statistical factors of the ingoing lines } k'+q/2,\ k'-q/2).\end{aligned} \tag{4.8}$$

The minus sign in front of this latter expression is justified because whenever we factorize a diagonal fragment we introduce a supplementary closed loop which is not present in the expansion of Fig. 9; also the statistical factors corresponding to the ingoing lines $(k'+q/2,\ k'-q/2)$ have to be suppressed because they are already counted in the destruction region.

We have thus

$$\begin{aligned}f_{q\omega}(k)=&[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega-i\alpha)]^{-1}\mathfrak{D}_{q\omega}(k;\beta)\\&+\sum_{k'}[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega-i\alpha)]^{-1}\psi_{q\omega}(k,k';\beta)[i(\epsilon_{k'+q/2}-\epsilon_{k'-q/2}-\omega-i\alpha)]^{-1}\mathfrak{D}_{q\omega}(k;\beta)+\cdots,\end{aligned} \tag{4.9}$$

or, in integral form

$$f_{q\omega}(k)=[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega-i\alpha)]^{-1}\mathfrak{D}_{q\omega}(k;\beta)+\sum_{k'}[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega-i\alpha)]^{-1}\psi_{q\omega}(k,k';\beta)f_{q\omega}(k'). \tag{4.10}$$

Multiplying on both sides by $[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega-i\alpha)]^{-1}$ and taking into account that we are interested in values of q which are small with respect to the Fermi momentum, we obtain

$$i(k^\alpha q^\alpha-\omega)f_{q\omega}(k)-\mathfrak{D}_{q\omega}(k;\beta)=\sum_{k'}\psi_{q\omega}(k,k';\beta)f_{q\omega}(k'). \tag{4.11}$$

From the rules given above, it is clear that $\mathfrak{D}_{q\omega}$ is a linear functional in the external field; we express this explicitly in writing

$$\mathfrak{D}_{q\omega}(k;\beta)=\mathcal{G}_{q\omega}{}^{\alpha}(k;\beta)E_{q\omega}{}^{\alpha} \tag{4.12}$$

which is a definition for the function $\mathcal{G}_{q\omega}{}^{\alpha}(k;\beta)$. Also, we may take into account that, in the limit of small q

and $\omega(q/k_F \ll 1; \omega/\epsilon_F \ll 1)$ we have,

$$\psi_{q\omega}(k,k';\beta)=\psi_{00}(k,k';\beta)+\omega\frac{\partial\psi_{q\omega}(k,k';\beta)}{\partial\omega}\bigg|_{0,0}+q^\alpha\frac{\partial\psi_{q\omega}(k,k';\beta)}{\partial q^\alpha}\bigg|_{0,0}+\cdots, \quad (4.13)$$

$$\mathcal{G}_{q\omega}{}^\alpha(k;\beta)=\mathcal{G}_{00}{}^\alpha(k;\beta)+\cdots. \quad (4.14)$$

The zero wave number and frequency operator $\psi_{00}(k,k';\beta)$ is, in this particular problem, identical to the linearized form of the asymptotic collision operator in a spatially homogeneous system, introduced by the author in the analysis of the approach to equilibrium.[20] We shall not prove this identity here, but we just want to mention that this operator has the equilibrium distribution function $\langle F_k\rangle$ as eigenfunction with eigenvalue zero. Also, in (4.14), we have limited ourselves to the lowest order approximation for $\mathcal{G}_{q\omega}{}^\alpha$ because, as may easily be checked, higher order contributions only give rise to negligible corrections to the one-particle d.f. $f_{q\omega}(k)$.

Inserting (4.13) and (4.14) into (4.11) we get the final result:

$$i(k^\alpha q^\alpha-\omega)f_{q\omega}(k)-\sum_{k'}\left(\omega\frac{\partial\psi_{q\omega}}{\partial\omega}\bigg|_{0,0}+q^\alpha\frac{\partial\psi_{q\omega}}{\partial q^\alpha}\bigg|_{0,0}\right)f_{q\omega}(k')-\mathcal{G}_{00}{}^\alpha(k;\beta)E_{q\omega}{}^\alpha=\sum_{k'}\psi_{00}(k,k';\beta)f_{q\omega}(k'). \quad (4.15)$$

This is the final expression for our transport equation of a Fermi system submitted to a weak external field. The physical interpertation of (4.15) is quite clear and may be given in complete analogy with the classical case[21]:

The first two terms of the left-hand side represents the free motion of the particles, with wave number q and frequency ω.

The next two terms, involving the derivative of $\psi_{q\omega}$, express the modification of the latter free motion due to the interactions; one sees indeed that these terms give rise to a "shift" in frequency and momentum. We want to come back with greater detail to these renormalization effects in a future publication, and we shall thus not comment about them any further here.

The last term on the left-hand side describes the acceleration of the particles by the external field; it generalizes to an interacting system the well-known free-particle expression[22]:

$$\beta eF_k{}^0(1-F_k{}^0)k^\alpha E_{q\omega}{}^\alpha. \quad (4.16)$$

It involves the interference effect between the collision processes and the external field. This term will also be discussed with great detail in future publications.

[20] See Ref. 4.
[21] See Ref. 19.
[22] R. Peierls, *Quantum Theory of Solids* (Oxford University Press, New York, 1955).

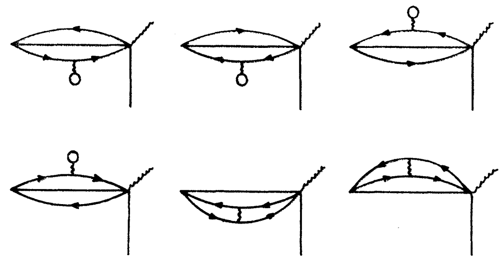


FIG. 10. The diagrams contributing to the Vlassov equation.

Finally, the right-hand side of (4.15) represents the usual collision term; it describes all dissipative effects in a finite temperature system and is responsible for the real part of the transport coefficients.

We would like to stress that an equation of the type (4.11) was derived previously by Konstantinov and Percel (see Ref. 11) using a technique rather similar to the one we have presented here. However, their definitions of the various quantities $\mathfrak{D}_{q\omega}(k;\beta)$, $\psi_{q\omega}(k,k';\beta)$ did not take in consideration the delicate question of anomalous diagrams, which play a very important role in the discussion of strongly coupled systems. This gives rise to divergence difficulties in the limit q, $\omega\to 0$ and makes it very difficult to apply their formalism to specific problems.

5. SIMPLE APPLICATIONS

In order to illustrate the method, we shall consider here two simple situations where Eq. (4.15) reduces to well-known results: The weakly interacting collision-free gas (quantum Vlassov equation) and the weakly coupled gas submitted to a stationary spatially homogeneous external field (Pauli transport equation). Of course, these cases may be treated by much more elementary methods and will serve only as examples.

In order to derive the Vlassov equation, we limit ourselves to the λ^0 approximation for describing the destruction region which is given by Fig. 4(a). Applying the rules of Sec. 3 and the definition (4.7), we obtain:

$$\mathfrak{D}_{00}{}^{(0)}(k;\beta)=(-1)e\int_0^\beta d\gamma(-F_k{}^0)(1-F_k{}^0)k^\alpha E_{q\omega}{}^\alpha \quad (5.1)$$

and thus

$$\mathcal{G}_{00}{}^{(0)\alpha}(k;\beta)=e\beta F_k{}^0(1-F_k{}^0)k^\alpha. \quad (5.2)$$

The diagonal fragment is approximated by the six λ contributions of Fig. 10 (we shall not write down the exchange terms explicitly). It is an easy matter to write down all these contributions and from (4.8), we obtain

$$\lambda\psi_{q\omega}{}^{(1)}(k,k';\beta)=-(\lambda/i)[-V_{\text{H.F.}}(k+q/2)+V_{\text{H.F.}}(k-q/2)]\delta_{kk'}{}^{kr}-(\lambda/i)[v(k-k')-v(0)](F_{k-q/2}{}^0-F_{k+q/2}{}^0), \quad (5.3)$$

where we have used the short-hand notation

$$V_{\mathrm{H.F.}}(k)=-\sum_{k'}[v(0)-v(k-k')](-F_{k'}{}^0). \quad (5.4)$$

It is easily verified in (5.3) that $\psi_{00}{}^{(1)}$ vanishes; expanding in q and ω, we get

$$\partial\psi_{q\omega}{}^{(1)}/\partial\omega=0, \quad (5.5)$$

$$\partial\psi_{q\omega}{}^{(1)}/\partial q^\alpha=-i\lambda(\partial V_{\mathrm{H.F.}}/\partial k^\alpha)\delta_{kk'}{}^{\mathrm{Kr}} +i\lambda[v(k-k')-v(0)]k^\alpha\beta F_k{}^0(1-F_k{}^0). \quad (5.6)$$

Inserting (5.2), (5.5), and (5.6) into (4.15), we obtain

$$-i\omega f_{q\omega}(k)+iq^\alpha\left[k^\alpha+\frac{\partial V_{\mathrm{H.F.}}}{\partial k^\alpha}\right]f_{q\omega}(k) -iq^\alpha k^\alpha\beta F_k{}^0(1-F_k{}^0)\sum_{k'}[v(k-k')-v(0)]f_{q\omega}(k') -e\beta F_k{}^0(1-F_k{}^0)k^\alpha E_{q\omega}{}^\alpha=0. \quad (5.7)$$

This equation is the low-temperature generalization of the Vlassov equation; indeed, in the classical limit, we may neglect the exchange term in (5.7) and we also assume

$$1\gg F_k{}^0=\exp-(\alpha+\beta k^2/2). \quad (5.8)$$

We obtain then

$$-i\omega f_{q\omega}(k)+iq^\alpha k^\alpha f_{q\omega}(k) -iq^\alpha v(0)(\partial F_k{}^0/\partial k^\alpha)\sum_{k'}f_{q\omega}(k') -eE_{q\omega}{}^\alpha(\partial F_k{}^0/\partial k^\alpha)=0 \quad (5.9)$$

which is nothing else than the Fourier transform (valid for small q) of the well-known Vlassov equation in an external field. Another interesting limit is the zero temperature case; neglecting the Hartree-Fock term in (5.7), which merely amounts to a redefinition of the energy of the free particles, and realizing that

$$\lim_{\beta\to\infty}\beta F_k{}^0(1-F_k{}^0)=\delta(\epsilon_k{}^0-\mu), \quad (5.10)$$

we obtain

$$-i\omega f_{q\omega}(k)+iq^\alpha k^\alpha[f_{q\omega}(k)+\delta(\epsilon_k{}^0-\mu)\sum_{k'}F_{kk'}f_{q\omega}(k')] =e\delta(\epsilon_k{}^0-\mu)k^\alpha E_{q\omega}{}^\alpha \quad (5.11)$$

with

$$F_{kk'}=v(k-k')-v(0). \quad (5.12)$$

This equation is the weak-coupling version of the Landau equation for Fermi quasiparticles.[23] However, it should not be inferred from this result that the function $f_{q\omega}(k)$ may be identified with the Landau quasiparticle distribution function $n_{q\omega}(k)$. Indeed, the equivalence obtained here is only valid in the lowest order and

[23] See Ref. 1.

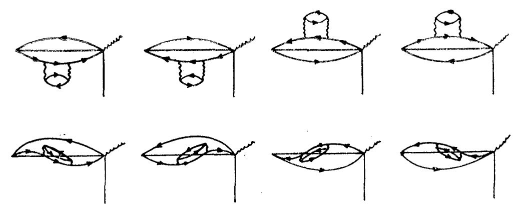


FIG. 11. The eight diagrams contributing to $\psi_{00}{}^{(2)}(k,k;\beta)$.

differences appear already at order λ^2, as we shall discuss in the next paper of this series (see Ref. 26 and App. C).

The second example corresponds to a weakly coupled homogeneous static system: Setting q, $\omega=0$ in (4.15), we obtain

$$-\mathcal{G}_{00}{}^\alpha(k,\beta)E_{00}{}^\alpha=\sum_{k'}\psi_{00}(k,k';\beta)f_{00}(k'). \quad (5.13)$$

The field term in the left-hand side of (5.13) is treated as in (5.2); the collision operator is treated in lowest order, which for an homogeneous system is λ^2 (Born approximation). As an example, we have indicated in Fig. 11 the eight diagrams (without exchange) corresponding to the direct term $k=k'$ in $\psi_{00}(k,k';\beta)$. The evaluation of these various contributions is easy although rather lengthy. For instance, the first diagram of Fig. 11, is associated with the contributions

$$C_{11a}=(-1)^2\left(-\frac{\lambda}{i}\right)^2\sum_{l,s}|v(l)|^2\frac{1}{i(\epsilon_s+\epsilon_k-\epsilon_{k-l}-\epsilon_{s+l}-i\alpha)} \times(1-F_k{}^0)(1-F_{k-l}{}^0)(1-F_{s+l}{}^0)(-F_s{}^0). \quad (5.14)$$

In order to obtain this result, one has to keep in mind that, by definition, the diagonal fragment does *not* include the current vertex operator on the temperature axis; also, by (4.8), its contribution is obtained by omitting the outgoing and the ingoing propagators as well as the statistical factors associated with the two ingoing pair hole k.

The other diagrams of Fig. 11 are computed similarly, with the result:

$$\lambda^2\psi_{00}(k,k;\beta)f_{q\omega}(k) =-2\pi\lambda^2\sum_{l,s}|v(l)|^2\delta(\epsilon_k+\epsilon_l-\epsilon_{k-l}-\epsilon_{s+l}) \times\{(1-F_s{}^0)F_{s+l}{}^0F_{k-l}{}^0 +(1-F_{k-l}{}^0)(1-F_{s+l}{}^0)F_s{}^0\}f_{00}(k). \quad (5.15)$$

In order to get the final answer, one still has to consider diagrams with $k'\neq k$, as typical examples are shown in Fig. 12. From a practical point of view, it should be pointed out that in drawing normal diagonal fragments, one is not obliged to place the dynamical vertices close to the time axis although they are all taken at temperature $\beta=0$; indeed, no temperature-dependent factor

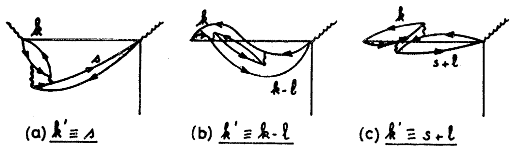


FIG. 12. Typical contribution to $\psi_{00}^{(2)}(k,k';\beta)$ $(k\neq k')$.

appears in the definition of a diagonal fragment and the current operator on the temperature axis in Figs. 11 and 12 is only introduced in order to make sure that the diagram is closed; in other words, the ordinate of the dynamical vertices with respect to the temperature axis is completely irrelevant in this case. The case of the destruction region is more delicate and should be treated with great care.

When these supplementary diagrams are evaluated, the following result is obtained:

$$\lambda^2 \sum_{k'} \psi_{00}^{(2)}(k,k';\beta) f_{00}(k')$$
$$= 2\pi\lambda^2 \mathfrak{L} \sum_{l,s} |v(l)+\text{exch}|^2 \{(1-f_k)(1-f_s)f_{k-l}f_{s+l} - (1-f_{k-l})(1-f_{s+l})f_k f_s\} \times \delta(\epsilon_k+\epsilon_s-\epsilon_{k-l}-\epsilon_{s+l}), \quad (5.16)$$

where we have set

$$f_k = F_k^0 + f_{00}(k) \quad (5.17)$$

and $\mathfrak{L}$ is a linearization operator which only keeps in (5.16) the terms which are linear in the deviations from equilibrium (i.e., proportional to f_{00}).

Inserting (5.2) and (5.16) into (5.13), we obtain the usual linearized transport equation for a weakly coupled gas in an external field.

From the two examples treated in this section, one realizes the great complexity which arises in the explicit evaluation of the various terms in the transport equation (4.15), even in the weak coupling case. This important drawback is common to all diagram techniques developed in the N-body problem and is still more apparent here because we have both a dynamical problem (along the time axis) and a purely statistical problem (along the temperature axis). However, as is usual, many important results may be obtained by considering very simple models; also, it may be hoped that many properties of low-temperature normal Fermi systems depend on the topological structure of the diagrams that contribute to their evolution and not on the explicit evaluation of these diagrams. It is known for instance that this is the case at zero temperatures as Landau, Luttinger, and Nozières have shown, using Green's function technique. Work in those two directions is now in progress and we hope to report about it soon. As a preliminary step in this direction, we discuss briefly in Appendix C the zero-temperature limit of our basic transport equation (4.15) in a very simple but nontrivial model and compare it to the Landau equation in the same approximation.

APPENDIX A. RESPONSE TO AN EXTERNAL CHARGE

For the sake of completeness, let us briefly derive (2.11). In the presence of the test charge (2.8), the Hamiltonian is

$$H_T = H + \mathcal{H}_t', \quad (A1)$$

where

$$\mathcal{H}_t' = \sum_i e\mathcal{V}(r_i,t) \quad (A2)$$

with

$$\mathcal{V}(r,t) = (4\pi e/q^2) r_q \exp[i(qr-\omega t)]. \quad (A3)$$

We may write

$$\mathcal{H}_t' = \mathcal{H}' \exp(-i\omega t), \quad (A4)$$
$$\mathcal{H}' = (4\pi e^2/q^2) r_q \rho_{-q}, \quad (A5)$$

where we have introduced the density fluctuation

$$\rho_q = \sum_i \exp(-iqr_i). \quad (A6)$$

The Von Neumann equation is

$$i\partial_t\rho = [H,\rho], \quad (\hbar = 1), \quad (A7)$$

and if we linearize the density matrix around equilibrium

$$\rho = \rho^{\text{eq}} + \Delta\rho, \quad (A8)$$

we obtain the well-known result

$$\Delta\rho(t) = \frac{1}{i}\int_0^t \exp[-iH(t-t')][\mathcal{H}_t,\rho^{\text{eq}}] \times \exp[iH(t-t')]dt'. \quad (A9)$$

We write the current per unit volume as

$$\langle j_{q\omega}^\alpha \rangle_t = (1/\Omega)\, \text{Tr} J_q^\alpha \Delta\rho_t \equiv \langle j_{q\omega}^\alpha \rangle e^{-i\omega t}, \quad (A10)$$

where J_q^α is the current operator

$$J_q^\alpha = (e/2)\sum_i (p_i e^{-iqr_i} + e^{-iqr_i} p_i). \quad (A11)$$

From (A9) and (A10), we obtain in the limit $t\to\infty$:

$$\langle j_{q\omega}^\alpha \rangle = \frac{1}{i\Omega}\int_0^\infty d\tau \exp(i\omega\tau) \times \text{Tr}\{J_q^\alpha \exp(-iH\tau)[\mathcal{H}',\rho^{\text{eq}}]\exp(iH\tau)\}. \quad (A12)$$

We now use the identities[24]

$$[\mathcal{H}',\rho^{\text{eq}}] = (4\pi e^2 r_q/q^2)[\rho_{-q},\rho^{\text{eq}}]$$
$$= -\frac{4\pi e^2 r_q}{q^2}\frac{\exp(-\beta H)}{Z}\int_0^\beta d\gamma \times \exp(\gamma H)[\rho_{-q},H]\exp(-\gamma H) \quad (A13)$$

[24] R. Kubo, J. Phys. Soc. Japan 12, 570 (1957).

and

$$[\rho_{-q},H]=-q^\beta J_{-q}^\beta/e. \qquad (A14)$$

Inserting (A13) and (A14), we get for (A12):

$$\langle j_{q\omega}{}^\alpha\rangle=-\frac{4\pi i r_q q^\beta}{q^2\Omega}\int_0^\infty d\tau\,\exp(i\omega\tau)\int_0^\beta d\gamma$$
$$\times\mathrm{Tr}\{J_q{}^\alpha\rho^{\mathrm{eq}}[\exp(-iH\tau)][\exp(\gamma H)]J_{-q}{}^\beta$$
$$\times[\exp(-\gamma H)]\exp(iH\tau)\}. \qquad (A15)$$

Using (2.9) and the cyclic invariance of the trace, we recover (2.11).

APPENDIX B. THE ANOMALOUS DIAGRAMS

We want to prove that the contributions of the anomalous diagrams are generated by the normal graphs if we replace in rule I′ the contraction $F_k{}^0$ by $\langle F_k\rangle$ as given by (4.6). As a preliminary result, let us consider

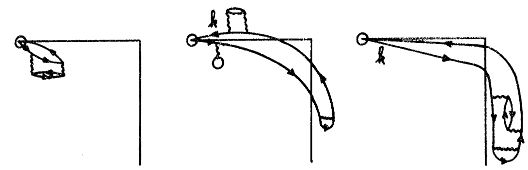


FIG. 13. Contributions to $\langle F_k(t)\rangle$.

the following average:

$$\langle F_k(t)\rangle=\mathrm{Tr}\{\exp(iH\tau)\hat{n}_k\exp(-iH\tau)\exp-\beta H\}/\Xi, \quad (B1)$$
$$\equiv\langle U(-i\beta;0)U(0,t)\hat{n}_kU(t,0)\rangle_{0,c}, \qquad (B2)$$

where we have used (2.21), (2.26), and the linked cluster expansion theorem.

However, we have also, using the cyclic invariance of the trace:

$$\langle F_k(t)\rangle=\langle F_k\rangle=\langle U(-i\beta;0)\hat{n}_k\rangle_{0,c}. \qquad (B3)$$

If we expand (B2) as was done for $\langle j_{q\omega}{}^\alpha\rangle$, we obtain

$$\langle F_k(t)\rangle=\sum_{n=0}^\infty\sum_{p=0}^\infty\sum_{r=0}^\infty(-\lambda)^n\left(-\frac{\lambda}{i}\right)^p\left(\frac{\lambda}{i}\right)^r\int_0^\beta d\gamma_n\cdots\int_0^{\gamma_2}d\gamma_1\sum_{\mathcal{P}}\int_0^t dt_{p+r}\cdots\int_0^{t_2}dt_1$$
$$\times\mathcal{P}\langle V(-i\gamma_n)\cdots V(-i\gamma_1)\{V(t_\alpha)\cdots V(t_\beta)\hat{n}_kV(t_\gamma)\cdots V(t_\delta)\}\rangle_{0,c}. \quad (B4)$$

The expansion for (B3) is similar to (B4) with the restriction that $p=r=0$; we conclude thus that the *sum* of all graphs in (B4) having a dynamical component (p, $r\neq0$) vanishes while the purely temperature-dependent part gives the correct equilibrium distribution $\langle F_k\rangle$.

The contributions to (B4) are obtained by drawing all connected graphs with an arbitrary number of dynamical interactions (above and below the time axis) and an arbitrary number of temperature-dependent interactions (along the vertical axis), with one supplementary vertex, indicated by a circle, taken at time t and corresponding to $\hat{n}_k=a_k{}^\dagger a_k$. Examples are given in Fig. 13. All these graphs are of the anomalous type because a pair particle-hole (k,k) is absorbed at time t. From what we have seen, the dynamical graphs (a) and (b) do not contribute to (B3) while (c) gives a nonvanishing result.

After these preliminaries, let us analyze the structure of anomalous graphs in the expression (3.8) for the current: We first notice that any anomalous graph may be generated from a normal graph (which we call its *skeleton*) by the replacement of an arbitrary number of contractions by anomalous insertions. An example is given in Figs. 5(a)–(b) of the text and the two possible types of insertions are exemplified in Figs. 14(b), (b′), and (c). Indeed the insertions may in principle be put on the left or on the right of the initial line. However, it is immediately seen that the sum of all left insertions gives zero, as a consequence of the left-multidentate structure theorem (see Refs. 10, 11): to each left insertion, we may associate another graph where the last vertex on the left is symmetrical with respect to the horizontal axis; from rules I′, this amounts merely to replacing a factor (λ/i) by $(-\lambda/i)$ and the sum vanishes [see Fig. 14]. As a consequence we have: *The only anomalous graphs which contribute to the current in (3.8) are such that the anomalous insertions are on the right of the line of the corresponding skeleton.*

The next step is to show that all right anomalous insertions with a dynamical part also vanish. The proof of this theorem makes use of a factorization property for independent parts of a given graph which was demonstrated previously in other contexts.[25] In order to avoid unnecessary repetition, we shall merely illustrate it by a simple example. Let us consider the anomalous graphs of Fig. 15(a) and (b); as is immediately seen, these two graphs only differ by the order of the interaction 1 (involving the anomalous lines) with respect to the interaction 2 in the main part of the diagram.

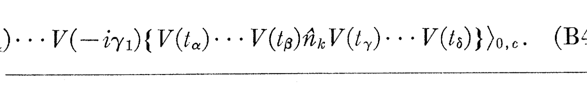


FIG. 14. A contraction and examples of anomalous insertions. (a) Normal contraction. (b)(b′) Left anomalous insertion. (c) Right anomalous insertion.

[25] See Ref. 9 and P. Résibois, Phys. Fluids 6, 817 (1963).

From the rules I, we have, respectively,

$$C_{15a}=\frac{E_{q\omega}{}^{\beta}}{\Omega}2^3(-1)^4\left(\frac{\lambda}{i}\right)\left(-\frac{\lambda}{i}\right)^2\int_0^{\beta}d\gamma\,\exp(-\gamma\Delta\epsilon_1{}')\int_0^{\infty}dt\,e^{i\omega t}k^{\beta}\int_0^{t}dt_3\int_0^{t_3}dt_2\int_0^{t_2}dt_1\,\exp[-i\Delta\epsilon_1(t-t_3)]$$
$$\times\exp[-i\Delta\epsilon_1(t_3-t_2)]\,\exp[-i\Delta\epsilon_1(t_2-t_1)]\,\exp[-i\Delta\epsilon_1 t_1]V_{\text{H.F.}}(k-q/2)V_{\text{H.F.}}(k-q/2)V_{\text{H.F.}}(k+q/2)$$
$$\times(-F_{k+q/2}{}^0)(1-F_{k+q/2}{}^0)(1-F_{k-q/2}{}^0)(1-F_{k-q/2}{}^0)(-F_{k-q/2}{}^0)\,,\qquad\text{(B5)}$$

$$C_{15b}=\frac{E_{q\omega}{}^{\beta}}{\Omega}2^3(-1)^4\left(\frac{\lambda}{i}\right)\left(-\frac{\lambda}{i}\right)^2\int_0^{\beta}d\gamma\,\exp(-\gamma\Delta\epsilon_1{}')\int_0^{\infty}dt\,\exp(i\omega t)k^{\beta}\int_0^{t}dt_3\int_0^{t_3}dt_2\int_0^{t_2}dt_1\,\exp[-i\Delta\epsilon_1(t-t_3)]$$
$$\times\exp[-i\Delta\epsilon_1(t_3-t_2)]\,\exp[-i\Delta\epsilon_1(t_2-t_1)]\,\exp[-i\Delta\epsilon_1 t_1]V_{\text{H.F.}}(k-q/2)V_{\text{H.F.}}(k+q/2)V_{\text{H.F.}}(k-q/2)$$
$$\times(-F_{k+q/2}{}^0)(1-F_{k+q/2}{}^0)(1-F_{k-q/2}{}^0)(-F_{k-q/2}{}^0)(1-F_{k-q/2}{}^0)\,,\qquad\text{(B6)}$$

where

$$\Delta\epsilon_1=\epsilon_{k+q/2}-\epsilon_{k-q/2}\,.\qquad\text{(B7)}$$

We may then write

$$C_{15a}+C_{15b}=\frac{E_{q\omega}{}^{\beta}}{\Omega}(2^3)(-1)^4\left(\frac{\lambda}{i}\right)\left(-\frac{\lambda}{i}\right)^2\int_0^{\beta}d\gamma\,\exp(-\gamma\Delta\epsilon_1{}')\int_0^{\infty}dt\,\exp(i\omega t)k^{\beta}\int_0^{t}dt_3\int_0^{t_3}dt_2$$
$$\times\exp[-i\Delta\epsilon_1(t-t_3)]\,\exp[-i\Delta\epsilon_1(t_3-t_2)]\,\exp[-i\Delta\epsilon_1 t_1]V_{\text{H.F.}}(k-q/2)V_{\text{H.F.}}(k+q/2)$$
$$\times(-F_{k+q/2}{}^0)(1-F_{k+q/2}{}^0)(1-F_{k-q/2}{}^0)\left\{\int_0^{t_3}dt_2{}'V_{\text{H.F.}}(k-q/2)(1-F_{k-q/2}{}^0)(-F_{k-q/2}{}^0)\right\}\,.\qquad\text{(B8)}$$

In the bracketed part of (B8), we have extracted a contribution to $\langle F_k(t_3)\rangle$, given by the diagram Fig. 16; as this contribution is purely dynamical, it gives a vanishing contribution (when properly associated with other graphs), which we wanted to show. The first part of theorem I is thus demonstrated for this simple example: The only anomalous graphs which contribute to $f_{q\omega}(k)$ are such that the interactions on this anomalous lines are purely temperature-dependent.

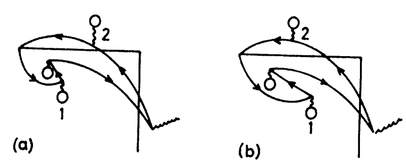


FIG. 15. Examples of anomalous contributions.

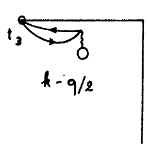


FIG. 16. The contribution to $\langle F_k(t_3)\rangle$ involved in (B8).

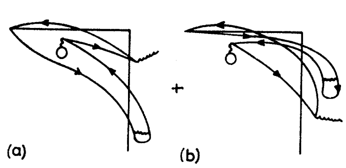


FIG. 17. Purely temperature-dependent anomalous insertion.

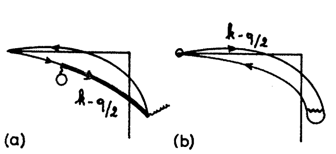


FIG. 18. The skeleton of Fig. 17 and the contribution to $\langle F_{k-q/2}\rangle$.

The proof of the second part follows the same scheme; however, it will not be reproduced here and we leave as an exercise to the reader to verify that the sum of the graphs of Fig. 17 is identical to the contribution of Fig. 18(a) where the heavy line is given by the graph 18(b) for $\langle F_k\rangle$.

Finally, we should mention that from these results a perturbation expansion for the diagonal fragment $\psi_{q\omega}(k,k')$ is easily obtained; in any normal diagonal fragment one has merely to replace the statistical factor $\langle F_k\rangle$ by its perturbation expansion, which, as we have just seen, is given by all purely temperature-dependent anomalous insertions.

APPENDIX C. A SIMPLE PERTURBATIVE MODEL

For equilibrium properties,[26] the relationship between the quasiparticle d.f. $\langle n_k\rangle$ and the momentum d.f. of bare particles $\langle F_k\rangle$ has been studied by Luttinger (see Ref. 5); in the zero-temperature limit, it was shown that

$$\langle F_k(\infty)\rangle=z_k\langle n_k(\infty)\rangle+\bar{F}_k\,,\qquad\text{(C1)}$$

where $\langle F_k(\beta)\rangle$ is given by (4.6) while

$$\langle n_k(\beta)\rangle=[\exp(\alpha+\beta E_k)+1]^{-1}.\qquad\text{(C2)}$$

[26] For more details, see P. Résibois, Bull. Acad. Sci. Belgique **51**, 1288 (1965).

In these formulas, z_k is a normalization coefficient (≤ 1), E_k is the quasiparticle energy and $\bar{F}_k$ is a function regular at the Fermi surface which does not contribute to characteristic equilibrium properties.

On the other hand, transport theory for quasiparticles is well known in the zero-temperature limit (see Refs. 1–3); Landau has shown that the quasiparticle d.f. $n_{q\omega}(k)$ obeys the following equation:

$$[iq^\alpha V_k{}^\alpha - i\omega]n_{q\omega}(k) + iq^\alpha V_k{}^\alpha \delta_k \sum_{k'} \Gamma_{kk'} n_{q\omega}(k') - E_{q\omega}{}^\alpha V_k{}^\alpha \delta_k = 0\,, \quad \text{(C3)}$$

where the first term represents the "free" motion of quasiparticle (with velocity $V_k{}^\alpha$ and frequency ω), the second term expresses the nondissipative interaction between the quasiparticles and the last one corresponds to the acceleration by the electric field. Moreover the factor

$$\delta_k = \delta(E_k - \mu) \quad \text{(C4)}$$

ensures that transport takes place only at the Fermi surface, i.e., we have

$$n_{q\omega}(k) = g_{q\omega}(k)\delta_k\,, \quad \text{(C5)}$$

where $g_{q\omega}(k)$ is a well-behaved function; the electric current is given by

$$j_{q\omega}{}^\alpha(k) = \frac{e}{8\pi^3}\int d^3k\, k^\alpha n_{q\omega}(k)\,. \quad \text{(C6)}$$

The question immediately arises to see what the connection is between the transport equation (C3) and the zero-temperature limit of Eq. (4.15). We have investigated this problem in the frame of low-order perturbation theory; indeed it is well known that most of the physical information of interest is contained in a second-order calculation for all "irreducible" quantities; for instance, Eq. (C1) is already nontrivial at this order.

We have thus analyzed Eq. (4.15) in this approximation: All irreducible operators are computed to order λ^2 and the limit $\beta \to \infty$ is taken. The result may be written as

$$\Bigg\{ -i\omega z_k{}^{-2} f_{q\omega}(k) - i\omega \sum_{k'} \gamma_{kk'} f_{q\omega}(k') \left[\frac{1}{z_k} + \frac{1}{z_{k'}}\right] + iq^\alpha \left[\frac{V_k{}^\alpha}{z_k{}^2} f_{q\omega}(k) + \frac{V_k{}^\alpha}{z_k}\delta_k \sum_{k'} z_k z_{k'} \Gamma_{kk'} \frac{f_{q\omega}(k')}{z_k{}'}\right]$$
$$+ iq^\alpha \sum_{k'} \gamma_{kk'} \left(\frac{V_{k'}{}^\alpha}{z_k{}'} + \frac{V_k{}^\alpha}{z_k}\right) f_{q\omega}(k') - E_{q\omega}{}^\alpha \left(\frac{V_k{}^\alpha \delta_k}{z_k} + \sum_{k'} \gamma_{kk'} V_{k'}{}^\alpha \delta_{k'}\right)\Bigg\}^{(0,1,2)} = 0\,, \quad \text{(C7)}$$

where $\{\ \}^{(0,1,2)}$ means that the bracketed expression should be computed up to second order only, and all other terms are negligible. The notation used here is the same as in Ref. 3 except for the new function $\gamma_{kk'}$:

$$\{\sum_{k'} \gamma_{kk'} f_{q\omega}(k')\}^{(0,1,2)} = -\lambda^2 \sum_{l,s} |v(l) + \text{exch}|^2 \frac{1}{(\epsilon_{k-l} + \epsilon_{s+l} - \epsilon_s - \epsilon_k)^2}$$
$$\times \{[(1-F_{k-l}{}^0)(1-F_{s+l}{}^0)(-F_k{}^0) - (-F_{k-l}{}^0)(-F_{s+l}{}^0)(1-F_k{}^0)] f_{q\omega}(s)$$
$$+ [(-F_k{}^0)(-F_s{}^0)(1-F_{s+l}{}^0) - (1-F_k{}^0)(1-F_s{}^0)(-F_{s+l}{}^0)] f_{q\omega}(k-l)$$
$$+ [(-F_k{}^0)(-F_s{}^0)(1-F_{k-l}{}^0) - (1-F_k{}^0)(1-F_s{}^0)(-F_{k-l}{}^0)] f_{q\omega}(s+l)\}\,. \quad \text{(C8)}$$

The remarkable feature of Eq. (C7) is that it is related to the Landau equation (C3) (computed to the same order) by a linear relation which is exactly of the form (C1), which was proved only for equilibrium.

More precisely, if we set

$$f_{q\omega}(k) = z_k n_{q\omega}(k) + \sum_{k'} \gamma_{kk'} f_{q\omega}(k') \quad \text{(C9)}$$

which defines a new function $n_{q\omega}(k)$, we have shown that Eq. (C7) for $f_{q\omega}(k)$ may be reduced to Eq. (C3) for $n_{q\omega}(k)$; moreover, as we have the relation

$$\sum_k k^\alpha \gamma_{kk'}{}^{(2)} = -k'^\alpha z_{k'}{}^{(2)}\,, \quad \text{(C10)}$$

formula (4.1) for the current reduces to Eq. (C6).

As the second term in (C9) is nonsingular through the Fermi surface, one sees that the general conclusions of equilibrium theory concerning the connection between quasiparticle d.f. and momentum d.f. may be extended to transport theory without any deep modifications.

We refer the reader interested in the details of the calculation to Ref. 26.