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Many-Body Problem for Composite Particles*

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A formalism for treating the many-body problem of composite particles is presented for the case of composite particles consisting of two fermions. Commutation relations for composite-particle operators are derived, as well as a sum rule satisfied by the composite-particle Green's function. In an approximation that shuts off the interactions between composite particles in a consistent manner, the dynamical equation for the one-composite-particle Green's function is solved and the distribution function for the composite particles obtained. Possible applications to real systems are discussed.

I. INTRODUCTION

THE many-body problem for a system of composite particles is one that has received attention from time to time in recent years. Most of the efforts that have been made to treat this problem attempt to formulate the problem in such a way that the deviation of the composite particle from true boson or fermion behavior can be handled in a perturbation-theoretic manner. There have also been some efforts, notably those of Dyson[1] and Girardeau,[2] to handle these deviations from simple statistics by the imposition of constraints on the system.

It is the purpose of this paper to describe yet another approach to the problem which is, in principle, exact. It will be shown that composite particles can exhibit Bose-like or Fermi-like behavior under certain circumstances and that physical significance can be attached to the deviations from pure Bose or Fermi character of the composite particles. These deviations from simple statistical behavior will be shown to occur in the commutation relations, the sum rule satisfied by the spectral weight function of the one-composite-particle Green's function, as well as the mass operator describing the interaction of the composite particle with the medium.

A dynamical equation for the one-composite-particle Green's function is derived and solved in what we call the noninteracting limit. The corrections to the Bose or Fermi distribution functions due to the composite nature of the particles are derived. These corrections are seen to vanish in the limit of low-density systems where there is little overlap of the wave functions of the elementary fermions composing the composite particle.

The main value of the present approach, we feel, lies in the utilization of the Green's function formalism in a space spanned by the eigenfunctions of the isolated composite particle. In this formalism the discussion of the "bound" states of a quasiparticle is straightforward. It is thus a particularly suitable method for the description of plasmas, where the bare composite particle (hydrogen atom, say) becomes, upon being introduced into the plasma, a dressed particle which will in general have a finite lifetime as well as shifted levels.

In addition, it will be seen that the present formalism effects in some measure a separation of dynamical effects from statistical effects. It is, therefore, anticipated that it will be a useful approach to adopt in treating those systems whose behavior is determined largely by statistics rather than dynamics, e.g., the two isotopes of helium in the liquid state.

II. COMPOSITE-PARTICLE OPERATORS

To describe the present method it will be necessary to define composite-particle creation and annihilation operators. For the sake of simplicity we shall restrict the present discussion to the problem of a composite particle consisting of two equal-mass fermions. It will be clear that the method may be generalized in a straightforward, albeit tedious, manner.

Let $a_s^\dagger(\mathbf{k})$ create a fermion with momentum $\mathbf{k}$. The letter s is taken as an index (two-valued for simplicity) whose value identifies the species of fermion under consideration. We then define the following operator:

$$\pi_{ss'}^\dagger(\mathbf{K},\alpha,t) = \int d\mathbf{k}\,\phi_\alpha(\mathbf{k})a_s^\dagger(\tfrac{1}{2}\mathbf{K}-\mathbf{k},\,t)a_{s'}^\dagger(\tfrac{1}{2}\mathbf{K}+\mathbf{k},\,t)\,. \quad \text{(II.1)}$$

This operator[3] is seen to create a pair of fermions (one s particle and one s' particle); the pair having center-of-mass momentum $\mathbf{K}$ and an internal state, either bound or scattering, identified by the quantum numbers α. The function $\phi_\alpha(\mathbf{k})$ is the properly antisymmeterized, if necessary, wave function in momentum space of the internal state of the pair of particles.

The commutation relations satisfied by the elementary creation and annihilation operators are

$$\begin{aligned}[a_s^\dagger(\mathbf{k}),a_{s'}^\dagger(\mathbf{k}')]_+ &= [a_s(\mathbf{k}),a_{s'}(\mathbf{k}')]_+ = 0\,, \quad \text{(II.2)}\\ [a_s^\dagger(\mathbf{k}),a_{s'}(\mathbf{k}')]_+ &= \delta_{ss'}\delta_{\mathbf{k}\mathbf{k}'}\,.\end{aligned}$$

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[1] F. J. Dyson, Phys. Rev. **102**, 1217 (1956).

[2] M. Girardeau, J. Math. Phys. **4**, 1096 (1963).

[3] It will be observed that the composite-particle creation operator so defined differs somewhat from those defined by Dyson and Blatt [in his book, for example, John M. Blatt, *Theory of Superconductivity* (Academic Press Inc., New York, 1964)] in that the center-of-mass momentum is not integrated over. Thus, the pair that is created is characterized by the six numbers, $\mathbf{P}$, α.

If one introduces the Fourier transforms of the elementary creation and destruction operators, it is not difficult to show that an operator analogous to $\pi^\dagger(\mathbf{K},\alpha,t)$ exists. In fact,[4]

$$\pi_{ss'}{}^\dagger(\mathbf{K},\alpha,t)=\int\frac{d\mathbf{R}}{(2\pi)^3}e^{i\mathbf{K}\cdot\mathbf{R}}\pi_{ss'}{}^\dagger(\mathbf{R},\alpha,t)\,,\qquad(\text{II.3})$$

where

$$\pi_{ss'}{}^\dagger(\mathbf{R},\alpha,t)=\int d\mathbf{r}\,\phi_\alpha(\mathbf{r})\Psi_s{}^\dagger(\mathbf{R}+\tfrac{1}{2}\mathbf{r},\,t)\Psi_{s'}{}^\dagger(\mathbf{R}-\tfrac{1}{2}\mathbf{r},\,t)\qquad(\text{II.4})$$

and

$$\phi_\alpha(r)=\int\frac{d\mathbf{p}}{(2\pi)^3}\phi_\alpha(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{r}}.\qquad(\text{II.5})$$

The pair creation operators satisfy the relationship

$$\pi_{ss'}{}^\dagger+(-)^{P_\alpha}\pi_{s's}{}^\dagger=0\,,\qquad(\text{II.6})$$

where P_α is the parity of the state α. Thus, for $s=s'$ we see that the $\pi^\dagger$ operator vanishes identically for symmetric states as is to be expected when the elementary particles are fermions.

In order to gain some insight into the nature of the composite particle created by the $\pi^\dagger$, we construct with some algebraic labor the commutation relations for these operators. They are given by

$$[\pi_{ss'}(\mathbf{K},\alpha,t),\pi_{\tau\tau'}(\mathbf{K},\alpha,t)]_-=[\pi_{ss'}{}^\dagger(\mathbf{K},\alpha,t),\pi_{\tau\tau'}{}^\dagger(\mathbf{K},\alpha,t)]_-=0\,,$$

$$\begin{aligned}&[\pi_{s's}(\mathbf{K},\alpha,t),\pi_{\tau\tau'}{}^\dagger(\mathbf{K}',\alpha',t)]_-\\&=\delta_{\mathbf{KK}'}\delta_{\alpha\alpha'}[\delta_{s\tau}\delta_{s'\tau'}-(-)^{P_\alpha}\delta_{s\tau'}\delta_{s'\tau}]+\int d\mathbf{k}\phi_\alpha{}^\dagger(\mathbf{k})[\phi_{\alpha'}(-\tfrac{1}{2}\mathbf{K}+\tfrac{1}{2}\mathbf{K}'-\mathbf{k})a_{\tau'}{}^\dagger(\mathbf{K}'-\tfrac{1}{2}\mathbf{K}-\mathbf{k})a_s(\tfrac{1}{2}\mathbf{K}-\mathbf{k})\delta_{s'\tau}\\&+\phi_{\alpha'}(\tfrac{1}{2}\mathbf{K}-\tfrac{1}{2}\mathbf{K}'-\mathbf{k})a_\tau{}^\dagger(\mathbf{K}'-\tfrac{1}{2}\mathbf{K}+\mathbf{k})a_{s'}(\tfrac{1}{2}\mathbf{K}+\mathbf{k})\delta_{s\tau'}-\phi_{\alpha'}(-\tfrac{1}{2}\mathbf{K}+\tfrac{1}{2}\mathbf{K}'+\mathbf{k})a_{\tau'}{}^\dagger(\mathbf{K}'-\tfrac{1}{2}\mathbf{K}+\mathbf{k})a_{s'}(\tfrac{1}{2}\mathbf{K}+\mathbf{k})\delta_{s\tau}\\&-\phi_{\alpha'}(\tfrac{1}{2}\mathbf{K}-\tfrac{1}{2}\mathbf{K}'+\mathbf{k})a_\tau{}^\dagger(\mathbf{K}'-\tfrac{1}{2}\mathbf{K}-\mathbf{k})a_s(\tfrac{1}{2}\mathbf{K}-\mathbf{k})\delta_{s'\tau'}].\qquad(\text{II.7})\end{aligned}$$

Thus, it is seen that the composite particle consisting of two fermions is neither a boson nor a fermion. The deviations from pure Bose behavior appear in the form of integrals on the right-hand side of Eq. (II.7). These terms are due to the internal structure of the composite particle. It is clear that the size of the corrections to the delta-function term depends on the degree of overlap of fermion wave functions and the restrictions that arise from the Pauli principle. Indeed, it is not difficult to show that if the internal wave function of the composite particle in configuration space is a delta function, then

$$\pi_{ss'}{}^\dagger(\mathbf{R},\alpha,t)$$

and

$$\pi_{s's}(\mathbf{R}',\alpha',t)$$

do in fact commute for $\mathbf{R}\neq\mathbf{R}'$. This means simply that the composite particle is contained entirely within a sphere of vanishingly small radius and there is, therefore, no opportunity for the fermion wave functions from one composite particle to overlap with those from another composite particle. It might therefore be considered plausible that tightly-bound composite particles will behave like bosons if it is unlikely for the centers of mass of two such particles to coincide. It is the case that the expectation value of the additional terms in the commutation relation rigorously vanish in the limit of low density for translationally invariant systems. We have thus recovered in a straightforward fashion an early result due to Ehrenfest and Oppenheimer.[5]

III. COMPOSITE-PARTICLE GREEN'S FUNCTIONS

In order to describe the behavior of physical systems at finite temperature we shall be interested in the expectation value of operators computed with the aid of the grand canonical ensemble. Thus, for example, for an operator S we have

$$\langle S\rangle=\mathrm{Tr}[\{\exp[-\beta(H-\textstyle\sum_i\mu_iN_i)]\}S]/\mathrm{Tr}[\exp[-\beta(H-\textstyle\sum_i\mu_iN_i)]]\,,\qquad(\text{III.1})$$

where the sum in the exponential is over all the species in the system, and $\beta=1/KT$. The one composite particle Green's function is defined by

$$\mathcal{G}_{ss'}(\mathbf{R},\alpha,t;\,\mathbf{R}',\alpha',t')=(1/i)^2\langle T(\pi_{s's}(\mathbf{R},\alpha,t)\pi_{ss'}{}^\dagger(\mathbf{R}',\alpha',t'))\rangle\,,\qquad(\text{III.2})$$

where T stands for the Wick time-ordering operation. It will also be convenient to introduce two auxiliary functions defined by

$$\mathcal{G}_{ss'}{}^>(\mathbf{R},\alpha,t;\,\mathbf{R}',\alpha',t')=(1/i)^2\langle\pi_{s's}(\mathbf{R},\alpha,t)\pi_{ss'}{}^\dagger(\mathbf{R}',\alpha',t')\rangle\,,\qquad(\text{III.3a})$$

$$\mathcal{G}_{ss'}{}^<(\mathbf{R},\alpha,t;\,\mathbf{R}',\alpha',t')=(1/i)^2\langle\pi_{ss'}{}^\dagger(\mathbf{R}',\alpha',t')\pi_{s's}(\mathbf{R},\alpha,t)\rangle\,.\qquad(\text{III.3b})$$

[4] We shall denote a function and its Fourier transform by the same symbol, distinguishing between them by the argument which will be stated explicitly.

[5] P. Ehrenfest and J. R. Oppenheimer, Phys. Rev. **37**, 333 (1931).

It will be noted that for $t>t'$, $\mathcal{G}=\mathcal{G}^>$ while for $t<t'$, $\mathcal{G}=\mathcal{G}^<$. It is clear that more complicated Green's functions may be defined in analogy with the formalism that deals with many-body systems of particles with no internal structure. We shall have no occasion, however, in the present paper to make use of such functions.

IV. THE BOUNDARY CONDITION AND THE SUM RULE

It is well known that the Green's functions for unstructured particles satisfy a quasiperiodic boundary condition in the imaginary time interval $0>t>-i\beta$. It will now be shown that a similar condition applies to the composite-particle Green's functions. From Eq. (III.3b), we have

$$\mathcal{G}_{ss'}^<(\mathbf{R},\alpha,0;\mathbf{R}',\alpha',t')=\frac{\mathrm{Tr}[(1/i)^2\{\exp[-\beta(H-\sum_i\mu_iN_i)]\}\pi_{ss'}^\dagger(\mathbf{R}',\alpha',t')\pi_{s's}(\mathbf{R},\alpha,0)]}{\mathrm{Tr}[\exp[-\beta(H-\sum_i\mu_iN_i)]]} \tag{IV.1}$$

and by using the cyclic invariance of the trace and judiciously inserting a factor of unity, we obtain

$$\mathcal{G}_{ss'}^<(\mathbf{R},\alpha,0;\mathbf{R}',\alpha',t')$$
$$=\frac{\mathrm{Tr}[(1/i)^2\{\exp[-\beta(H-\sum_i\mu_iN_i)][\exp(\beta(H-\sum_i\mu_iN_i))]\pi_{s's}(\mathbf{R},\alpha,0)[\exp(-\beta(H-\sum_i\mu_iN_i))]\pi_{ss'}^\dagger(\mathbf{R}',\alpha',t')\}]}{\mathrm{Tr}\{\exp[-\beta(H-\sum_i\mu_iN_i)]\}}. \tag{IV.2}$$

Further, it is not difficult to show by operating on an eigenstate of the number operators that

$$[\exp(-\beta\sum\mu_iN_i)]\pi_{s's}(\mathbf{R},\alpha,0)\exp(\beta\sum\mu_iN_i)$$
$$=\exp(\beta\sum\mu_i)\pi_{s's}(\mathbf{R},\alpha,0), \tag{IV.3}$$

which, in conjunction with the prescription for converting Schrödinger operators into Heisenberg operators, yields the result

$$\mathcal{G}_{ss}^<(\mathbf{R},\alpha,0;\mathbf{R}',\alpha',t')$$
$$=[\exp(\beta\sum\mu_i)]\mathcal{G}_{ss}^>(\mathbf{R},\alpha,-i\beta;\mathbf{R}',\alpha',t'). \tag{IV.4}$$

The plus or minus sign that generally appears in this relationship is absent here because we have used the ordinary Wick time-ordering operator rather than the one which introduces sign changes according to the number of fermion permutations. Because we shall be dealing largely with systems possessing both spatial and temporal translational invariance, the Green's functions introduced above depend on R and R' only through the combination $\mathbf{r}=\mathbf{R}-\mathbf{R}'$ and upon t and t' only through the combination $\tau=t-t'$. We shall be able to make use of the Fourier transform of the Green's function defined by

$$\mathcal{G}_{ss'}^{>,<}(\mathbf{P},\omega,\alpha,\alpha')$$
$$=\int d\mathbf{r}\int d\tau\, e^{-i\mathbf{P}\cdot\mathbf{r}+i\omega\tau}\mathcal{G}_{ss}^{>,<}(\mathbf{r},\tau,\alpha,\alpha'). \tag{IV.5}$$

Taking the Fourier transform of the boundary condition, we obtain

$$\mathcal{G}_{ss'}^<(\mathbf{P},\omega,\alpha,\alpha')$$
$$=\exp[-\beta(\omega-\sum\mu_i)]\mathcal{G}_{ss'}^>(\mathbf{P},\omega,\alpha,\alpha'). \tag{IV.6}$$

If we define a spectral weight function $A(\mathbf{P},\omega,\alpha,\alpha')$ by

$$A_{ss'}(\mathbf{P},\omega,\alpha,\alpha')\equiv\mathcal{G}_{ss'}^>(\mathbf{P},\omega,\alpha,\alpha')-\mathcal{G}_{ss'}^<(\mathbf{P},\omega,\alpha,\alpha'), \tag{IV.7}$$

then the boundary condition may be written as

$$\mathcal{G}_{ss'}^>(\mathbf{P},\omega,\alpha,\alpha')=[1+f(\omega)]A_{ss'}(\mathbf{P},\omega,\alpha,\alpha'), \tag{IV.8a}$$

$$\mathcal{G}_{ss'}^<(\mathbf{P},\omega,\alpha,\alpha')=f(\omega)A_{ss'}(\mathbf{P},\omega,\alpha,\alpha'), \tag{IV.8b}$$

where

$$f(\omega)=[\exp(\beta(\omega-\sum\mu_i))-1]^{-1}. \tag{IV.9}$$

The spectral-weight function $A(\mathbf{P},\omega,\alpha,\alpha')$ is found to satisfy a sum rule which casts additional light on the deviations from pure Bose statistics on the part of the composite particles. From Eq. (IV.7) with the aid of Eq. (IV.5), we have

$$A_{ss'}(\mathbf{P},\omega,\alpha,\alpha')$$
$$=\int d\tau\, e^{i\omega\tau}\{\mathcal{G}_{ss'}^>(\mathbf{P},\tau,\alpha,\alpha')-\mathcal{G}_{ss'}^<(\mathbf{P},\tau,\alpha,\alpha')\}. \tag{IV.10}$$

If we integrate this expression over all ω and interchange orders of integration on the right-hand side, we obtain

$$\int\frac{d\omega}{2\pi}A_{ss'}(\mathbf{P},\omega,\alpha,\alpha')$$
$$=-\langle\pi_{ss'}^\dagger(\mathbf{P},\alpha',t'=0)\pi_{s's}(\mathbf{P},\alpha,t=0)\rangle$$
$$+\langle\pi_{s's}(\mathbf{P},\alpha,t=0)\pi_{ss'}^\dagger(\mathbf{P},\alpha',t'=0)\rangle. \tag{IV.11}$$

Making use of the commutation relations [Eq. (II.2)] and the definition of the elementary single-particle

Green's functions, we obtain finally

$$\mathcal{G}_{ss'}^{>}(\mathbf{P},\tau=0,\alpha,\alpha')-\mathcal{G}_{ss'}^{<}(\mathbf{P},\tau=0,\alpha,\alpha')=\delta_{\alpha\alpha'}+i\int d\mathbf{k}\,\phi_\alpha^*(\mathbf{k})\phi_{\alpha'}(\mathbf{k})[G_s^{<}(\tfrac{1}{2}\mathbf{P}-\mathbf{k},t=0)+G_{s'}^{<}(\tfrac{1}{2}\mathbf{P}+\mathbf{k},t=0)]$$
$$=\delta_{\alpha\alpha'}-\int d\mathbf{k}\,\phi_\alpha^*(\mathbf{k})\phi_{\alpha'}(\mathbf{k})[n_s(\tfrac{1}{2}\mathbf{P}-\mathbf{k})+n_{s'}(\tfrac{1}{2}\mathbf{P}+\mathbf{k})]. \quad \text{(IV.12)}$$

We have here made use of the definition of the Green's functions of the elementary particles

$$G_s(\mathbf{p},t)=(1/i)\langle T(a_s(\mathbf{p},t)a_s^\dagger(\mathbf{p},0))\rangle$$

and the relationship between the single-fermion density and the zero-time-difference one-particle Green's function for that species of particle.[6] If we set $\alpha=\alpha'$ and interpret the composite-particle Green's functions for zero-time difference in the same manner that the zero-time-difference elementary-particle Green's functions are interpreted, then the physical content of Eq. (IV.12) is that the composite particles are bosons subject to certain restrictions. The substance of the restrictions is that, for a given center-of-mass momentum P and internal state α, one must take into account the fractional occupation by the separate fermion species of the momentum states used in the construction of the internal state α.

V. COMPOSITE-PARTICLE DYNAMICS

In order to extract the information contained in the Green's functions, we require a procedure for their determination and toward this end we shall develop dynamical equations from whose solution the one-composite-particle Green's function may be obtained.

The mixed (in the sense of particle species) two-particle two-time Green's function can be expressed in terms of the one-composite-particle Green's function as follows:

$$\begin{aligned}G_2^{ss'}(\mathbf{r}_1t,\mathbf{r}_2t;\mathbf{r}_1't',\mathbf{r}_2't')&=(1/i)^2\langle T(\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle\\&=\sum_{\alpha\alpha'}\phi_{\alpha'}^*(\mathbf{r}')\phi_\alpha(\mathbf{r})(1/i)^2\langle T(\pi_{s's}(\mathbf{R},\alpha,t)\pi_{ss'}^\dagger(\mathbf{R}',\alpha',t'))\rangle\\&=\sum_{\alpha\alpha'}\phi_{\alpha'}^*(\mathbf{r}')\phi_\alpha(\mathbf{r})\phi_{ss'}(\mathbf{R},\alpha,t;\mathbf{R}',\alpha',t'),\end{aligned}\quad \text{(V.1)}$$

where

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2;\quad \mathbf{r}'=\mathbf{r}_1'-\mathbf{r}_2'$$

and

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2);\quad \mathbf{R}'=\tfrac{1}{2}(\mathbf{r}_1'+\mathbf{r}_2').$$

This relationship may be inverted using the orthonormality of the ϕ's to yield

$$\mathcal{G}_{ss'}(\mathbf{R},\alpha,t;\mathbf{R}',\alpha',t')=\int d\mathbf{r}d\mathbf{r}'\phi_{\alpha'}(\mathbf{r}')\phi_\alpha^*(\mathbf{r})G_2^{ss'}(\mathbf{r}_1t,\mathbf{r}_2t;\mathbf{r}_1't',\mathbf{r}_2't'). \quad \text{(V.2)}$$

Thus, the one-composite-particle Green's function is determined once the ordinary two-particle, two-species Green's function is obtained. We shall now proceed to derive the equation of motion for the desired function, and exhibit some of the formal properties possessed by it.[7] The Hamiltonian for a system consisting of two species of equal-mass interacting fermions is given by

$$\begin{aligned}H-\mu_sN_s-\mu_{s'}N_{s'}&=\sum_\tau\int\Psi_\tau^\dagger(x)\left(-\frac{\nabla^2}{2m}+\mu_\tau\right)\Psi_\tau(x)dx\\&+\tfrac{1}{2}\sum_{\tau\tau'}\int\Psi_\tau^\dagger(x)\Psi_{\tau'}^\dagger(x')v(x,x')\Psi_{\tau'}(x')\Psi_\tau(x)dxdx',\\&\tau,\tau'=s,s',\quad s\neq s',\end{aligned}\quad \text{(V.3)}$$

where we have assumed for simplicity the species independence of the interaction potential. The time rate of change of the Green's function

$$G_2^{ss'}(\mathbf{r}_1t,\mathbf{r}_2t;\mathbf{r}_1't',\mathbf{r}_2't')$$

we are interested in is given by

$$\begin{aligned}dG_2^{ss'}/dt=&(1/i)^2\langle T(\dot\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle+(1/i)^2\langle T(\Psi_{s'}(\mathbf{r}_1t)\dot\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle\\&+(1/i)^2\delta(t-t')[\langle\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't')\rangle-\langle\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't')\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\rangle].\end{aligned}\quad \text{(V.4)}$$

To evaluate this quantity we require expressions for

$$(id/dt)\Psi_\sigma(x)=[\Psi_\sigma,H],\quad \sigma=s,s'. \quad \text{(V.5)}$$

[6] We have throughout this paper, when dealing with the Green's functions of unstructured particles, made use of the definitions and notations employed by L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin and Company, Inc., New York, 1962).

[7] The dynamical equation satisfied by the two-species, two-particle Green's function derived here appears in a recent article by L. E. Pargamanik and G. M. Pyatigorskii, Zh. Eksperim. i Teor. Fiz. 44, 2029 (1963) [English transl.: Soviet Phys.—JETP **17**, 1365 (1963)] as well as in an elegant unpublished work of J. C. Garrison and J. Wong, in a Bethe-Salpeter type approximation. These authors assume the existence of a density-independent part of the vertex operator. The present derivation is given in order to illustrate how this occurs in a straightforward fashion. In fact, it can be shown that this is a general feature of the Green's function formalism valid for even more complex composite particles than those considered here.

A straightforward calculation of $[\Psi_\sigma,H]$ yields

$$[\Psi_s(x),H]=\left(-\frac{\nabla^2}{2m}+\mu_s\right)\Psi_s(x)+\int dz\, v(x,z)\{\Psi_s^\dagger(z)\Psi_s(z)\Psi_s(x)+\Psi_{s'}^\dagger(z)\Psi_{s'}(z)\Psi_s(x)+\Psi_{s'}^\dagger(z)\Psi_{s'}(z)\Psi_{s'}(x)\}\,. \quad \text{(V.6)}$$

If one takes into account the fact that the expectation values of the form

$$\langle T(\Psi_{s'}^\dagger\Psi_{s'}\Psi_{s'}\Psi_{s'}\Psi_{s'}^\dagger\Psi_s^\dagger)\rangle \quad \langle T(\Psi_s\Psi_s^\dagger\Psi_s\Psi_s\Psi_{s'}^\dagger\Psi_s^\dagger)\rangle$$

vanish when evaluated in eigenstates of particle number, one obtains the following dynamical equation for $G_2^{ss'}(r_1t,r_2t;r_1't',r_2't')$:

$$\left[\frac{i\partial}{\partial t}+\frac{\nabla_1^2+\nabla_2^2}{2m}-\mu_s-\mu_{s'}\right]G_2^{ss'}(\mathbf{r}_1t,\mathbf{r}_2t;\mathbf{r}_1't',\mathbf{r}_2't')$$
$$=-i\delta(t-t')[\langle\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't')\rangle-\langle\Psi_s^\dagger(\mathbf{r}_2t')\Psi_{s'}^\dagger(\mathbf{r}_1t')\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\rangle]$$
$$+\int d\mathbf{r}_3\, v(\mathbf{r}_1-\mathbf{r}_3)\{\langle T(\Psi_{s'}^\dagger(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle$$
$$+\langle T(\Psi_s^\dagger(\mathbf{r}_3t)\Psi_s(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_{2'}t')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle\}$$
$$+\int d\mathbf{r}_3\, v(\mathbf{r}_2-\mathbf{r}_3)\{\langle T(\Psi_{s'}(\mathbf{r}_1t)\Psi_s^\dagger(\mathbf{r}_3t)\Psi_s(\mathbf{r}_3t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle$$
$$+\langle T(\Psi_{s'}(\mathbf{r}_1t)\Psi_{s'}^\dagger(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_3t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle\}\,. \quad \text{(V.7)}$$

A judicious application of fermion anticommutation relations allows one to express the first term on the right-hand side of Eq. (V.7) as

$$-\delta(t-t')[i\delta(\mathbf{r}_1-\mathbf{r}_1')\delta(\mathbf{r}_2-\mathbf{r}_2')-G_s^<(\mathbf{r}_1t,\mathbf{r}_1't)\delta(\mathbf{r}_2-\mathbf{r}_2')-G_{s'}^<(\mathbf{r}_2t,\mathbf{r}_2't)\delta(\mathbf{r}_1-\mathbf{r}_1')]\,.$$

Finally, the last term on the right-hand side

$$\int d\mathbf{r}_3\, v(\mathbf{r}_2-\mathbf{r}_3)\langle T(\Psi_{s'}(\mathbf{r}_1t)\Psi_{s'}^\dagger(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_3t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle$$

can, by the use of the equal-time commutation relations, be expressed as

$$+v(\mathbf{r}_2-\mathbf{r}_1)G_2^{ss'}(\mathbf{r}_1t,\mathbf{r}_2t,\mathbf{r}_1't',\mathbf{r}_2't')+\int d\mathbf{r}_3\, v(\mathbf{r}_2-\mathbf{r}_3)\langle T(\Psi_{s'}^\dagger(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle\,.$$

The resulting equation of motion for $G_2^{ss'}(\mathbf{r}_1t,\mathbf{r}_2t;\mathbf{r}_1't',\mathbf{r}_2't')$ is given by

$$\left\{\frac{i\partial}{\partial t}-\left[\frac{-\nabla_1^2-\nabla_2^2}{2m}+v(\mathbf{r}_1-\mathbf{r}_2)\right]-\mu_s-\mu_{s'}\right\}G_2^{ss'}(\mathbf{r}_1t,\mathbf{r}_2t;\mathbf{r}_1't',\mathbf{r}_2't')$$
$$=-i\delta(t-t')\delta(\mathbf{r}_1-\mathbf{r}_1')\delta(\mathbf{r}_2-\mathbf{r}_2')+\delta(t-t')[G_s^<(\mathbf{r}_1t,\mathbf{r}_1't)\delta(\mathbf{r}_2-\mathbf{r}_2')+G_{s'}^<(\mathbf{r}_2t,\mathbf{r}_2't)\delta(\mathbf{r}_1-\mathbf{r}_1')]$$
$$+\int d\mathbf{r}_3\{\langle T(\Psi_{s'}^\dagger(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle$$
$$+\langle T(\Psi_s^\dagger(\mathbf{r}_3t)\Psi_s(\mathbf{r}_3t)\Psi_{s'}(\mathbf{r}_1t)\Psi_s(\mathbf{r}_2t)\Psi_s^\dagger(\mathbf{r}_2't')\Psi_{s'}^\dagger(\mathbf{r}_1't'))\rangle\}\{v(\mathbf{r}_1-\mathbf{r}_3)+v(\mathbf{r}_2-\mathbf{r}_3)\}\,. \quad \text{(V.8)}$$

There are several prominent features of this equation which we shall now discuss. To begin with, we observe that if all but the delta-function term on the right-hand side is ignored, we have a description of a system of noninteracting pairs of particles. We shall demonstrate that this is the low-density limit of a system of compound particles. The last two terms under the integral signs are proportional to three-particle Green's functions and describe the interactions between the composite particles and the medium. We have already seen how these interaction terms yield the potential term that now appears on the left-hand side of Eq. (V.8) and describes the direct interaction between the two particles. It might be pointed out that in a similar calculation done for 3-body objects, the three pair potentials that describe the direct, medium-independent interaction of the composite particle (which may be in a 3-body scattering state) also separate out exactly as in the present case.

This interpretation of the separation of the interactions is strengthened by the following observation. The equation of motion is the expectation value in a grand canonical ensemble of an operator equation. The expectation value could just as well have been taken in the vacuum in which case the resulting equation of motion is simply the Schrödinger equation for the two-fermion system.

Moreover, we observe that there are two terms on the right-hand side of Eq. (V.8) that persist even when interactions with the medium may be neglected. These terms are due to the Pauli principle and have a particularly appealing interpretation which will become evident in the next section.

VI. NONINTERACTING COMPOSITE PARTICLES

We shall first solve Eq. (V.8) neglecting the interaction terms on the right-hand side of the equation.

From Eq. (V.2) we have

$$G_2^{ss'}(\mathbf{r}_1t,\mathbf{r}_2t;\mathbf{r}_1't',\mathbf{r}_2't')=\sum_{\alpha\beta}\phi_\alpha(\mathbf{r})\phi_\beta^*(\mathbf{r}')\mathcal{G}_{ss'}(\mathbf{R},\alpha,t;\mathbf{R}',\beta,t')$$
$$\equiv\sum_{\alpha\beta}\int\frac{d\mathbf{P}d\omega}{(2\pi)^4}e^{i\mathbf{P}\cdot(\mathbf{R}-\mathbf{R}')-i\omega(t-t')}\phi_\alpha(\mathbf{r})\phi_\beta^*(\mathbf{r}')\mathcal{G}_{ss'}(\mathbf{P},\alpha,\beta,\omega)\,. \qquad \text{(VI.1)}$$

A Fourier decomposition of the first three terms on the right-hand side of Eq. (V.8) leads to

$$i\delta(t-t')\delta(\mathbf{r}_1-\mathbf{r}_1')\delta(\mathbf{r}_2-\mathbf{r}_2')-G_s^<(\mathbf{r}_1t,\mathbf{r}_1't)\delta(t-t')\delta(\mathbf{r}_2-\mathbf{r}_2')-G_{s'}^<(\mathbf{r}_2t,\mathbf{r}_2't)\delta(t-t')\delta(\mathbf{r}_1-\mathbf{r}_1')$$
$$=i\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d\mathbf{p}_1d\mathbf{p}_2}{(2\pi)^6}e^{i\mathbf{p}_1\cdot(\mathbf{r}_1-\mathbf{r}_1')+i\mathbf{p}_2\cdot(\mathbf{r}_2-\mathbf{r}_2')}[1-iG_s^<(\mathbf{p}_1t=t')-iG_{s'}^<(\mathbf{p}_2t=t')]$$
$$=i\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d\mathbf{P}d\mathbf{p}}{(2\pi)^6}e^{i\mathbf{P}\cdot(\mathbf{R}-\mathbf{R}')+i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}[1-n_s(\tfrac{1}{2}\mathbf{P}+\mathbf{p})-n_s(\tfrac{1}{2}\mathbf{P}-\mathbf{p})]\,. \qquad \text{(VI.2)}$$

We insert Eqs. (VI.1) and (VI.2) into Eq. (V.8) and neglect the interaction term on the right-hand side. Equating the coefficients of $e^{i\mathbf{P}\cdot(\mathbf{R}-\mathbf{R}')-i\omega(t-t')}$, one obtains

$$\sum_{\alpha\beta}\left[\omega-\frac{P^2}{2M}-\epsilon_\alpha-\mu_s-\mu_{s'}\right]\phi_\alpha(\mathbf{r})\phi_\beta^*(\mathbf{r}')\mathcal{G}_{ss'}(\mathbf{P},\alpha,\beta,\omega)$$
$$=+i\left[\delta(\mathbf{r}-\mathbf{r}')-\int\frac{d\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}[n_s(\tfrac{1}{2}\mathbf{P}+\mathbf{p})+n_{s'}(\tfrac{1}{2}\mathbf{P}-\mathbf{p})]\right]. \qquad \text{(VI.3)}$$

Multiplying by $\phi_{\alpha'}^*(\mathbf{r})\phi_{\beta'}(\mathbf{r}')$ and integrating over all $\mathbf{r}$ and $\mathbf{r}'$, one obtains finally

$$\mathcal{G}_{ss'}(\mathbf{P},\alpha,\beta,\omega)=+\frac{i\left[\delta_{\alpha\beta}-\int\frac{d\mathbf{p}}{(2\pi)^3}\phi_\alpha(\mathbf{p})\phi_\beta^*(\mathbf{p})[n_s(\tfrac{1}{2}\mathbf{P}+\mathbf{p})+n_{s'}(\tfrac{1}{2}\mathbf{P}-\mathbf{p})]\right]}{\omega-(P^2/2M)-\epsilon_\alpha}\equiv\frac{i\Delta_{\alpha\beta}(\mathbf{P})}{\omega-(P^2/2M)-\epsilon_\alpha}\,. \qquad \text{(VI.4)}$$

As in the case of Green's functions for unstructured particles, we have

$$A_{ss'}(\mathbf{P},\omega,\alpha,\beta)=i[\mathcal{G}(\mathbf{P},\,\omega+i\epsilon,\,\alpha,\,\beta)-\mathcal{G}(\mathbf{P},\,\omega-i\epsilon,\,\alpha,\,\beta)]\,, \qquad \text{(VI.5)}$$

which yields in the present approximation

$$A_{ss'}(\mathbf{P},\omega,\alpha,\beta)=2\pi\Delta_{\alpha\beta}(\mathbf{P})\delta(\omega-P^2/2M-\epsilon_\alpha)\,. \qquad \text{(VI.6)}$$

We have therefore from Eq. (IV.8b)

$$\mathcal{G}_{ss'}^<(\mathbf{P},\omega,\alpha,\beta)=\frac{2\pi\Delta_{\alpha\beta}(\mathbf{P})\delta(\omega-(P^2/2M)-\epsilon_\alpha)}{e^{\beta(\omega-\Sigma\mu_i)}-1}\,. \qquad \text{(VI.7)}$$

Finally, we obtain the density of composite particles with center-of-mass momentum $\mathbf{P}$ and internal state α:

$$\mathcal{G}_{ss'}^<(\mathbf{P},\,\tau=0,\,\alpha,\,\alpha)=\lim_{\tau\to0+}\int\frac{d\omega}{2\pi}e^{i\omega\tau}\mathcal{G}_{ss'}^<(\mathbf{P},\omega,\alpha,\alpha)$$
$$=\frac{\Delta_{\alpha\alpha}(\mathbf{P})}{\exp[\beta(P^2/2M+\epsilon_\alpha-\sum\mu_i)]-1}\,. \qquad \text{(VI.8)}$$

Thus, we obtain the result that the composite particle consisting of two unlike fermions has a Bose-like distribution function with an altered occupation number in the numerator. It will be observed in fact that the quantity $\Delta_{\alpha\beta}(\mathbf{P})$ is essentially the expectation value of the right-hand side of the commutation relations [Eq. (II.7)] for $s\approx\tau$, $s'=\tau'$, $s\neq s'$. The physical significance of the diagonal element of $\Delta_{\alpha\beta}(\mathbf{P})$ is that the occupation of the state α must be decreased by the extent to which the separate species in their one-particle behavior occupy the state. It is not difficult to see that in the limit of low density

$$\Delta_{\alpha\beta}(\mathbf{P})\rightarrow\delta_{\alpha\beta},$$

and one has a description of the propagation of independent noninteracting composite particles.

VII. SELF-ENERGIES AND INTERACTIONS WITH THE MEDIUM

We have constructed a theory that describes in a low-density noninteracting limit a collection of independent composite particles. On purely physical grounds we known this to be a reasonable description of certain types of systems. The question that we must now address ourselves to is how to modify this theory for densities that are not small and in the case where interactions between composite particles may not be ignored.

In the approximation in which the interactions that involve the medium may be ignored Eq. (VI.4) may be used to define a self-energy whose origin lies primarily with the Pauli principle and is present even when the interactions with the medium are small. We require that

$$\mathcal{G}_{ss'}(\mathbf{P},\alpha,\beta,\omega)=\frac{i\Delta_{\alpha\beta}(\mathbf{P})}{\omega-(P^2/2M)-\epsilon_\alpha}\equiv\frac{i}{\omega-(P^2/2M)-\epsilon_\alpha-\Sigma_0(\mathbf{P},\alpha,\beta)}. \quad \text{(VII.1)}$$

In this form we find the Pauli principle giving rise to a density- and temperature-dependent level shift but no width because of the reality of Σ_0. It should also be observed that the "statistical" self-energy Σ_0 couples the internal states of the composite particle even in the absence of interactions with the medium.

When it becomes necessary to consider interactions with the medium the last term in the equation of motion [Eq. (V.8)] may be used to define a self-energy due to interactions with the medium and all of the procedures used to handle the analogous problem with unstructured particles may be brought to bear. We shall defer a more detailed discussion of this matter to future papers where the present formalism will be applied to particular problems since the nature of the appproximations to be used in treating the dynamical self-energy will depend on the system under consideration.

VIII. CONCLUSION

One may legitimately ask what advantage is offered by the present formalism in contrast to other ways of looking at the many-body problem for composite particles. In this concluding section we shall attempt to answer this question.

To begin with we feel that a certain insight is afforded by the method into the physics of the composite particle in the medium. It can be shown that the appearance of the Schrödinger operator for the composite particle in vacuum on the left-hand side of the equation of motion [Eq. (V.8)] is a general feature of the formulation and occurs even when one considers composite particles consisting of more than two particles. This means that we have succeeded in relating an N-body problem in a medium to the corresponding N-body problem in vacuum. As a consequence of this fact, one has managed to separate the interactions in the system into two classes: those that occur directly between the particles in the composite particles, and those that are mediated by the surrounding matter. It is now possible, therefore, to consider in a consistent fashion the treatment of some interactions as small while others are not. Moreover, it seems to us that certain questions of interest are more easily formulated and answered by the method described here. For example, in the case of a hydrogen plasma, one might be interested in the degree of ionization of a gas at a given temperature and density. Since a solution to the problem in the present formalism yields a density of composite particles as a function of center-of-mass momentum and internal quantum numbers, one simply sums these densities over all states of negative internal energy and divides by a suitable normalization.

In addition, the separation of dynamical and statistical effects that occurs in the present procedure allows one to consider the possible application of these methods to problems in which statistics may be playing a greater role than the interactions between composite particles. Thus, a generalized form of this present method is now being used to investigate those differences in the thermodynamic behavior of the two isotopes of helium at low temperatures that might arise by virtue of the different statistics.

In a totally different application, the reader will recognize that what is presented here can be construed as a formalism for describing the propagation of a deuteron in infinite nuclear matter. The self-energy operators now take on the significance of optical potentials. In this context the statistical self-energy is particularly interesting and calculations are under way to calculate its effect on the scattering of low-energy deuterons from heavy nuclei.

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