P- and *D*-State Contributions to the Charge Form Factors of H^3 and He^3†

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The method of Sachs is used to obtain explicit expressions for the *P*- and *D*-state components of the ground-state wave functions of H^3 and He^3, under the assumption that they have $T=\frac{1}{2}$ and $J^P=\frac{1}{2}^+$. The effect of a reasonable admixture of these components on the electric charge form factors of the two nuclei is calculated, and it is found that the striking difference between the observed charge form factors cannot be accounted for in this way. It seems most likely that a combination of S'-state and $T=\frac{3}{2}$-state admixture might provide an explanation of this effect without leading to disagreement with other experimental observations.

I. INTRODUCTION

INFORMATION obtained from experiments in which high-energy electrons are scattered elastically from H^3 and He^3 may be expressed in terms of electric charge and magnetic moment form factors for the two nuclei, by making use of the Rosenbluth equation for spin-$\frac{1}{2}$ systems. The basic formulas, which take account of the charge, mass, and anomalous magnetic moments of these nuclei,[1] have been used to calculate $F_{ch}(H^3)$, $F_{ch}(He^3)$, $F_{mag}(H^3)$, and $F_{mag}(He^3)$ as functions of q^2, the square of the four-momentum transfer.[2] Since radiative (soft-photon) corrections are included in this calculation, an ambiguity could arise only from processes in which two or more photons are exchanged between electron and nucleus. While a careful estimate of these contributions has not been made in the case of three-body nuclei, it seems likely from other work[3] that they are not more than a few percent, and hence less than the present experimental uncertainties.

Several attempts have been made to understand these four experimental form factors on the basis of various assumptions concerning the three-nucleon system. In all of these, nonrelativistic nuclear wave functions are used to calculate the expectation value of an appropriate nonrelativistic charge-current operator, and nucleon recoil is neglected. The error thus incurred is probably of order q^2/M^2, where M is the nucleon mass,[4] and hence is only of minor importance for $q^2<5\ \mathrm{F}^{-2}$. The dominant component of the ground-state nuclear wave function is the fully space-symmetric $^2S_{1/2}$ state with $T=\frac{1}{2}$ (denoted here by S).[5,6] A detailed treatment of the effect of a small admixture of the mixed-symmetry $^2S_{1/2}$ state with $T=\frac{1}{2}$ (denoted by S') has been given.[5] The only other admixed states that have been considered at all carefully are one of the three $^4D_{1/2}$ states,[7] and the $^2S_{1/2}$ state with $T=\frac{3}{2}$,[8] which is expected to appear because of the repulsion between protons in He^3 and perhaps also because of charge asymmetry of nuclear forces.[9,10] Various other effects have been included phenomenologically: interference between the S and D states,[5,7] an isovector exchange magnetic moment,[5,7,11] an isoscalar exchange magnetic moment,[12,13] and an isovector exchange charge density.[13] Model calculations have also been made of the isovector exchange terms.[13,14]

All of the last-named effects are extremely difficult to calculate in a reliable fashion, and there are too many of them to determine empirically from the four measured form factors. It is inevitable that sources of information other than elastic electron scattering will have to be exploited. This is already being done with respect to experiments[15] and calculations[16] on the breakup of He^3 by electrons, on the radiative capture of thermal neutrons by deuterons,[17] on the β decay of H^3,[18] on the capture of negative muons by He^3 to form H^3,[19] and on the muon-catalyzed fusion of protons and deuterons to form He^3.[20] In addition, independent information on the ground-state wave function is provided by recent variational calculations of the binding energy of the three-body nuclei[21] and by calculations and experiments on their low-energy photodisintegration.[22]

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In the present paper a start is made on the extension of the detailed nonrelativistic analysis of Ref. 5, which was limited to the S and S' states, so as to include the P and D states with $T=\frac{1}{2}$. Only the charge form factors are considered here, and effects of exchange currents are ignored. It is expected that the analysis will later be extended to a similar calculation of $F_{\text{mag}}(\text{H}^3)$ and $F_{\text{mag}}(\text{He}^3)$. There are ten even parity states with $T=J=\frac{1}{2}$.[23] Of the eight not already considered (i.e., other than S and S'), three will be neglected for the following reasons. Doublet spin states can interfere with the dominant S state in the charge form factors, but two of them are expected to have such small amplitude as to be unimportant; these are the fully space-antisymmetric $^2S_{1/2}$ and $^2P_{1/2}$ states, which are not included in Sachs' classification.[24] The remaining two $^2P_{1/2}$ states are considered in the present paper so far as SP interference is concerned, but are assumed to have such small probabilities that P^2 effects may be ignored. Quartet spin states cannot interfere with the dominant S state in the charge form factors, and one of them is expected to have such small probability as to be unimportant; this is the $^4P_{1/2}$ state.

The remaining states all appear in Sachs' classification, and we have found it more convenient to work with this formulation of the states than with that of Derrick and Blatt.[23] There are then two $^2P_{1/2}$ states (Sachs' numbers 3 and 4) and three $^4D_{1/2}$ states (Sachs' numbers 6, 7, and 8).[24] For the convenience of the reader, Sachs' formalism is rederived in the next three sections in a form that corresponds more closely to the notation of Ref. 5. Expressions for the charge form factors are obtained in the following three sections, and Sec. VIII presents some numerical results.

II. SYMMETRY PROPERTIES

Since we must deal repeatedly with quantities that have mixed symmetry with respect to interchanges of the three nucleons, we start with the permutation table given in Eqs. (3) of Ref. 5. Any of the quantities introduced below that carries subscripts 1 and 2 ($\phi,\chi,\eta,\mathbf{R},v,S,\boldsymbol{\pi},D,u$) transforms in the following way under interchange of nucleons:

$$\begin{aligned} &P_{23}\phi_1=\phi_1, \quad P_{12}\phi_1=\tfrac{1}{2}(3^{1/2}\phi_2-\phi_1), \quad P_{13}\phi_1=-\tfrac{1}{2}(3^{1/2}\phi_2+\phi_1), \\ &P_{23}\phi_2=-\phi_2, \quad P_{12}\phi_2=\tfrac{1}{2}(\phi_2+3^{1/2}\phi_1), \quad P_{13}\phi_2=\tfrac{1}{2}(\phi_2-3^{1/2}\phi_1). \end{aligned} \tag{1}$$

It can then be shown that the following combinations of any quantities that transform in accordance with Eqs. (1) also transform in accordance with Eqs. (1):

$$\phi_1=\chi_2\eta_2-\chi_1\eta_1, \quad \phi_2=\chi_2\eta_1+\chi_1\eta_2. \tag{2}$$

Furthermore, the combination

$$\phi_a=\chi_2\eta_1-\chi_1\eta_2 \tag{3}$$

is fully antisymmetric (i.e., changes sign when operated on by any of the three permutations), and the combination

$$\phi_s=\chi_2\eta_2+\chi_1\eta_1 \tag{4}$$

is fully symmetric (i.e., remains unchanged when operated on by any of the three permutations). We shall consistently use the subscripts a and s to denote antisymmetric and symmetric quantities.

The internal space coordinates of the three-nucleon system are chosen as in Appendix A of Ref. 5: $\mathbf{r}$ is the vector from nucleon 2 to nucleon 3, and $\boldsymbol{\varrho}$ is the vector from the midpoint of nucleons 2 and 3 to nucleon 1. Then the vectors

$$\mathbf{R}_1=(\tfrac{4}{3})^{1/2}\boldsymbol{\varrho}, \quad \mathbf{R}_2=-\mathbf{r} \tag{5}$$

satisfy the permutation table (1). Where the vector character of the coordinates does not enter, space functions may be defined as in Eqs. (4) of Ref. 5. Let $g(12,3)$ be a scalar function of the internal space coordinates that is symmetric under interchange of nucleons 1 and 2, but neither symmetric nor antisymmetric under interchange of 1 and 3 or 2 and 3; then

$$\begin{aligned} v_1&=6^{-1/2}[g(12,3)+g(13,2)-2g(23,1)], \\ v_2&=2^{-1/2}[g(12,3)-g(13,2)] \end{aligned} \tag{6}$$

transform in accordance with Eqs. (1). However, for simplicity in constructing the P- and D-state functions, we shall not exploit the full arbitrariness of the function $g(12,3)$, but rather choose the form that results from combination of Eqs. (5) according to Eqs. (2), where multiplication is interpreted as the scalar product:

$$S_1=R_2^2-R_1^2, \quad S_2=2\mathbf{R}_1\cdot\mathbf{R}_2. \tag{7}$$

It is easily seen that Eqs. (7) are the same as Eqs. (6) if we choose $g(12,3)=-(8/3)^{1/2}r_{12}^2$, where r_{12} is the distance between nucleons 1 and 2. Equations (3) and (4) in combination with (5) show that S_a is identically zero, and that

$$S_s=R_1^2+R_2^2=\tfrac{2}{3}(r_{12}^2+r_{13}^2+r_{23}^2). \tag{8}$$

It should be noted that in spite of the above restriction on the choice of $g(12,3)$, there remains a great deal of freedom in choosing a fully symmetric multiplying function of the space coordinates, $f(S_s)$, for each wave function.

Doublet spin functions are chosen as in Eqs. (1) of Ref. 5:

$$\begin{aligned} \chi_1&=6^{-1/2}[(++-)+(+-+)-2(-++)], \\ \chi_2&=2^{-1/2}[(++-)-(+-+)]; \end{aligned} \tag{9}$$

again, they satisfy the permutation table (1). $A+$ (or $-$) in, say, the second position of a parenthesis means that

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nucleon 2 has spin up (or down). The χ's of Eqs. (9) have total spin component $+\frac{1}{2}$ in the "up" direction; interchange of $+$ and $-$ in their arguments changes this spin component to $-\frac{1}{2}$. The doublet isospin functions η_1 and η_2 also have the form of Eqs. (9), where now a $+$ (or $-$) means that the nucleon is a proton (or neutron). These η's describe He^3; interchange of $+$ and $-$ in their arguments gives functions describing H^3.

While we shall not be interested in quartet isospin functions in this paper, we shall require quartet spin functions. However, rather than write them down explicitly, it is more convenient to generate them by operating on the χ's with $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$, where the components of $\boldsymbol{\sigma}_i$ are the three Pauli spin matrices (with unit elements) that act on the ith nucleon. Furthermore, it is not necessary to use both of the χ's, since χ_1 can be generated from χ_2: $\chi_1=(12)^{-1/2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_{23})\chi_2$, where $\boldsymbol{\sigma}_{23}=\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3$. If χ_2 is used to start with, only two of the three $\boldsymbol{\sigma}$'s, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_{23}$, need be used since $(\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)\chi_2=0$.

III. *P*-STATE WAVE FUNCTIONS

A wave function with $J=\frac{1}{2}$ and total angular-momentum component $+\frac{1}{2}$ in the "up" direction can be obtained by operating on χ_2 with a spherically symmetric operator. For a P-wave function this operator must contain a space vector, which must have even parity since all constituents of the ground state have even parity. The only such vector is $\mathbf{R}_1\times\mathbf{R}_2$, which is fully antisymmetric. The required spherically symmetric operator is then constructed by taking the scalar product of this vector with a spin vector. There are only three spin vectors: $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_{23}$, and $\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_{23}$; higher powers can be reduced to these or to constants. After they operate on χ_2, it is convenient to group them into two linear combinations

$$\boldsymbol{\pi}_1=(12)^{-1/2}[\boldsymbol{\sigma}_{23}+i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_{23})]\chi_2\,,\quad \boldsymbol{\pi}_2=\boldsymbol{\sigma}_1\chi_2 \qquad (10)$$

which satisfy Eqs. (1), and a third combination

$$\boldsymbol{\pi}_s=[\boldsymbol{\sigma}_{23}-\tfrac{1}{2}i(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_{23})]\chi_2 \qquad (11)$$

which is fully symmetric.

Any symmetric spin function of three nucleons, such as Eq. (11), must represent a quartet state. Then $\boldsymbol{\pi}_s\cdot(\mathbf{R}_1\times\mathbf{R}_2)$ is antisymmetric, and must be multiplied by a space-isospin function that is a scalar with respect to space rotations, and is also symmetric with respect to permutations of the nucleons in order to satisfy the Pauli principle. From Eqs. (4) and (6), such a function is $v_2\eta_2+v_1\eta_1$. As remarked in Sec. II, we replace Eqs. (6) by Eqs. (7) in the interests of simplicity, and obtain

$$\psi_5=(S_2\eta_2+S_1\eta_1)[\boldsymbol{\pi}_s\cdot(\mathbf{R}_1\times\mathbf{R}_2)]f_5(S_s)\,. \qquad (12)$$

Equation (12) describes the ${}^4P_{1/2}$ state, and is equivalent to Sachs' number 5. As stated in Sec. I, we shall neglect this state.

The spin functions $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ of Eqs. (10) have the permutation symmetry of doublet states. The symmetric combination (4) of these $\boldsymbol{\pi}$'s and the η's, multiplied by $\mathbf{R}_1\times\mathbf{R}_2$, satisfies the Pauli principle, and gives a ${}^2P_{1/2}$ state that is equivalent to Sachs' number 3:

$$\psi_3=(\boldsymbol{\pi}_2\eta_2+\boldsymbol{\pi}_1\eta_1)\cdot(\mathbf{R}_1\times\mathbf{R}_2)f_3(S_s)\,. \qquad (13)$$

Another symmetric combination may be formed if the v's are used along with the η's, and the $\boldsymbol{\pi}$'s of Eqs. (10). This may be done by combining any two of the three pairs into a pair of mixed symmetry in accordance with Eqs. (2), and then combining these with the third pair in accordance with Eq. (4). The result is independent of which two pairs are used to start with, and yields a ${}^2P_{1/2}$ state that is equivalent to Sachs' number 4:

$$\psi_4=[(\boldsymbol{\pi}_2S_1+\boldsymbol{\pi}_1S_2)\eta_2+(\boldsymbol{\pi}_2S_2-\boldsymbol{\pi}_1S_1)\eta_1]\cdot(\mathbf{R}_1\times\mathbf{R}_2)f_4(S_s)\,. \qquad (14)$$

Again, the v's have been replaced by S's for simplicity.

Finally, as pointed out by Derrick and Blatt,[23] a third ${}^2P_{1/2}$ state can be formed by multiplying the antisymmetric combination of $\boldsymbol{\pi}$'s and η's given by Eq. (3), $\boldsymbol{\pi}_2\eta_1-\boldsymbol{\pi}_1\eta_2$, by a fully antisymmetric S state, the existence of which was not appreciated by Sachs. As remarked in Sec. I, we shall neglect this state.

IV. *D*-STATE WAVE FUNCTIONS

As with the P states, both of the space vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ must be used to construct D states. If $\mathbf{A}$ and $\mathbf{B}$ are any two of the three spin vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_{23}$, and $\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_{23}$, then the space-spin functions $(\mathbf{A}\cdot\mathbf{R}_i)(\mathbf{B}\cdot\mathbf{R}_j)\chi_2$, where i and j may equal 1 or 2, are combinations of S, P, and D states that have even parity, $J=\frac{1}{2}$, and total angular momentum component $+\frac{1}{2}$ in the "up" direction. For a particular choice of $\mathbf{A}$ and $\mathbf{B}$, the four space-spin functions may be grouped into two combinations

$$\begin{aligned}\phi_1&=[(\mathbf{A}\cdot\mathbf{R}_2)(\mathbf{B}\cdot\mathbf{R}_2)-(\mathbf{A}\cdot\mathbf{R}_1)(\mathbf{B}\cdot\mathbf{R}_1)]\chi_2\,,\\ \phi_2&=[(\mathbf{A}\cdot\mathbf{R}_2)(\mathbf{B}\cdot\mathbf{R}_1)+(\mathbf{A}\cdot\mathbf{R}_1)(\mathbf{B}\cdot\mathbf{R}_2)]\chi_2\end{aligned} \qquad (15)$$

that transform in accordance with Eqs. (1) under permutations of *only* the space coordinates of the nucleons, and two combinations

$$\begin{aligned}\phi_a&=[(\mathbf{A}\cdot\mathbf{R}_2)(\mathbf{B}\cdot\mathbf{R}_1)-(\mathbf{A}\cdot\mathbf{R}_1)(\mathbf{B}\cdot\mathbf{R}_2)]\chi_2\,,\\ \phi_s&=[(\mathbf{A}\cdot\mathbf{R}_2)(\mathbf{B}\cdot\mathbf{R}_2)+(\mathbf{A}\cdot\mathbf{R}_1)(\mathbf{B}\cdot\mathbf{R}_1)]\chi_2\end{aligned} \qquad (16)$$

that are, respectively, antisymmetric and symmetric with respect to permutations of the nucleon space coordinates. It is easily seen that $\phi_a=(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{R}_1\times\mathbf{R}_2)\chi_2$, and hence yields the P-state functions of Sec. III when $\mathbf{A}$ and $\mathbf{B}$ are suitably chosen.

The remaining three ϕ's are combinations of D and S states. Since higher powers of the $\boldsymbol{\sigma}$'s introduce nothing new, it is sufficient to choose $\mathbf{A}=\boldsymbol{\sigma}_1$, $\mathbf{B}=\boldsymbol{\sigma}_{23}$. We then convert ϕ_1, ϕ_2, and ϕ_s given by Eqs. (15) and (16) into pure D states by subtracting their S parts, which are the averages over orientations of the space triangle defined by $\mathbf{R}_1$ and $\mathbf{R}_2$. The results are

$$\begin{aligned} D_1 &= [(\boldsymbol{\sigma}_1\cdot\mathbf{R}_2)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_2)-(\boldsymbol{\sigma}_1\cdot\mathbf{R}_1)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_1) \\ &\quad -\tfrac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_{23})(R_2^2-R_1^2)]\chi_2, \\ D_2 &= [(\boldsymbol{\sigma}_1\cdot\mathbf{R}_2)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_1)+(\boldsymbol{\sigma}_1\cdot\mathbf{R}_1)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_2) \\ &\quad -\tfrac{2}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_{23})(\mathbf{R}_1\cdot\mathbf{R}_2)]\chi_2, \quad (17) \\ D_s &= [(\boldsymbol{\sigma}_1\cdot\mathbf{R}_2)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_2)+(\boldsymbol{\sigma}_1\cdot\mathbf{R}_1)(\boldsymbol{\sigma}_{23}\cdot\mathbf{R}_1) \\ &\quad -\tfrac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_{23})(R_1^2+R_2^2)]\chi_2. \end{aligned}$$

Since Eqs. (17) are D states with $J=\frac{1}{2}$, they must be ${}^4D_{1/2}$ states. The quartet character of their spin dependence means that they are fully symmetric with respect to permutations of the nucleon spins. Thus the D's have the symmetries indicated by their subscripts with respect to permutations of all (space and spin) coordinates of the three nucleons.

The D's must now be multiplied by or combined with isospin functions, and perhaps also spherically symmetric space functions, so as to form fully antisymmetric wave functions that obey the Pauli principle. D_s can only be multiplied by $v_2\eta_1-v_1\eta_2$, which is antisymmetric. Again replacing the v's by S's, we obtain

$$\psi_6=D_s(S_2\eta_1-S_1\eta_2)f_6(S_s), \tag{18}$$

which is equivalent to Sachs' number 6. D_1 and D_2 may be combined with the η's in accordance with Eq. (3) to give

$$\psi_7=(D_2\eta_1-D_1\eta_2)f_7(S_s), \tag{19}$$

which is equivalent to Sachs' number 7. Finally, mixed-symmetry combinations of the v's and η's, or for simplicity the S's and η's, may be combined with the D's in accordance with Eq. (3) to yield a wave function that is equivalent to Sachs' number 8; as in the construction of Eq. (14), the three pairs may be combined in any order, and the result is

$$\psi_8=[(D_2S_1+D_1S_2)\eta_1-(D_2S_2-D_1S_1)\eta_2]f_8(S_s). \tag{20}$$

V. GENERAL STRUCTURE OF THE CHARGE FORM FACTORS

The P- and D-state wave functions obtained in the last two sections must be supplemented by the dominant S-state function (Sachs' number 1):

$$\psi_1=(\chi_2\eta_1-\chi_1\eta_2)f_1(S_s). \tag{21}$$

Our wave function is then the sum of Eqs. (13), (14), (18), (19), (20), and (21), which we write in the form

$$\psi=\sum\psi_i=u_2\eta_1-u_1\eta_2, \tag{22}$$

where the summation extends over $i=1, 3, 4, 6, 7, 8$. The u's in Eq. (22) are space-spin functions that satisfy Eqs. (1). As shown by Sachs,[24] time-reversal invariance requires that the functions $f_i(S_s)$ be real.

We assume that the electric charge density operator is that given in Eq. (7) of Ref. 5:

$$\rho_C(\mathbf{r},\mathbf{r}_i)=\sum_{i=1}^{3}[\tfrac{1}{2}(1+\tau_{iz})f_{\mathrm{ch}}{}^p(\mathbf{r}-\mathbf{r}_i) +\tfrac{1}{2}(1-\tau_{iz})f_{\mathrm{ch}}{}^n(\mathbf{r}-\mathbf{r}_i)]. \tag{23}$$

Here the τ's are Pauli isospin matrices (with unit elements), and the f's are nucleon spatial charge densities. The charge form factor of He^3 is then given by Eq. (9) of Ref. 5. With the help of Eqs. (22) and (23), and the properties of the τ's and η's, this may be written

$$\begin{aligned} 2F_{\mathrm{ch}}(\mathrm{He}^3) &= \int\int \exp(i\mathbf{q}\cdot\mathbf{r})\psi^*\rho_C(\mathbf{r},\mathbf{r}_i)\psi d^3r d^3r_i \\ &= \tfrac{1}{2}(F_{\mathrm{ch}}{}^p+F_{\mathrm{ch}}{}^n)\sum_{i=1}^{3}\int \exp(i\mathbf{q}\cdot\mathbf{r}_i)(u_2{}^*u_2+u_1{}^*u_1)d^3r_i+\tfrac{1}{2}(F_{\mathrm{ch}}{}^p-F_{\mathrm{ch}}{}^n)\int\{\exp(i\mathbf{q}\cdot\mathbf{r}_1)(-\tfrac{1}{3}u_2{}^*u_2+u_1{}^*u_1) \\ &\quad +\exp(i\mathbf{q}\cdot\mathbf{r}_2)[\tfrac{2}{3}u_2{}^*u_2-3^{-1/2}(u_2{}^*u_1+u_1{}^*u_2)]+\exp(i\mathbf{q}\cdot\mathbf{r}_3)[\tfrac{2}{3}u_2{}^*u_2+3^{-1/2}(u_2{}^*u_1+u_1{}^*u_2)]\}d^3r_i. \end{aligned} \tag{24}$$

$F_{\mathrm{ch}}{}^p$ and $F_{\mathrm{ch}}{}^n$ are the charge form factors of the proton and the neutron.

From Eq. (4), $u_2{}^*u_2+u_1{}^*u_1$ is symmetric with respect to permutations of the nucleons, so that the three terms in the summation are equal. Also, Eqs. (1) show, on permutation of the nucleon indices, that the three terms in the last integral are equal. Thus Eq. (24) may be rewritten

$$\begin{aligned} 2F_{\mathrm{ch}}(\mathrm{He}^3) &= \tfrac{1}{2}(F_{\mathrm{ch}}{}^p+F_{\mathrm{ch}}{}^n)\int \exp(i\mathbf{q}\cdot\mathbf{r}_1)(3u_2{}^*u_2+3u_1{}^*u_1)d^3r_i \\ &\quad +\tfrac{1}{2}(F_{\mathrm{ch}}{}^p-F_{\mathrm{ch}}{}^n)\int \exp(i\mathbf{q}\cdot\mathbf{r}_1)(-u_2{}^*u_2+3u_1{}^*u_1)d^3r_i. \end{aligned} \tag{25}$$

Equation (25) and its counterpart for H^3 are conveniently written in the form of the first and third of Eqs. (17) of Ref. 5:

$$\begin{aligned} 2F_{\mathrm{ch}}(\mathrm{He}^3) &= 2F_{\mathrm{ch}}{}^pF_L+F_{\mathrm{ch}}{}^nF_O, \\ F_{\mathrm{ch}}(\mathrm{H}^3) &= 2F_{\mathrm{ch}}{}^nF_L+F_{\mathrm{ch}}{}^pF_O. \end{aligned} \tag{26}$$

The body form factors for the like pair of nucleons (F_L) and for the odd nucleon (F_O) may then be expressed in terms of F_a and F_b as follows[25]:

$$F_L=\tfrac{1}{2}(3F_a+F_b)\,, \qquad F_O=2F_b\,,$$

$$F_a=\int \exp(i\mathbf{q}\cdot\mathbf{r}_1)u_1{}^*u_1d^3r_i\,,\quad F_b=\int \exp(i\mathbf{q}\cdot\mathbf{r}_1)u_2{}^*u_2d^3r_i\,. \tag{27}$$

Since $\mathbf{r}_1$ is the vector from the center of mass of the nucleus to nucleon 1, it may be written

$$\mathbf{r}_1=\tfrac{2}{3}\boldsymbol{\rho}=3^{-1/2}\mathbf{R}_1\,. \tag{28}$$

VI. *P*-STATE CONTRIBUTION TO THE CHARGE FORM FACTORS

As discussed toward the end of Sec. I, the expected probabilities of the ${}^2P_{1/2}$ states ψ_3 and ψ_4 are so small that they can only be significant through interference with ψ_1. From Eqs. (22) and (27), the contribution to the factor $u_1{}^*u_1$ in the integrand of F_a is then

$$-\chi_1{}^*\boldsymbol{\pi}_2\cdot(\mathbf{R}_1\times\mathbf{R}_2)f_1f_3-\chi_1{}^*(\boldsymbol{\pi}_2S_1+\boldsymbol{\pi}_1S_2)\cdot(\mathbf{R}_1\times\mathbf{R}_2)f_1f_4+\text{c.c.}\,, \tag{29}$$

where substitutions have been made from Eqs. (13), (14), and (21). Similarly, the *SP* interference contribution to the factor $u_2{}^*u_2$ in the integrand of F_b is

$$\chi_2{}^*\boldsymbol{\pi}_1\cdot(\mathbf{R}_1\times\mathbf{R}_2)f_1f_3+\chi_2{}^*(\boldsymbol{\pi}_2S_2-\boldsymbol{\pi}_1S_1)\cdot(\mathbf{R}_1\times\mathbf{R}_2)f_1f_4+\text{c.c.} \tag{30}$$

The following quantities, which are required for the evaluation of Eqs. (29) and (30), are easily calculated from Eqs. (9) and (10):

$$\chi_1{}^*\boldsymbol{\pi}_2=\chi_2{}^*\boldsymbol{\pi}_1=0\,,$$
$$\chi_1{}^*\boldsymbol{\pi}_1=\chi_2{}^*\boldsymbol{\pi}_2=(0,0,1)\,.$$

The last expression denotes a vector whose component in the "up" or z direction is unity, and whose other two rectangular components are zero. Substitution into Eqs. (29) and (30) shows that the first of these is equal to $-(\mathbf{R}_1\times\mathbf{R}_2)_zS_2f_1f_4$, and the second is the same with opposite sign. In view of Eq. (28), it follows that integration over the direction of $\mathbf{R}_2$ in Eqs. (27) causes F_a and F_b to vanish. Thus there is no *P*-state interference contribution to the charge form factors. It is apparent that this result would be unchanged if the more general v's were used in place of S's in Eq. (14).

VII. *D*-STATE CONTRIBUTION TO THE CHARGE FORM FACTORS

It is apparent from Eqs. (26) and (27) that there is no *SD* interference contribution to the charge form factors, since doublet and quartet spin states are orthogonal to each other. It is possible that the *D*-state probability is large enough to make a significant contribution, so we now calculate the D^2 parts of $u_1{}^*u_1$ and $u_2{}^*u_2$ for substitution into Eqs. (27).

We first require the following spin products of the D's defined in Eqs. (17), which may be obtained in a straightforward manner:

$$\begin{aligned}
D_1{}^*D_1&=S_s{}^2+\tfrac{1}{3}S_1{}^2-S_2{}^2\,,\\
D_2{}^*D_2&=S_s{}^2-S_1{}^2+\tfrac{1}{3}S_2{}^2\,,\\
D_s{}^*D_s&=\tfrac{1}{3}S_s{}^2+S_1{}^2+S_2{}^2\,,\\
D_1{}^*D_2+D_2{}^*D_1&=(8/3)S_1S_2\,,\\
D_1{}^*D_s+D_s{}^*D_1&=(8/3)S_sS_1\,,\\
D_2{}^*D_s+D_s{}^*D_2&=(8/3)S_sS_2\,.
\end{aligned} \tag{31}$$

The S's are defined in Eqs. (7) and (8).

Next we consider the orthonormality properties of ψ_6, ψ_7, and ψ_8 given by Eqs. (18), (19), and (20). The normalization integral for the D part of ψ given by Eq. (22) is

$$\int(\psi_6+\psi_7+\psi_8)^*(\psi_6+\psi_7+\psi_8)d^3r_i=\int(u_{1D}{}^*u_{1D}+u_{2D}{}^*u_{2D})d^3r_i\,, \tag{32}$$

where u_{1D} and u_{2D} are the D parts of u_1 and u_2 in (22).

The volume element of integration may be written, with the help of Eqs. (5),

$$d^3r_i=d^3rd^3\rho=(\tfrac{3}{4})^{3/2}d^3R_1d^3R_2=(3^{3/2}/8)R_1{}^2dR_1d\Omega_1R_2{}^2dR_2d\Omega_2\,,$$

where we have dropped the integration over the nuclear center of mass. The angle integrations over $d\Omega_1$ and $d\Omega_2$ are easily performed by first fixing $\mathbf{R}_1$ and integrating over the directions of $\mathbf{R}_2$, and then integrating over the directions of $\mathbf{R}_1$. The first angle integration, either in the normalization integral (32), or in the form factor integrals (27), multiplies everything that does not involve S_2 by 4π, causes odd powers of S_2 to vanish, and replaces $S_2{}^2$ by $(4\pi/3)4R_1{}^2R_2{}^2$ and $S_2{}^4$ by $(4\pi/5)16R_1{}^4R_2{}^4$. The second angle integration simply multiplies the normalization integral by 4π, and replaces $\exp(i\mathbf{q}\cdot\mathbf{r}_1)$ in the form factors integrals by [see Eq. (28)]

$$(4\pi/qr_1)\sin qr_1=(4\pi/3^{-1/2}qR_1)\sin(3^{-1/2}qR_1)\,. \tag{33}$$

The remaining integrations over the magnitudes of $\mathbf{R}_1$ and $\mathbf{R}_2$ may be evaluated by using the Irving transformation,[26] according to which they are regarded as

[25] The analysis of this paragraph is equivalent to but somewhat simpler than that in the corresponding part of Sec. II of Ref. 5.

[26] J. Irving, Phil. Mag. 42, 338 (1951).

the rectangular components of a single new two-dimensional vector, and the integration is performed in polar coordinates. Thus we put $R_1=R\cos\theta$, $R_2=R\sin\theta$, and replace the integral

$$\int_0^\infty\int_0^\infty dR_1dR_2 \quad\text{by}\quad \int_0^\infty\int_0^{1/2\pi} RdRd\theta;$$

note that from Eq. (8), $S_s=R^2$.

A straightforward application of the procedure outlined in the preceding paragraph to the normalization integral (32) shows that ψ_8 is orthogonal to ψ_6 and ψ_7, but that ψ_6 and ψ_7 are not orthogonal to each other. We therefore redefine ψ_6 as a linear combination of Eqs. (18) and (19). It is also convenient to redefine ψ_7 by multiplying Eq. (19) by S_s in order to make it homogeneous with the other two. From this point on we retain the definition (20) of ψ_8, but replace Eqs. (18) and (19) by

$$\psi_6=[(5D_sS_2-2D_2S_s)\eta_1 -(5D_sS_1-2D_1S_s)\eta_2]f_6(S_s), \quad (34)$$

$$\psi_7=(D_2S_s\eta_1-D_1S_s\eta_2)f_7(S_s).$$

The expressions for the D parts of $u_1^*u_1$ and $u_2^*u_2$ are then

$$u_{1D}^*u_{1D}=(4S_s^4+25S_1^4-17S_s^2S_1^2-4S_s^2S_2^2+25S_1^2S_2^2)f_6^2+S_s^2(S_s^2+\tfrac{1}{3}S_1^2-S_2^2)f_7^2 +[S_s^2(S_1^2+S_2^2)+\tfrac{1}{3}(S_1^4+S_2^4)-(14/3)S_1^2S_2^2]f_8^2+4S_s^2(-S_s^2+3S_1^2+S_2^2)f_6f_7 +4S_sS_1(S_s^2-3S_1^2+S_2^2)f_6f_8+2S_sS_1[-S_s^2-\tfrac{1}{3}S_1^2+(7/3)S_2^2]f_7f_8, \quad (35)$$

$$u_{2D}^*u_{2D}=(4S_s^4+25S_2^4-4S_s^2S_1^2-17S_s^2S_2^2+25S_1^2S_2^2)f_6^2+S_s^2(S_s^2-S_1^2+\tfrac{1}{3}S_2^2)f_7^2 +[S_s^2(S_1^2+S_2^2)-(S_1^4+S_2^4)+(10/3)S_1^2S_2^2]f_8^2+4S_s^2(-S_s^2+S_1^2+3S_2^2)f_6f_7 +4S_sS_1(-S_s^2+S_1^2+5S_2^2)f_6f_8+2S_sS_1[S_s^2-S_1^2+(5/3)S_2^2]f_7f_8. \quad (36)$$

Each of the six terms in the sum of Eqs. (35) and (36) should be fully symmetric with respect to nucleon permutations; it is easily verified by application of Eqs. (2) and (4) to S_1 and S_2 that this is the case.

The normalization integrals for the new ψ's are the integrals of the sums of the first terms, second terms, and third terms of (35) and (36). The results of these integrations are

$$\int\psi_6^*\psi_6d^3r_i=\int[8S_s^4-21S_s^2(S_1^2+S_2^2)+25(S_1^2+S_2^2)^2]f_6^2d^3r_i=(3^{1/2}35\pi^3/16)\int_0^\infty f_6^2R^{13}dR,$$

$$\int\psi_7^*\psi_7d^3r_i=\int[2S_s^4-(2/3)S_s^2(S_1^2+S_2^2)]f_7^2d^3r_i=(3^{1/2}5\pi^3/8)\int_0^\infty f_7^2R^{13}dR, \quad (37)$$

$$\int\psi_8^*\psi_8d^3r_i=\int[2S_s^2(S_1^2+S_2^2)-(2/3)(S_1^2+S_2^2)^2]f_8^2d^3r_i=(3^{1/2}7\pi^3/24)\int_0^\infty f_8^2R^{13}dR.$$

Finally we calculate the D^2 parts of the form factors F_a and F_b defined in Eqs. (27). In accordance with (33), these are given by

$$F_{aD}=\int(3^{1/2}/qR_1)\sin(3^{-1/2}qR_1)u_{1D}^*u_{1D}d^3r_i,$$

and a similar expression for F_{bD} with u_{1D} replaced by u_{2D}. Evaluation of the integral over θ makes use of the formula

$$\int_0^{\frac{1}{2}\pi}\sin(z\cos\theta)\cos n\theta d\theta=\tfrac{1}{2}\pi(-1)^{\frac{1}{2}(n-1)}J_n(z),$$

and combinations of the Bessel functions may be reduced by using

$$J_{n-1}(z)+J_{n+1}(z)=(2n/z)J_n(z).$$

The results for the form factors are

$$F_{aD}=(3^{1/2}\pi^3/24)\int_0^\infty(R^{13}/z^2)[3f_6^2(70J_2+93J_6+125J_{10})+12f_7^2(5J_2+3J_6) +4f_8^2(7J_2+6J_6+11J_{10})+576f_6f_7J_6-96f_6f_8(J_4+5J_8)-16f_7f_8(7J_4+5J_8)]dR,$$

$$F_{bD}=(3^{1/2}\pi^3/24)\int_0^\infty(R^{13}/z^2)[15f_6^2(14J_2-9J_6-5J_{10})+60f_7^2(J_2-J_6) +4f_8^2(7J_2+6J_6-13J_{10})+96f_6f_8(J_4-J_8)+112f_7f_8(J_4-J_8)]dR. \quad (38)$$

The argument of each of the Bessel functions is $z=3^{-1/2}qR$.

The form factors (38) may be expanded in powers of q by making use of the power series for the Bessel functions. Since there is a common factor $1/z^2$ in the integrand, the terms with J_2 give the values of the F's at $q=0$, the terms with J_2 and J_4 give the coefficients of q^2 in the series expansions of the F's, etc. It is easily seen in this way that $F_{aD}(0)$ and $F_{bD}(0)$ are equal, that their sum is in agreement with the normalization integrals (37), and that the three ψ's are orthogonal.

VIII. NUMERICAL RESULTS

The simplest way of relating Eqs. (38) to the experimental observations is to ignore possible S'-state contributions and attempt to fit the charge form factors with Eqs. (26), assuming that $F_{\mathrm{ch}}{}^p$ and $F_{\mathrm{ch}}{}^n$ are known.[27] The proton charge form factor is indeed known quite well,[28] and it is reasonable to assume that the neutron charge form factor lies somewhere between zero and $0.02q^2$, the latter being the extrapolation of the Foldy expression.[29] It then follows that the coefficient of q^2 in the difference between F_L and F_O, and hence in the difference between F_a and F_b, is significantly different from zero. It is apparent from Eqs. (38) that a difference term proportional to q^2 can arise only from the f_6f_8 and f_7f_8 parts of the F's. Thus ψ must contain ψ_8, and also at least one of the other two functions.[30,31]

The procedure outlined in the last paragraph leads to the experimental result

$$F_O-F_L\rightarrow(0.05\pm0.01)q^2\,,\quad\text{as}\quad q^2\rightarrow 0\,. \qquad (39)$$

The upper limit corresponds to $F_{\mathrm{ch}}{}^n=0$ and the lower limit to $F_{\mathrm{ch}}{}^n=0.02q^2$. We shall assume in the remainder of this paper that the three f's have the same form, and differ only in their numerical coefficients:

$$f_6=\alpha f\,,\quad f_7=\beta f\,,\quad f_8=f\,.$$

It then follows that the total D-state probability is

$$P_D=(3^{1/2}\pi^3/48)(105\alpha^2+30\beta^2+14)\int_0^\infty f^2R^{13}dR\,,$$

and that the mean-square distance of a nucleon from the center of mass of the nucleus, computed for the D-state wave functions by themselves, is

$$\rho^2=\int_0^\infty f^2R^{15}dR\bigg/\left(6\int_0^\infty f^2R^{13}dR\right).$$

Then in the limit of small q^2:

$$F_O-F_L\rightarrow q^2\rho^2P_D(6\alpha+7\beta)/(210\alpha^2+60\beta^2+28)\,. \qquad (40)$$

The coefficient of $q^2\rho^2P_D$ on the right side of Eq. (40) has a rather flat maximum at about 0.09 as α and β are varied. At this maximum point, comparison of Eqs. (39) and (40) shows that $\rho=3.64$ F when $P_D=0.04$; this appears to be a lower limit for the D state probability.[21] The value for ρ thus obtained is about twice that which corresponds to the main S-state part of the three-nucleon wave function and is perhaps not unreasonable for the D-state part.

The numerical result obtained in the last paragraph is independent of the form of the radial function $f(R)$. Only the first two terms in power-series expansions of the integrands of Eqs. (38) were required. Further terms in these series are easily calculated, but unfortunately show that convergence is very slow unless q^2 is less than about 0.2. Thus the integrals in (38) must be evaluated numerically. The results are not sensitive to the values chosen for α and β so long as they are positive and not too small. They were assumed to correspond to equal probabilities for the three D states: $\alpha=+(2/15)^{1/2}$, $\beta=+(7/15)^{1/2}$. A number of numerical calculations were then performed for various choices of $f(R)$ of the Irving-Gunn form e^{-aR}/R^n. No fit to the experimental results for F_O-F_L over the range of q^2 from 0 to 8 could be obtained. The theoretical computations lead in each case to values that are too small and that (unlike the experiments) change sign for q^2 much less than 8.

IX. CONCLUSIONS

We conclude that reasonable P- and D-state admixtures in the wave functions of H^3 and He^3 cannot account for the striking difference between the charge form factors of these two nuclei. As pointed out earlier,[5] an S'-state admixture of about 4% can account for this difference. However, it seems likely that not more than about 2% S'-state admixture is compatible with inelastic electron scattering from these nuclei,[16] with the rate of slow neutron capture in deuterium,[17] or with the nuclear binding energy.[21] The remaining discrepancy might be accounted for by insufficient accuracy of the assumptions that underlie the use of Eq. (23), by an exchange charge density,[13] or by admixtures of still other states. The last possibility seems to us most likely to be correct, and the $T=\frac{3}{2}$ state[8] appears to be the most promising candidate.

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[27] Computer time was supported by National Science Foundation Grant No. NSF-GP948.

[28] The experiments are summarized in four papers presented at the 1963 International Conference on Nucleon Structure. See *Nucleon Structure* (Stanford University Press, Stanford, California, 1964): K. Berkelman, p. 45; K. W. Chen, A. Cone, J. Dunning, N. F. Ramsey, J. K. Walker, and R. Wilson, p. 55; T. A. Griffy, R. Hofstadter, E. B. Hughes, T. Janssens, and M. R. Yearian, p. 61; B. Dudelzak, B. Grossetête, and P. Lehmann, p. 76.

[29] L. L. Foldy, Revs. Mod. Phys. **30**, 471 (1958).

[30] This possibility was first discussed by N. T. Meister, T. K. Radha, and L. I. Schiff, Phys. Rev. Letters **12**, 509 (1964). However, it was incorrectly stated there that only interference between ψ_7 and ψ_8 would give rise to this q^2 behavior.

[31] The only previous paper in which any of the D states is considered at all carefully is that of Krueger and Goldberg (Ref. 7). Since they limited themselves to consideration of ψ_7 alone, the effect described here does not appear in their work.