case of $1^+(27)$, A_1 is necessarily the $T=2$ member and the remaining members are above A_1.

We can easily see that the X meson ($\eta\pi\pi$) is not $0^-(1)$ as in the vector-pseudoscalar system. It will not be interpreted as a VP_8 resonance.

Another $K\pi\pi$ resonance which is not firmly established was reported to be around 1270 MeV.[37] If $T=\frac{1}{2}$, it will constitute $2^+(8_F)$ together with f^0 and A_2. The Gell-Mann–Okubo mass formula holds almost exactly for them. If $T=\frac{3}{2}$, it may be a member of $1^+(27)$.

Finally, some evidence has been reported for the isoscalar $\pi\rho$ resonance around 975 MeV.[38] From the Dalitz plot of the decay products, the spin-parity seems to be 1^+ or 2^-. This is the lowest VP_8 resonance that has ever been observed. It is quite welcome for the present level schemes. Indeed, a low-lying $1^+(1)$ has been predicted in both of the models. It is one of the most striking characteristics of the present work.

ACKNOWLEDGMENT

The author expresses his sincere thanks to Professor H. Miyazawa for valuable suggestions and discussions.

[37] For not firmly established resonances which were recently observed, see Proceedings of the 1964 International Conference on High-Energy Physics at Dubna, USSR, 1964 (to be published).

[38] J. Bartsch, L. Bondar, W. Brauneck, M. Deutschmann, K. Eikel *et al.*, Phys. Letters **11**, 167 (1964).

PHYSICAL REVIEW VOLUME 138, NUMBER 1B 12 APRIL 1965

High-Energy Limit of Pion-Nucleon Total Cross Section*

Masao Sugawara and Arnold Tubis
Department of Physics, Purdue University, Lafayette, Indiana
(Received 16 November 1964)

A simple estimate of the high-energy limit of the pion-nucleon total cross section is given in terms of the mass and the width of the 33 resonance and the S-wave pion-nucleon scattering lengths. The underlying assumptions are that the forward elastic (charge-nonexchange) scattering amplitude satisfies the usual (once-subtracted) dispersion relation, and that the scattering becomes dominantly absorptive in the high-energy region. This estimate gives 23 mb as the limit of the total cross section, which is in close agreement with what one obtains from a simple extrapolation of the available high-energy cross sections. The present analysis strongly suggests that a simple correlation exists between a pronounced low-energy resonance and the high-energy limit of the scattering amplitude.

IN dispersion theory, the high-energy behavior of the scattering amplitude is uniquely correlated to the low-energy behavior. Therefore, a detailed knowledge of low-energy scattering should yield some information about high-energy scattering as a consequence of the underlying analyticity assumption. If there is a pronounced low-energy resonance, it is possible that this resonance plays a dominant role in the discussion of high-energy scattering. An explicit assumption of such a correlation is made in the Regge-pole theory of strongly interacting particles.[1]

The purpose of this note is to show that one can actually obtain an estimate of the high-energy limit of the pion-nucleon total cross section in terms of the mass and the width of the 33 resonance and also the S-wave scattering lengths. The basic assumptions to be made are that the forward (charge-nonexchange) pion-nucleon scattering amplitude satisfies the usual dispersion relation and that this amplitude rapidly approaches a pure imaginary limit in the high-energy region. The latter of these may be regarded as the assumption that pion-nucleon scattering becomes dominantly absorptive in the high-energy limit. This estimate is then substantiated by a careful numerical analysis. This analysis also enables one to compute the high-energy limit of the total cross section if the high-energy phase of the scattering amplitude is given. This computed limit becomes 23 mb if one assumes the high-energy phase predicted by a reasonable optical potential model[2] of high-energy scattering.

Major conclusions are summarized at the end of this note, together with important remarks, the last of which concerns the question whether the same analysis can be applied to pion-pion scattering.

Let $T(\omega)$ be the forward (charge-nonexchange) pion-nucleon scattering amplitude as a function of the laboratory pion energy ω. We normalize $T(\omega)$ as $\mathrm{Im}T(\omega)=q\sigma(\omega)$, where q is the laboratory pion momentum and

* Work supported by the National Science Foundation and the U. S. Air Force.

[1] According to G. F. Chew and V. L. Teplitz, Phys. Rev. **136**, B1154 (1964), one expects in the Regge-pole theory a reasonable value of the high-energy limit of the pion-pion total cross section from a knowledge of the low-energy pion-pion resonances.

[2] Y. Nambu and M. Sugawara, Phys. Rev. Letters **10**, 304 (1963), and Phys. Rev. **132**, 2724 (1963). The high-energy phase is derived in the second of these references.

$\sigma(\omega)$ is the average of the two total cross sections, $\sigma_{\pi^\pm p}(\omega)$. In this normalization, $T(\mu)$ is given by

$$T(\mu)=4\pi[1+(\mu/M)]\tfrac{1}{3}(a_1+2a_3)\,, \tag{1}$$

where μ and M are the pion and nucleon masses, respectively, and a_1 and a_3 are the S-wave scattering lengths in the channels with total isospin $\frac{1}{2}$ and $\frac{3}{2}$, respectively. The usual dispersion relation for $T(\omega)$ reads

$$T(\omega)=T(\mu)+\left(\frac{2\omega_0 g^2}{\mu^2-\omega_0^2}\right)\left(\frac{\omega^2-\mu^2}{\omega^2-\omega_0^2}\right)+\frac{2(\omega^2-\mu^2)}{\pi}\int_0^\infty\frac{\sigma(\omega')dq'}{\omega'^2-\omega^2}\,, \tag{2}$$

where $\omega_0=\mu^2/2M$ and $g^2/4\pi$ is the pion-nucleon coupling constant. According to Hamilton and Woolcock,[3]

$$g^2/4\pi=0.081\pm0.002\,,$$
$$\tfrac{1}{3}(a_1+2a_3)=(-0.0017\pm0.004)\mu^{-1}. \tag{3}$$

We, therefore, assume $T(\mu)=0$ in the following analysis. The corrections due to the uncertainties in (3) are discussed in (*c*) of the concluding remarks. In the case of $T(\mu)=0$, the phase representation[4] of $T(\omega)$ becomes

$$T(\omega)=c\frac{(\omega^2-\mu^2)(\omega^2+a^2)}{\mu(\omega^2-\omega_0^2)}\exp\left\{\frac{2\omega^2}{\pi}\int_\mu^\infty\frac{\delta(\omega')d\omega'}{\omega'(\omega'^2-\omega^2)}\right\}\,, \tag{4}$$

where a and c are real constants and the phase $\delta(\omega)$ of $T(\omega)$ along the cut, $\infty>\omega\geq\mu$, satisfies $\delta(\mu)=0$, $\pi>\delta(\omega)>0$ for $\infty>\omega>\mu$, and $\delta(\omega)$ approaches $\delta(\infty)=\pi/2$ as $\omega\to\infty$ according to the second of the aforementioned assumptions.

The high-energy limit of $\sigma(\omega)$ can be obtained from the high-energy limit of $T(\omega)$ given by (4). The latter can be found most easily by separating $\delta(\omega')$ into $\delta(\infty)$ and $\delta(\omega')-\delta(\infty)$, thus splitting the integral in the exponent into two parts. The first of these can be evaluated exactly and the limit of the second as $\omega\to\infty$ can easily be inferred.[5] One thus derives our basic equation,

$$\sigma(\infty)=c\exp\left\{-(2/\pi)\int_\mu^\infty[\delta(\omega)-\delta(\infty)]d\omega/\omega\right\}\,. \tag{5}$$

We show in the following that c in (5) can be estimated if one knows the mass and the width of the 33 resonance. We then argue that the exponential factor in (5) is roughly unity.

The phase representation (4) must exhibit the correct residue at $\omega=\omega_0$, which yields

$$-g^2=c\frac{(\omega_0^2-\mu^2)(\omega_0^2+a^2)}{2\mu\omega_0}\times\exp\left\{\frac{2\omega_0^2}{\pi}\int_\mu^\infty\frac{\delta(\omega')d\omega'}{\omega'(\omega'^2-\omega_0^2)}\right\}\,. \tag{6}$$

The upper limit of the exponent can be obtained by replacing $\delta(\omega')$ by π and dropping ω_0^2 in the denominator. This upper limit, though obviously overestimated, amounts only to $\omega_0^2/\mu^2=0.0056$. Therefore, one may ignore this exponential factor in (6). Since $T(\omega)$ has a zero at $\omega=ia$, it follows from the dispersion relation (2) with $T(\mu)=0$, that

$$\left(\frac{2\omega_0 g^2}{\mu^2-\omega_0^2}\right)\left(\frac{a^2+\mu^2}{a^2+\omega_0^2}\right)=\frac{2(a^2+\mu^2)}{\pi}\int_0^\infty\frac{\sigma(\omega)dq}{\omega^2+a^2}\,. \tag{7}$$

Combining (7) with (6) without the exponential factor, one obtains

$$c=(2\mu/\pi)\int_0^\infty\sigma(\omega)dq/(\omega^2+a^2)\,. \tag{8}$$

The 33 resonance appears at $\omega=\omega_{33}=2.40\,\mu$. According to (8), c is determined primarily by the 33 resonance provided a^2 is small compared with ω_{33}^2. One can determine a and also c if one estimates the integral in (8) without a^2. Since the 33 resonance surely dominates in this integral, one assumes a simple resonance formula, $\sigma(\omega)\approx(2/3)\sigma_{33}(\omega)\approx4\pi^2\Gamma_{33}\delta(E-M_{33})/p_{33}^2$, expressed in terms of the mass M_{33} and the full width Γ_{33} of the 33 resonance, the c.m. momentum p_{33} at the 33 resonance, and the total c.m. energy $E=[2\omega M+M^2+\mu^2]^{1/2}$. This estimate yields 1.47 μ^{-2} for the integral in (8) without a^2, using $M_{33}=1237$ MeV and $\Gamma_{33}=90$ MeV. The right-hand side of (7) assumes this number (1.47 μ^{-1}) at $a^2=0$ and increases monotonically as a^2 increases. On the other hand, the left-hand side of (7) is nearly 27 μ^{-1} at $a^2=0$ and decreases very rapidly as a^2 increases. In fact, the left-hand side of (7) becomes only 1.58 μ^{-1} at $a^2=0.1\,\mu^2$, still decreasing fast as a^2 further increases. Therefore, one finds $a^2\approx0.1\,\mu^2$. This figure of a^2 is actually small compared with ω_{33}^2. This also permits one to estimate the integral in (8) by multiplying the same integral without a^2 by $\omega_{33}^2/(\omega_{33}^2+a^2)$. One thus finally estimates c as 1.44 μ^{-2}.

In order to see how accurate the above estimate of c is, we have carried out a careful numerical determination of a and c, using the table of the pion-nucleon total cross sections compiled by Höhler *et al.*[6] We found that

$$a^2=0.103\,\mu^2\,,\; c=1.40\,\mu^{-2}=27.9\text{ mb}. \tag{9}$$

The close agreement of these figures with the above estimates is somewhat accidental. However, it certainly justifies the statement that one can estimate c in (5) by knowing the mass and the width of the 33 resonance.

[3] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[4] M. Sugawara and A. Tubis, Phys. Rev. Letters **9**, 355 (1962); and Phys. Rev. **130**, 2127 (1963). As is shown in these references, no more assumptions are implied in the phase representation (4) than those in the usual dispersion relation (2).

[5] It is assumed here that the phase approaches its limit sufficiently rapidly as $\omega\to\infty$ so as to guarantee a finite limit for this second integral.

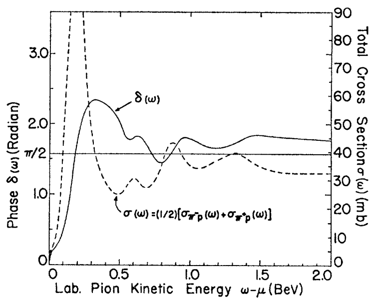

FIG. 1. Pion-nucleon total cross section $\sigma(\omega)$, taken from Ref. 6, and the computed phase $\delta(\omega)$ using the dispersion relation (2), plotted against laboratory pion kinetic energy, ω-μ. Both $\sigma(\omega)$ and $\delta(\omega)$ exhibit very small humps (invisible in this plot) outside this figure and then decrease slowly as ω increases.

We add that the unknown cross sections above 20 BeV hardly affect the figures in (9).

The exponential factor in (5) is much harder to estimate. However, one can argue very plausibly that this factor is likely to be roughly unity if there is a pronounced low-energy resonance. The argument is as follows: The phase $\delta(\omega)$, starting from zero at $\omega=\mu$, rises very steeply towards the low-energy resonance, passes $\frac{1}{2}\pi$ which is equal to $\delta(\infty)$ near the resonance, and is expected to begin to decrease slowly towards $\frac{1}{2}\pi$ beyond the resonance, but to stay around $\frac{1}{2}\pi$ all the way to infinite energy because of strong absorption. Therefore, a strong cancellation is expected to occur in the phase integral in (5) between the energy regions below and above the resonance. The very-high-energy contribution to the phase integral in (5) is also expected to be rather small because the high-energy phase is likely to approach the limit $\delta(\infty)$ quite rapidly. It is, therefore, quite plausible that the exponential factor in (5) is roughly unity.

In order to verify the above argument and also to estimate the phase integral in (5) precisely, we have carried out a numerical determination of the phase $\delta(\omega)$ in terms of the dispersion relation (2). We have used the total cross sections compiled by Höhler *et al.*[6] up to 5 BeV and the following two extrapolations[7] above 5 BeV,

$$\begin{aligned}&\text{(A)}\ \sigma(\omega)=20.94+16.51/q^{0.5},\\&\text{(B)}\ \sigma(\omega)=23.63+24.24/q\quad ,\end{aligned}\tag{10}$$

expressed in terms of mb and BeV units. These are the empirical formulas which fit equally well all the observed cross sections for 20 BeV$\geq\omega\geq$4.5 BeV. We have used these two extrapolations to estimate the effect of the unknown total cross sections above 20 BeV in the determination of the phase.[8] The computed phases are plotted in Fig. 1, together with the cross sections used, up to 2 BeV. The difference between the two cases in (10) is very small almost up to 5 BeV (certainly not visible in Fig. 1), as is exemplified by

$$\begin{aligned}&\delta(5\text{ BeV})-\tfrac{1}{2}\pi=0.204\text{ and }0.199,\\&\delta(10\text{ BeV})-\tfrac{1}{2}\pi=0.167\text{ and }0.154,\end{aligned}\tag{11}$$

corresponding to the cases (A) and (B) of (10), respectively. The phase integral in (5) was carried out numerically up to 5 BeV, yielding

$$-(2/\pi)\int_{\mu}^{5\text{ BeV}}[\delta(\omega)-\delta(\infty)]d\omega/\omega=0.0654\quad\text{and}\quad 0.0675,\tag{12}$$

corresponding to the cases (A) and (B) of (10), respectively. These small figures in (12) actually demonstrate that nearly complete cancellation takes place between the energy regions below and above the 33 resonance. The high-energy contribution of this phase integral can be estimated if one assumes

$$\delta(\omega)-\delta(\infty)=[\delta(5\text{ BeV})-\delta(\infty)](5\text{ BeV}/\omega)^{\alpha},\quad\text{for }\omega\geq5\text{ BeV},\tag{13}$$

with the computed values in (11) for δ(5 BeV). Since the high-energy phase is associated with the high-energy cross section at least in the asymptotic region,[9] we have chosen $\alpha=0.5$ and 1.0, respectively, in the cases of (A) and (B) of (10). Our final estimates of the exponential factor in (5) and the computed limits $\sigma(\infty)$ are as follows:

$$\exp\left\{-(2/\pi)\int_{\mu}^{\infty}[\delta(\omega)-\delta(\infty)]d\omega/\omega\right\}=0.823\quad\text{and}\quad 0.943,\quad\sigma(\infty)=23.0\text{ mb}\quad\text{and}\quad 26.3\text{ mb},\tag{14}$$

corresponding to the cases (A) and (B) of (10), respectively. The figures in (14) demonstrate that the exponential factor in (5) is actually close to unity. It is also clear that this exponential factor is essentially a high-energy quantity in the sense that the small deviation of this factor from unity in (14) is mainly due to the high-energy phase which is close to $\frac{1}{2}\pi$, yet decreasing only slowly towards $\frac{1}{2}\pi$ even in the very-high-energy region.

The major conclusions and important remarks of this note are summarized below:

(a) Our crude estimate of $\sigma(\infty)$, based upon the mass and the width of the 33 resonance is 28 mb in (9). The correction to this estimate depends primarily on the high-energy phase. A reasonable optical potential model[2] of high-energy scattering predicts the high-energy phase (13) with $\alpha=0.5$, and also the high-energy

[6] G. Höhler, G. Ebel, and J. Giesecke, Z. Physik **180**, 430 (1964).

[7] G. von Dardel, D. Dekkers, R. Mermod, M. Vivargent, G. Weber, and K. Winter, Phys. Rev. Letters **8**, 173 (1962).

[8] The extrapolation (A) of (10) is used also by Höhler *et al.* in completing the table of Ref. 6. We have reproduced in this case all the figures in this table, excepting those referring to the lowest energy point. However, the discrepancy is rather small and certainly cannot affect the figures in (12).

[9] The second paper cited in Ref. 2 actually demonstrates this association in the case of $\alpha=0.5$. One can show similarly that an analogous association exists also in the case of $\alpha=1$.

cross section which is essentially the case (A) of (10). Assuming this behavior, our final estimate of $\sigma(\infty)$ becomes 23 mb of (14). This is slightly different from that (21 mb) implied in (10) for the case (A). However, the figures in (10) were obtained[7] by fitting $\sigma_{\pi^\pm p}(\omega)$ individually in terms of the same energy dependence, thus imposing an extra condition on $\sigma_{\pi^-p}(\omega)-\sigma_{\pi^\pm p}(\omega)$. If only $\sigma_{\pi^-p}(\omega)+\sigma_{\pi^\pm p}(\omega)$ is fitted in terms of the same energy dependence, one finds $\sigma(\infty)\simeq 23$ mb.[2] This is in close agreement with our final estimates. In this respect, the optical potential model of the type of Ref. 2 is actually consistent with pion-nucleon scattering above 5 BeV.

(b) The S-wave scattering lengths do not show up explicitly in the preceding analysis since they cancel almost completely in $T(\mu)$ as is seen in (3). However, suppose that these scattering lengths had the same sign. This would make $T(\mu)$ as large as 1.68 μ^{-1}. This figure is even larger than 1.40 μ^{-1} for the integral in (8) which dominates the preceding analysis. Therefore, one must not overlook the importance of the S-wave scattering lengths in the analysis of this type.[10] The importance of the S-wave scattering lengths is evident in the case of pion-pion scattering as is discussed in (e) below.

(c) The corrections due to the uncertainties in (3) were also estimated. The largest correction comes from the uncertainty in $T(\mu)$, which amounts to modifying the figure of c in (9) by roughly 4%.

(d) It is important to observe that the computed phases are very nearly independent of the unknown total cross sections above 20 BeV. One may say that the phase is almost completely determined up to, say, 5 BeV and fairly well determined even at 10 BeV, as is seen in (11). It is interesting to compare these computed values with the direct experimental value. The experimental determination of the phase consists of measuring the forward differential cross section and the total cross section at the same energy. To our knowledge, one of the most accurate measurements of this type in the high-energy region is the one due to Cocconi *et al.*[11] at 10 BeV. Their data give the experimental phase, $\delta(10\ \text{BeV})-\frac{1}{2}\pi=\pm(0.135\pm0.135)$, which should be compared with the computed values in (11). One thus sees that the dispersion method of determining the phase is far more accurate than the direct experimental method. It is also pointed out that the dispersion method consists of using a large number of total cross sections and, therefore, should yield a reliable phase unless all the cross sections used are systematically deviated.

(e) The preceding analysis should be applicable without essential change to pion-pion scattering, even though almost nothing is known in this case except that a pronounced P-wave resonance[12] occurs at $M_\rho=760$ MeV with the full width $\Gamma_\rho=120$ MeV. However, one should observe that this ρ resonance may not dominate an analysis of this type. The importance of the ρ resonance depends on the magnitude of the contribution of this resonance to the integral in (8) without a^2. If one assumes a simple resonance formula of the type used in the previous analysis for the ρ resonance, one estimates this contribution as 0.145 μ^{-2}, which is only one-tenth of the corresponding figure in the case of the 33 resonance. This difference is due to a large difference in the laboratory pion energies at these resonances: $\omega_\rho=13.7\ \mu$ and $\omega_{33}=2.40\ \mu$. In other words, the ρ resonance is not a low-energy resonance from the point of view of this analysis. On the other hand, an S-wave pion-pion scattering length as small as 0.1 μ^{-1} gives rise to $T(\mu)=2.5\ \mu^{-1}$ according to (1) in which $\mu=M$ in this case. This figure is much larger than the figure 1.40 μ^{-1} in (9) which dominates the preceding analysis. Therefore, it is evident that the S-wave scattering lengths play dominant roles in the corresponding analysis of pion-pion scattering. It is possible that a strong correlation exists between the high-energy limit of the pion-pion total cross section and the pion-pion S-wave scattering lenths.

Part of this work was completed while one of the authors (M. S.) was visiting the Lawrence Radiation Laboratory, Berkeley, during the summer of 1964. The hospitality extended to him during this period is greatly appreciated. He wishes to thank, in particular, Professor G. F. Chew and Professor F. E. Low for valuable comments on this work. The present authors also wish to thank G. D. Doolen and Mrs. S. F. Tuan for performing most of the numerical work described in this note.

Note added in proof. Further analysis of pion-pion scattering [see (e) of the concluding remarks] indicates that one can actually estimate the high-energy limit of the pion-pion total section in terms of the ρ resonance, assuming sufficiently weak S-wave interactions.

[10] This point was first brought to our attention by G. F. Chew.

[11] S. Brandt, V. T. Cocconi, D. R. O. Morrison, A. Wroblewski, P. Fleury, G. Kayas, F. Muller, and C. Pelletier, Phys. Rev. Letters **10**, 413 (1963). A possibility of a larger real part of the scattering amplitude implied in this letter should now be discarded because there is no longer such an evidence.

[12] We disregard here the possibility that there is also an S-wave resonance, because the latest experimental evidence is rather against such a possibility.