Approximate Formula for a Regge Trajectory with Right-Hand Cut and Its Application*

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We derive a general ansatz for the imaginary part of a Regge trajectory, with only a right-hand cut, using conformal mapping and threshold behavior. Next we use the dispersion relation and a three-parameter approximation of this ansatz to obtain an analytical formula for the trajectory. If the Pomeranchuk trajectory has no left-hand cut, then application of this formula shows that f^0 does not lie on this trajectory. Further, we find $\alpha_P(\pm\infty)>-1$ and $\gtrsim 0$. Applying the formula to f^0 and ρ, we conclude that, if our approximation is good, then the Regge trajectories corresponding to these resonances must have left-hand cuts.

I. INTRODUCTION

It has long been realized that if the concept of Regge poles[1–4] is to be fruitful in our understanding of particles and resonances and in describing high-energy scattering, then calculation of Regge trajectories is of great physical interest. A large number of theoretical methods[5–8] had been outlined for calculating Regge trajectories self-consistently. However, actual calculations of trajectories have so far produced not very satisfactory results.[9,10] Phenomenological calculations have also been done,[11] assuming certain forms for $\mathrm{Im}\alpha$.

In this paper, we derive an approximate analytical formula for a Regge trajectory, with only a right-hand cut, in terms of hypergeometric functions. Our first step consists in using conformal mapping[12] and threshold behavior to obtain a general ansatz for $\mathrm{Im}\alpha$. Using a three-parameter approximation of this ansatz in the dispersion relation for α, we derive the analytical formula. Next we apply it to the Pomeranchuk, f^0 and ρ trajectories and draw some conclusions on the basis of numerical calculations.

II. DERIVATION OF REGGE TRAJECTORY FORMULA

We shall only consider Regge trajectories having the following properties[5,13]

(i) $\mathrm{Im}\alpha(\nu)\geq 0 \quad \text{for} \quad \infty>\nu>0\,,$

(ii) $\mathrm{Im}\alpha(\nu)\propto \nu^{\alpha_0+\frac{1}{2}} \quad \text{for} \quad \nu\to 0 \quad \text{and} \quad \alpha_0+\frac{1}{2}>0$ $[\text{here} \quad \alpha_0\equiv\alpha(\nu=0)]\,,$

(iii) $$\alpha(\nu)=\alpha_0+\frac{\nu}{\pi}\int_0^\infty \frac{\mathrm{Im}\alpha(\nu')}{(\nu'-\nu)\nu'}d\nu'\,,$$

where ν = square of c.m. momentum. Of these, the first property has been proved for potential scattering only. However, it is expected to hold in general corresponding to the requirement that resonances decay rather than grow in time. The second property has been proved by Barut and Zwanziger.[14] The third property follows from the analyticity of $\alpha(\nu)$ and from the assumption that it is bounded everywhere and does not intersect with any other trajectory.

The Regge trajectory $\alpha(\nu)$ is regular in the ν plane with only a right-hand cut from $\nu=0$ to ∞. Let us consider the conformal transformation

$$\zeta=[1+i(\nu)^{1/2}]/[1-i(\nu)^{1/2}] \tag{1}$$

which maps the whole ν plane into the interior of the unit circle ($|\zeta|=1$). It transforms the cut $\nu=0$ to ∞ into the boundary of the unit circle, with the upper branch of the cut going into the upper semicircle. If we write

$$\alpha(\nu)=\xi(\zeta)\,, \tag{2}$$

then $\xi(\zeta)$ is regular inside the unit circle in the ζ plane and therefore, can be expanded in a uniformly convergent series

$$\xi(\zeta)=\sum_{n=0}^{\infty} a_n\zeta^n\,,\quad (|\zeta|<1)\,. \tag{3}$$

In the Appendix, we shall show that the series (3) can actually be continued to the limit $|\zeta|\to 1$ and the series so obtained will be uniformly convergent and will give the value of $\alpha(\nu)$ on the cut. Our method of proof is similar to that given by Atkinson,[15] with some more details and with some extension of his results.

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From Eq. (1) we find for $\nu>0$, i.e., for ν on the cut, $\zeta=e^{i\theta}$, where

$$\sin\theta=2\nu^{1/2}/(1+\nu)\,, \tag{4a}$$

$$\cos\theta=(1-\nu)/(1+\nu)\,. \tag{4b}$$

Since the series (3) can be used to obtain the value of $\alpha(\nu)$ on the cut, so putting $\zeta=e^{i\theta}$, we get

$$\mathrm{Re}\alpha(\nu)=a_0+a_1\cos\theta+a_2\cos2\theta+\cdots\,, \tag{5}$$

$$\mathrm{Im}\alpha(\nu)=a_1\sin\theta+a_2\sin2\theta+\cdots\,. \tag{6}$$

We now try to find out how the right-hand side in Eq. (6) can reproduce the threshold behavior $\mathrm{Im}\alpha(\nu)\propto\nu^{\alpha_0+1/2}$ when $\nu\to0$. Equation (6) shows that $\mathrm{Im}\alpha(\nu)$ is a function of $\sin\theta$ and $\cos\theta$ alone. From Eq. (4) we therefore deduce that $\mathrm{Im}\alpha(\nu)$ should behave as $\sin\theta\times(\sin^2\theta)^{\alpha_0}$ as $\nu\to0$ to reproduce the right threshold behavior. Let us write

$$\mathrm{Im}\alpha(\nu)=F(\sin\theta,\cos\theta)\sin\theta(\sin^2\theta)^{\alpha_0}\,, \tag{7}$$

where the function $F(\sin\theta,\cos\theta)$ should behave as a constant when $\theta\to0$.

We now want to see how the series (6) can be cast into the form (7). For this purpose, we need the following trigonometric relations:

If n is a positive integer ($n=1,2,3,\cdots$), then

$$\cos2n\theta=P(\cos^2\theta)\,, \tag{8a}$$

$$\sin2n\theta=\sin2\theta P(\cos^2\theta)\,, \tag{8b}$$

$$\sin(2n+1)\theta=\sin\theta P(\cos^2\theta)\,, \tag{8c}$$

where $P(\cos^2\theta)$ denotes a polynominal in $\cos^2\theta$. Using the above relations, we find

$$a_1\sin\theta+a_3\sin3\theta+a_5\sin5\theta+\cdots=\sin\theta P(\cos^2\theta)\,, \tag{9}$$

$$a_2\sin2\theta+a_4\sin4\theta+a_6\sin6\theta+\cdots=\sin2\theta P(\cos^2\theta)\,. \tag{10}$$

From (6), (9), and (10), we have

$$\mathrm{Im}\alpha(\nu)=\sin\theta[P_1(\cos^2\theta)+2\cos\theta P_2(\cos^2\theta)]\,, \tag{11}$$

where $P_1(\cos^2\theta)$ and $P_2(\cos^2\theta)$ are two power series in $\cos^2\theta$.

Now, from (7) we have

$$\frac{\mathrm{Im}\alpha(\nu)}{\sin\theta(\sin^2\theta)^{\alpha_0}}=F(\sin\theta,\cos\theta)=\text{const}\quad\text{as}\quad\theta\to0\,. \tag{12}$$

Also,

$$(\sin^2\theta)^{\alpha_0}=(1-\cos^2\theta)^{\alpha_0}$$
$$=1-\alpha_0\cos^2\theta+[\alpha_0(\alpha_0-1)]/(2!)\cos^4\theta-\cdots$$

is a power series in $\cos^2\theta$. Therefore, in order that (11) satisfy (12), we must have

$$\mathrm{Im}\alpha(\nu)=\sin\theta(\sin^2\theta)^{\alpha_0}[P_3(\cos^2\theta)+2\cos\theta P_4(\cos^2\theta)]\,, \tag{13}$$

where P_3 and P_4 are two new power series. Equation (13) is a general ansatz for $\mathrm{Im}\alpha$ based on the following properties of α: (i) analyticity, (ii) continuity, (iii) limited total fluctuation, and (iv) threshold behavior. Now the last property requires that, for $\theta\to0$,

$$[P_3(\cos^2\theta)+2\cos\theta P_4(\cos^2\theta)]=g\,, \tag{14a}$$

where g is a finite constant. We shall further assume that, for $\theta\to\pi$,

$$[P_3(\cos^2\theta)+2\cos\theta P_4(\cos^2\theta)]=h\,, \tag{14b}$$

where h is a finite constant. Had we known the asymptotic behavior of $\mathrm{Im}\alpha$, then this assumption would not have been necessary.[16] Replacing θ by $\pi-\theta$ in (14b), we get, from (14a) and (14b)

$$\lim_{\theta\to0}P_3(\cos^2\theta)=(g+h)/2$$

and

$$\lim_{\theta\to0}P_4(\cos^2\theta)=(g-h)/4\,.$$

Let us write

$$P_3(\cos^2\theta)=C_1^{(0)}+C_1^{(1)}\cos^2\theta+C_1^{(2)}\cos^4\theta+\cdots \tag{15a}$$

$$P_4(\cos^2\theta)=C_2^{(0)}+C_2^{(1)}\cos^2\theta+C_2^{(2)}\cos^4\theta+\cdots\,. \tag{15b}$$

We first want to show that the series (15a) and (15b) are uniformly converging.

If we call $P_3(\cos^2\theta)=a$ and $P_4(\cos^2\theta)=b$ when $\theta\to0$, then from (15a) and (15b) we have

$$a=C_1^{(0)}+C_1^{(1)}+C_1^{(2)}+\cdots \tag{16}$$

$$b=C_2^{(0)}+C_2^{(1)}+C_2^{(2)}+\cdots\,. \tag{17}$$

Since both a and b are finite constants, therefore, the two infinite series

$$\sum_{n=0}^{\infty}C_1^{(n)}\quad\text{and}\quad\sum_{n=0}^{\infty}C_2^{(n)}$$

occurring in (16) and (17) should converge. Writing $u_n=(\cos^2\theta)^n$, we have, for the series (15a) and (15b)

$$\text{(i)}\qquad\sum_{n=1}^{\infty}C_1^{(n)}\quad\text{and}\quad\sum_{n=1}^{\infty}C_2^{(n)}\quad\text{convergent}$$

$$\text{(ii)}\qquad u_n>u_{n+1} \tag{18}$$

$$\text{(iii)}\qquad u_n<1\quad\text{independent of }n.$$

[16] For example, let us suppose we know $\mathrm{Im}\alpha\propto\nu^{-\beta}$ when $\nu\to\infty$. Now, for $\nu\to\infty$, $\sin\theta\propto\nu^{-1/2}$ and $(1+\cos\theta)\propto\nu^{-1}$. Therefore, $\mathrm{Im}\alpha$ will behave as $\nu^{-\beta}$ if $[P_3(\cos^2\theta)+2\cos\theta P_4(\cos^2\theta)]$ behaves as $(1+\cos\theta)^{-(\alpha_0+1/2)+\beta}$. Let us write

$$[P_3(\cos^2\theta)+2\cos\theta P_4(\cos^2\theta)]$$
$$=[P_5(\cos^2\theta)+2\cos\theta P_6(\cos^2\theta)](1+\cos\theta)^{-(\alpha_0+1/2)+\beta},$$

where $P_5(\cos^2\theta)$ and $P_6(\cos^2\theta)$ are two other power series in $\cos^2\theta$. From the above equation, we find that $[P_3(\cos^2\theta)+2\cos\theta\times P_4(\cos^2\theta)]$ can tend to infinity or zero when $\theta\to\pi$, depending on whether $(\alpha_0+\frac{1}{2})-\beta$ is positive or negative. However, the threshold behavior and the asymptotic behavior of $\mathrm{Im}\alpha$, in this case, will require that $[P_5(\cos^2\theta)+2\cos\theta P_6(\cos^2\theta)]$ behaves as a finite constant when $\theta\to0$ as well as when $\theta\to\pi$.

By Abel's test for convergence[17] the above conditions show that both

$$\sum_{n=1}^{\infty} C_1^{(n)}(\cos^2\theta)^n \quad \text{and} \quad \sum_{n=1}^{\infty} C_2^{(n)}(\cos^2\theta)^n \quad \text{converge.}$$

Physically these results imply that we can keep a few terms in (15a) and (15b) and obtain from Eq. (13) an expression for Imα in terms of a limited number of parameters. Now, in Eq. (15a) we notice that the term $[C_1^{(1)}\cos^2\theta+C_1^{(2)}\cos^4\theta+\cdots]$ is most important relative to the constant term $C_1^{(0)}$ when θ is near zero or π. However, in Eq. (13) both terms are multiplied by $\sin\theta(\sin^2\theta)^{\alpha_0}$. Therefore, for θ near zero or π, the contribution of the term $[C_1^{(1)}\cos^2\theta+C_1^{(2)}\cos^4\theta+\cdots]$ relative to $C_1^{(0)}$ is suppressed in the expression for Imα. On the other hand, $\sin\theta(\sin^2\theta)^{\alpha_0}$ is large for θ near $\pi/2$, where the term $[C_1^{(1)}\cos^2\theta+C_1^{(2)}\cos^4\theta+\cdots]$ is small. Thus, as a first approximation, we can consider $P_3(\cos^2\theta)$ as a constant ($\sim C_1^{(0)}$). Similarly, we can consider $P_4(\cos^2\theta)$ as a constant ($\sim C_2^{(0)}$). We are, therefore, led to the following three-parameter approximation of Imα:

$$\mathrm{Im}\alpha\approx\sin\theta(\sin^2\theta)^{\alpha_0}[c_1+2c_2\cos\theta]. \tag{19}$$

At this point, one may ask why application of Eq. (19) will be of interest, when calculations have been done by Ahmadzadeh and Sakmar[15] and by Domokos,[18] using other forms for Imα. We feel there are two good reasons:

(i) The other forms which have been used are completely phenomenological, while Eq. (19) has been arrived at from theoretical considerations.

(ii) Form (19) gives an asymptotic behavior

$$\mathrm{Im}\alpha\propto\nu^{-\alpha_0-1/2} \quad \text{for} \quad \nu\to\infty$$

which is very different from the asymptotic behavior assumed by the above authors. It will therefore be interesting to see the change in the nature of a Regge trajectory with the change in asymptotic behavior of Imα.

Expressing $\sin\theta$ and $\cos\theta$ as functions of ν, we get from (19)

$$\mathrm{Im}\alpha(\nu)\approx[c_1+2c_2-4c_2\nu/(1+\nu)]\times[2\nu^{1/2}/(1+\nu)]^{2\alpha_0+1}. \tag{20}$$

Inserting this in the dispersion relation

$$\alpha(\nu)=\alpha_0+\frac{\nu}{\pi}\int_0^\infty\frac{\mathrm{Im}\alpha(\nu')d\nu'}{\nu'(\nu'-\nu)}$$

we get

$$\alpha(\nu)=\alpha_0+\frac{\nu}{\pi}(2)^{2\alpha_0+1}[(c_1+2c_2)I_1(\nu)-4c_2I_2(\nu)], \tag{21}$$

where[19]

$$I_1(\nu)=\int_0^\infty\frac{\nu'^{\alpha_0-\frac{1}{2}}}{(\nu'-\nu)(1+\nu')^{2\alpha_0+1}}d\nu' =B(\alpha_0+\tfrac{1}{2},\alpha_0+\tfrac{3}{2})F(1,\alpha_0+\tfrac{3}{2};2\alpha_0+2;1+\nu) \tag{22}$$

and

$$I_2(\nu)=\int_0^\infty\frac{\nu'^{\alpha_0+\frac{1}{2}}}{(\nu'-\nu)(1+\nu')^{2\alpha_0+2}}d\nu' =B(\alpha_0+\tfrac{3}{2},\alpha_0+\tfrac{3}{2})F(1,\alpha_0+\tfrac{3}{2};2\alpha_0+3;1+\nu). \tag{23}$$

In (22) and (23) the B's are beta functions and the F's are hypergeometric functions. In deriving (22) and (23), we have to use $|\arg(-\nu)|<\pi$. Notice that, since the hypergeometric functions F's in (22) and (23) are analytic functions of ν with a right-hand cut from $\nu=0$ to ∞, so the formula (21) explicitly shows the analyticity of $\alpha(\nu)$.

For $\nu>0$, we can separate the real and imaginary parts of $I_1(\nu)$ and $I_2(\nu)$, by using the following relations[20]:

$$F(1,\alpha_0+\tfrac{3}{2};2\alpha_0+2;1+\nu) =\frac{\Gamma(2\alpha_0+2)\Gamma(\alpha_0-\frac{1}{2})}{\Gamma(2\alpha_0+1)\Gamma(\alpha_0+\frac{1}{2})}F(1,\alpha_0+\tfrac{3}{2};\tfrac{3}{2}-\alpha_0;-\nu) +\frac{\Gamma(2\alpha_0+2)\Gamma(\frac{1}{2}-\alpha_0)}{\Gamma(\alpha_0+\frac{3}{2})}(e^{-i\pi}\nu)^{\alpha_0-1/2}(1+\nu)^{-2\alpha_0-1} \tag{24}$$

$$F(1,\alpha_0+\tfrac{3}{2};2\alpha_0+3;1+\nu) =\frac{\Gamma(2\alpha_0+3)\Gamma(\alpha_0+\frac{1}{2})}{\Gamma(2\alpha_0+2)\Gamma(\alpha_0+\frac{3}{2})}F(1,\alpha_0+\tfrac{3}{2};\tfrac{1}{2}-\alpha_0;-\nu) +\frac{\Gamma(2\alpha_0+3)\Gamma(-\alpha_0-\frac{1}{2})}{\Gamma(\alpha_0+\frac{3}{2})}(e^{-i\pi}\nu)^{\alpha_0+1/2}\times(1+\nu)^{-2\alpha_0-2}. \tag{25}$$

In (24) and (25), we have used $-\nu=e^{-i\pi}\nu$ corresponding to the restriction that $|\arg(-\nu)|<\pi$. Using (24) and (25), we get

$$\frac{1}{2i}[I_1(\nu_+)-I_1(\nu_-)]=\pi\nu^{\alpha_0-1/2}(1+\nu)^{-2\alpha_0-1}, \tag{26}$$

$$\frac{1}{2i}[I_2(\nu_+)-I_2(\nu_-)]=\pi\nu^{\alpha_0+1/2}(1+\nu)^{-2\alpha_0-2}. \tag{27}$$

The corresponding real parts of $I_1(\nu)$ and $I_2(\nu)$ are

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[17] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1962), Chap. II, p. 17.

[18] G. Domokos, in *1962 International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962).

[19] *Tables of Integral Transforms, Bateman Manuscript Project*, edited by H. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1954), Vol. 1, p. 310.

[20] *Higher Transcendental Functions, Bateman Manuscript Project*, edited by H. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, Chap. II.

given by

$$\mathrm{Re}I_1(\nu)=-\frac{[\Gamma(\alpha_0+\frac{1}{2})]^2}{\Gamma(2\alpha_0+1)}\frac{1}{(1+\nu)}\times F\left(1,-2\alpha_0;\tfrac{1}{2}-\alpha_0;\frac{1}{1+\nu}\right)-\pi\tan\pi\alpha_0\frac{\nu^{\alpha_0-1/2}}{(1+\nu)^{2\alpha_0+1}},\quad(\nu>0)\quad(28)$$

$$\mathrm{Re}I_2(\nu)=-\frac{[\Gamma(\alpha_0+\frac{1}{2})]^2}{2\Gamma(2\alpha_0+1)}\frac{1}{(1+\nu)}\times F\left(1,-1-2\alpha_0;\tfrac{1}{2}-\alpha_0;\frac{1}{1+\nu}\right)-\pi\tan\pi\alpha_0\frac{\nu^{\alpha_0+1/2}}{(1+\nu)^{2\alpha_0+2}},\quad(\nu>0).\quad(29)$$

For ν large and negative, (22) and (23) are not convenient for numerical calculations. For this case, we can use the following equations:

$$I_1(\nu)=\frac{[\Gamma(\alpha_0+\frac{1}{2})]^2}{\Gamma(2\alpha_0+1)}\frac{1}{(-\nu)}F\left(1,\alpha_0+\tfrac{1}{2};\tfrac{1}{2}-\alpha_0;\frac{1}{-\nu}\right)-\frac{\pi}{\cos\pi\alpha_0}\frac{1}{(-\nu)^{\alpha_0+3/2}}\left(1+\frac{1}{\nu}\right)^{-2\alpha_0-1},\quad(\nu<-1)\quad(30)$$

$$I_2(\nu)=\frac{[\Gamma(\alpha_0+\frac{1}{2})]^2}{2\Gamma(2\alpha_0+1)}\frac{1}{(-\nu)}F\left(1,\alpha_0+\tfrac{3}{2};\tfrac{1}{2}-\alpha_0;\frac{1}{-\nu}\right)-\frac{\pi}{\cos\pi\alpha_0}\frac{1}{(-\nu)^{\alpha_0+3/2}}\left(1+\frac{1}{\nu}\right)^{-2\alpha_0-2}\quad(\nu<-1).\quad(31)$$

Equations (30) and (31) are derived from (22) and (23) by using the analytic continuation of hypergeometric functions.[20]

We are now in a position to derive the asymptotic value of $\alpha(\nu)$. From Eq. (21) and Eqs. (28)–(31), we get

$$\alpha(\infty)=\alpha_0-\frac{1}{\pi}(2)^{2\alpha_0+1}\frac{[\Gamma(\alpha_0+\frac{1}{2})]^2}{\Gamma(2\alpha_0+1)}c_1=\alpha(-\infty).\quad(32)$$

Two quantities of physical interest are $\alpha(\nu)$ at $\nu=-1$ and $(d\alpha/ds)$ at $s=0$, where $s=4\nu+4$. They are respectively given by

$$\alpha(-1)=\alpha_0-\frac{1}{\pi}(2)^{2\alpha_0+1}\frac{[\Gamma(\alpha_0+\frac{1}{2})]^2}{2\Gamma(2\alpha_0+1)}\left[c_1+c_2\frac{1}{\alpha_0+1}\right]\quad(33)$$

and

$$(d\alpha/ds)_{s=0}=[\alpha_0-\alpha(\infty)](\alpha_0+\tfrac{1}{2})/[16(\alpha_0+1)].\quad(34)$$

Before ending this section, we would like to point out that since $\mathrm{Im}\alpha(\nu)\geq 0$ for $\infty>\nu>0$, then from (20) we find

$$\text{(a) for } c_2 \text{ positive},\quad c_1-2c_2>0$$
$$\text{(b) for } c_2 \text{ negative},\quad c_1-2|c_2|>0.\quad(35)$$

Thus, in actual calculation, (35) can be used to find acceptable solutions.

III. APPLICATION TO P, f^0, AND ρ TRAJECTORIES

If the Pomeranchuk trajectory $\alpha_P(\nu)$ has no left-hand cut, then formula (21) can be applied to make predictions about this trajectory. The first problem we have investigated about $\alpha_P(\nu)$ is whether f^0 lies on it. Chew and Frautschi[1] pointed out, on the basis of linear Regge trajectories with slope $\sim(50\mu^2)^{-1}$, that there may occur a spin-2 particle of mass $\sim 7\mu$ on the Pomeranchuk (or vacuum) trajectory. The subsequent discovery of the π-π, $I=0$ resonance[21] f^0 of mass 1250 MeV and spin ≥ 2 led to the conjecture that this may be the predicted particle on the Pomeranchuk trajectory. Ahmadzadeh and Sakmar,[11] using a phenomenological Regge trajectory formula, pointed out that f^0 does not lie on α_P, if one at the same time uses the analysis of the experimental results of Diddens *et al.*[22] and Baker *et al.*[23] But they concluded that the new experimental results on p–p scattering of Foley *et al.*[24] were not in conflict with f^0 lying on α_P. Pignotti[25] has also investigated this problem using dispersion relation and inequality arguments. On the basis of then available experimental results, he concluded that f^0 does not lie on α_P. However, his arguments became inconclusive when analysis[4,26] of the new experimental results of Foley *et al.* indicated that there are other Regge poles which give important contributions.

One way of investigating the above problem using our formula (21) will be to fix the three parameters α_0, c_1, c_2 by three conditions and then see whether $\mathrm{Re}\alpha_P(\nu_r)$ can take the value 2 at the f^0 mass (here ν_r is the value of ν corresponding to f^0). However, we have only one condition,[1] $\alpha_P(-1)=1$. Though we do not have any more conditions, we know[27] $\alpha_P(-\infty)\geq -1$ and also, since $\alpha_P(\nu)$ is monotonic for $\nu<0$, $\alpha_P(-\infty)<1$. Thus $\alpha_P(-\infty)$ lies in the range

$$1>\alpha_P(-\infty)\geq -1.\quad(36)$$

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[25] A. Pignotti, Phys. Rev. Letters **10**, 416 (1963).
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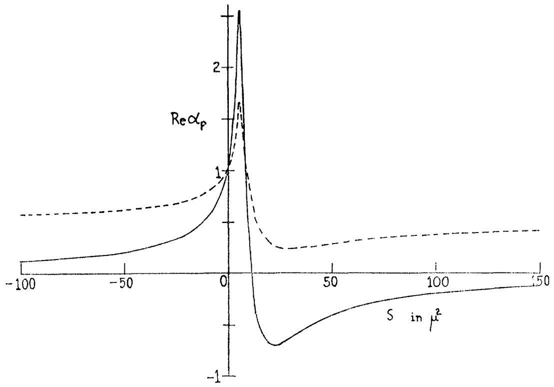

FIG. 1. Possible Pomeranchuk trajectories with only right-hand cuts. Solid line corresponds to $\alpha_P(-\infty)=0$ and $\alpha_0=1.85$; dashed line corresponds to $\alpha_P(-\infty)=0.5$ and $\alpha_0=1.35$.

Again, if f^0 lies on the Pomeranchuk trajectory, then

$$2>\alpha_0>1. \qquad (37)$$

What we can now do is to fix $\alpha_P(-\infty)$ at any point in the range (36) and calculate $\mathrm{Re}\alpha_P(\nu_r)$ for values of α_0 varying in the range (37). Then we can repeat with other values of $\alpha_P(-\infty)$. We have carried out this calculation program on the computer and have found no $\mathrm{Re}\alpha_P(\nu_r)$ equal to 2 or near it. In fact, all the values obtained were less than 1. This shows that f^0 does not lie on $\alpha_P(\nu)$.

Next we have attempted to draw some possible Pomeranchuk trajectories with $\alpha_P(-\infty)$ and α_0 lying in the ranges (36) and (37). We have found that for

$$\alpha_P(-\infty)=-1,\ -0.5,\ -0.25$$

and

$$\alpha_0=1.85,\ 1.60,\ 1.35,\ 1.10$$

no c_1 and c_2 were obtained so that $\mathrm{Im}\alpha_P(\nu)\geq 0$ for all values of $\nu>0$. On the other hand, for the above values of α_0 and for

$$\alpha_P(-\infty)=-0.1,\ 0,\ 0.25,\ 0.5$$

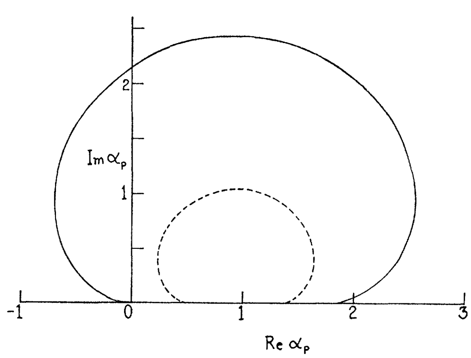

FIG. 2. Calculated Pomeranchuk trajectories in the complex l plane. Solid line corresponds to $\alpha_P(-\infty)=0$, $\alpha_0=1.85$; dashed line corresponds to $\alpha_P(-\infty)=0.5$, $\alpha_0=1.35$.

we find acceptable trajectories. This indicates that $\alpha_P(-\infty)>-1$ and $\gtrsim 0$. Two of the possible trajectories are shown in Fig. 1. The solid line corresponds to a trajectory with $\alpha_P(-\infty)=0$ and $\alpha_0=1.85$; the dashed line corresponds to $\alpha_P(-\infty)=0.5$ and $\alpha_0=1.35$. The corresponding trajectories in the complex l plane are given in Fig. 2. In Fig. 3, we have plotted $\mathrm{Im}\alpha$ for these trajectories against $s(s=4\nu+4)$. If the phase shift due to a Regge pole is taken approximately as $\delta_l\approx\arctan[\mathrm{Im}\alpha/(l-\mathrm{Re}\alpha)]$, then Fig. 2 indicates certain interesting features. For example, for the solid-line trajectory, the s-wave phase shift starts at π at threshold, falls through $\pi/2$ at $s=11.5\ \mu^2$, and goes to zero at infinite energy. It will be interesting to see whether the falling through $\pi/2$ of the s-wave phase shift simulates a resonance of energy $3.6\ \mu=476$ MeV, and in particular, the σ meson.[28]

Let us now apply our formula (21) to the f^0 trajectory assuming, for the moment that this trajectory has only

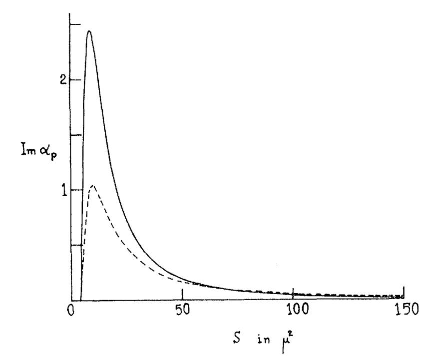

FIG. 3. $\mathrm{Im}\alpha(\nu)$ versus s for the calculated Pomeranchuk trajectories; solid line for $\alpha_P(-\infty)=0$, $\alpha_0=1.85$. dashed line for $\alpha_P(-\infty)=0.5$, $\alpha_0=1.35$.

a right-hand cut. In this case, the spin $(=2)$ and the width of f^0 ($\sim$150 MeV) together with its experimental mass furnish two conditions. Though we do not have the one more condition needed to specify the parameters completely, still we know the range of variation of α_0:

$$2>\alpha_0>-0.5.$$

So what we can do is to take various values of α_0 in the above range and for each value applying the two known conditions, calculate the parameters c_1, c_2. After that, we can examine whether any acceptable trajectory is obtained. We have carried out this calculation program and have found no trajectory for which $\mathrm{Im}\alpha\geq 0$, for all positive values of ν. We have also obtained for all these trajectories $(d\alpha/ds)_{s=0}<0$. These results indicate that for f^0 trajectory, a formula with only a right-hand cut is not possibly adequate and we should have a left-hand cut.

[28] L. Brown and P. Singer, Phys. Rev. 133, B812 (1964).

The above calculation program was also applied to the ρ, where again, the spin ($=1$), width (≈ 100 MeV), mass ($=750$ MeV) and the range of variation of $\alpha_0(1>\alpha_0>-0.5)$ are known. Numerical results similar to those for f^0 were obtained. We found no trajectory that guaranteed $\text{Im}\alpha(\nu)\geq 0$ for $\nu>0$. Thus, for the ρ trajectory, also, we should have a left-hand cut.

IV. CONCLUSION

Our important conclusion is that for relativistic Regge trajectories, such as those of f^0 and ρ, we should have left-hand cuts in addition to the right-hand cuts. This means that these trajectories intersect with other trajectories[29] for $\nu<0$. For the Pomeranchuk trajectory, if there is no left-hand cut, then f^0 does not lie on this trajectory. Further, we find $\alpha_P(-\infty)>-1$ and $\gtrsim 0$. If $\alpha_P(-\infty)$ is positive or zero, then $\alpha_P(\nu)$ never crosses the $l=0$ line and the problem of ghost states[4] disappears. Our calculated Pomeranchuk trajectories can give large π-π s-wave phase shift and it will be worth investigating whether this is connected with the ABC enhancement.[30]

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APPENDIX

The function $\xi(\zeta)=\alpha(\nu)$ is regular inside the unit circle in the ζ plane and can, therefore, be expanded in a uniformly convergent series

$$\xi(\zeta)=\sum_{n=0}^{\infty} a_n\zeta^n = \sum_{n=0}^{\infty} a_n\rho^n e^{in\theta} \quad (\text{putting}\quad \zeta=\rho e^{i\theta},\ \rho<1). \tag{A1}$$

The problem we want to solve is whether the series (A1) can be continued to the limit $\rho\to 1$ and whether the series so obtained is uniformly convergent and can give the value of the function $\alpha(\nu)$ on its cut.

First notice that if ν is on the cut,[31] then the function $\alpha(\nu)$ on the cut is defined by the following limiting process:

$$\alpha(\nu)=\alpha(z) \quad \text{as} \quad z\to\nu. \tag{A2}$$

Since the limit of $\alpha(z)$ when z approaches a point on the cut exists, the limit of $\xi(\rho e^{i\theta})$ when $\rho\to 1$ exists. Let us call this limit $\chi(e^{i\theta})$; that is,

$$\chi(e^{i\theta})=\xi(\rho e^{i\theta}) \quad \text{as} \quad \rho\to 1. \tag{A3}$$

Here we notice that, if the limit $\rho\to 1$ could be taken, then we would obtain a Fourier series from (A1). Let us, therefore, examine under what conditions the Fourier series of the function $\chi(e^{i\theta})$ is uniformly convergent and represents the function itself. According to Fourier's theorem,[32] for a function $f(t)$ that is periodic, so that $f(t+2\pi)=f(t)$, (i) if the integral $\int_{-\pi}^{\pi} f(t)dt$ exists and, in the case of an improper integral, is absolutely convergent, and in any interval, (ii) if the function has limited total fluctuation (i.e., it has a finite number of maxima and minima), and (iii) if the function is continuous, then the Fourier series of the function converges uniformly to its value, at any point in the interval.

The Fourier series for $f(t)$ is given by

$$c_0+\sum_{n=1}^{\infty} (c_n \cos nt+d_n \sin nt), \tag{A4}$$

where

$$c_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt, \qquad c_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt dt, \tag{A5}$$

and

$$d_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt dt.$$

From physical considerations, $\alpha(\nu)$ is a continuous function of ν for all values of ν on the cut. Therefore, $\chi(e^{i\theta})$ is a continuous function of θ for all values of θ, i.e., $-\pi\leq\theta<\pi$. This in turn means that $\int_{-\pi}^{\pi}\chi(e^{i\theta})d\theta$ exists. Thus the function $\chi(e^{i\theta})$ satisfies conditions (i) and (iii). Since for $\alpha(\nu)$ we do not expect an infinite number of maxima and minima to occur, then $\chi(e^{i\theta})$ can be considered to obey condition (ii) as well. Thus, the Fourier series for $\chi(e^{i\theta})$ converges uniformly to the function itself for all values of $\theta(-\pi\leq\theta<\pi)$.

Let us now find the Fourier coefficients of $\chi(e^{i\theta})$:

$$c_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\chi(e^{i\theta}) \cos n\theta d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{Re}\chi(e^{i\theta}) \cos n\theta d\theta \tag{A6.1}$$

$$d_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\chi(e^{i\theta}) \sin n\theta d\theta = \frac{i}{\pi}\int_{-\pi}^{\pi} \text{Im}\chi(e^{i\theta}) \sin n\theta d\theta \qquad (n=1, 2, 3, \cdots). \tag{A6.2}$$

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[29] In nonrelativistic scattering by a superposition of Yukuwa potentials, it is known that crossing of trajectories happens for an infinite number of Regge trajectories; see H. Cheng, Phys. Rev. **130**, 1283 (1963). It is worth mentioning that from the Mandelstam representation one would expect not only a right-hand cut, but also a left-hand cut for a Regge trajectory; see Ref. 14.

[30] A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters **5**, 258 (1960); N. E. Booth and A. Abashian, Phys. Rev. **132**, 2314 (1963).

[31] By this we mean $\nu=\nu\pm i\epsilon$ and $\infty>\nu>0$.

[32] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1962), Chap. IX.

Relations (A6.1) and (A6.2) follow from the fact that

$$\chi(e^{-i\theta})=\text{Re}\chi(e^{i\theta})-i\,\text{Im}\chi(e^{i\theta}).$$

Again,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\chi(e^{i\theta})e^{in\theta}d\theta=\frac{1}{\pi i}\int_{-\pi}^{\pi}\chi(e^{i\theta})e^{i(n-1)\theta}d(e^{i\theta})$$

$$=\frac{1}{\pi i}\int_{c}\chi(\zeta)\zeta^{(n-1)}d\zeta, \tag{A7}$$

where the integration is taken on the inner boundary of the circle. Since the integrand is an analytic function regular inside the circle, then the right-hand side in (A7) is zero. This gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\text{Re}\chi(e^{i\theta})\cos n\theta d\theta-\frac{1}{\pi}\int_{-\pi}^{\pi}\text{Im}\chi(e^{i\theta})\sin n\theta d\theta=0,$$

that is, $d_n=ic_n$. Thus, from (A4), we get,

$$\chi(e^{i\theta})=\sum_{n=0}^{\infty}c_ne^{in\theta}, \tag{A8}$$

where the series is uniformly converging. If we can now show that the c_n's are the same as the a_n's occurring in (A1), then our problem is solved.

For this purpose, we take ν on the cut. Now, the continuity of the function $\alpha(z)$ means that, given any positive number ϵ, however small, we can find another positive number $\gamma(\epsilon,\nu)$ such that

$$|\alpha(\nu)-\alpha(z)|<\epsilon \quad \text{for} \quad |\nu-z|<\gamma(\epsilon,\nu). \tag{A9.1}$$

Correspondingly, in the ζ plane we have

$$|\chi(e^{i\theta})-\xi(\rho e^{i\theta})|<\epsilon \quad \text{for} \quad 1-\rho<\delta(\epsilon,\theta). \tag{A9.2}$$

Statement (A9.2) implies that, for a given ϵ, with each value of θ, a positive number $\delta(\epsilon,\theta)$ is associated, such that the inequality in (A9.2) is satisfied. If we now take the smallest of these positive numbers $\delta(\epsilon,\theta)$ and call it $\delta(\epsilon)$, then

$$|\chi(e^{i\theta})-\xi(\rho e^{i\theta})|<\epsilon \quad \text{for} \quad 1-\rho<\delta(\epsilon) \tag{A10}$$

for any θ. Thus (A9.1) and (A10) lead us to the conclusion that $\xi(\rho e^{i\theta})$ tends to $\chi(e^{i\theta})$ uniformly for $\rho\to 1$. Notice that we have arrived at this result from continuity alone and did not have to postulate it, as has been done by Atkinson.[15]

Subtracting (A1) from (A8), we have

$$\chi(e^{i\theta})-\xi(\rho e^{i\theta})=\sum_{n=0}^{\infty}(c_n-a_n\rho^n)e^{in\theta}. \tag{A11}$$

The series (A11) is uniformly convergent for $0<1-\rho$, so that we can integrate it term by term and obtain

$$c_n-a_n\rho^n=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\chi(e^{i\theta})-\xi(\rho e^{i\theta})]e^{-in\theta}d\theta. \tag{A12}$$

From (A12), we get, for $0<1-\rho$

$$|c_n-a_n\rho^n|<\frac{1}{2\pi}\int_{-\pi}^{\pi}|\chi(e^{i\theta})-\xi(\rho e^{i\theta})|d\theta.$$

Therefore, using (A10),

$$|c_n-a_n\rho^n|<\epsilon \quad \text{for} \quad 0<1-\rho<\delta. \tag{A13}$$

Hence

$$c_n=a_n\rho^n \quad \text{as} \quad \rho\to 1; \quad \text{i.e.,} \quad c_n=a_n. \tag{A14}$$

Thus, the series

$$\xi(\rho e^{i\theta})=\sum_{n=0}^{\infty}a_n\rho^ne^{in\theta}$$

can be continued to the limit $\rho\to 1$ and the series so obtained is

$$\xi(e^{i\theta})=\sum_{n=0}^{\infty}a_ne^{in\theta},$$

which is uniformly convergent and which gives the value of $\alpha(\nu)$ on the cut.