C6. H. R. Rugge and O. T. Vik, Phys. Rev. **129**, 2300 (1963)—7.
C7. R. E. Hill, N. E. Booth, R. J. Esterling, D. L. Jenkins, N. H. Lipman, H. R. Rugge, and O. T. Vik, Bull. Am. Phys. Soc. **9**, 410 (1964) [superseded by R. E. Hill (private communication)]—8.
C8. R. D. Eandi, T. J. Devlin, P. G. McManigal, and B. J. Moyer, Phys. Rev. **136**, B536 (1964)—6, 7.
C9. O. Chamberlain, C. D. Jeffries, C. H. Schultz, G. Shapiro, and L. Van Rossum, Phys. Letters **7**, 293 (1963) [superseded by C. Schultz, Lawrence Radiation Laboratory (Berkeley) Report UCRL-11149, 1964 (unpublished)]—6.

PHYSICAL REVIEW VOLUME 138, NUMBER 1B 12 APRIL 1965

Nuclear Recoil in the Decay of Bound Muons; A Possible Tool in Muonic-Atom Studies

RICHARD M. WEINER
Labirint 28, Bucharest, Rumania
(Received 27 October 1964)

The nuclear-recoil spectrum in the decay of a bound muon is calculated using sudden perturbation theory and with neglect of the final Coulomb interaction between the μ-decay electron and the nucleus and of relativistic corrections to the muon wave function. In this approximation the recoil momentum distribution is identical with the square of the Bessel-Hankel transform of the bound muon wave function. By measuring the recoil spectrum one could gain important information about the muonic atom.

INTRODUCTION

THE muon, besides its intrinsic interest as an elementary particle, is also an important tool in nuclear and atomic physics. Due to the relatively large mass of the muon compared to that of the electron, the atomic system nucleus+muon has a radius more than 200 times smaller than the system nucleus+electron. That is why muonic atoms are very sensitive to the nuclear charge distribution and why muonic molecules can be formed which lead to fusion reactions with the muon playing the role of a catalyst which reduces the Coulomb potential barrier.

These facts, among others, explain the great interest in muonic atoms and molecules and the increasing number of works, both theoretical and experimental, dedicated to this subject.

Up to the present, information about muonic atoms has been gained by measuring the energies of the x and γ rays emitted in atomic transitions. However, this information is limited for several reasons:

(1) The energy does not specify completely the bound state of the muon.

(2) One measures only energy differences.

(3) The measured x-ray yields are still in disagreement with the theoretical predictions,[1] especially at low Z.

Some progress concerning item (1) has been achieved recently with the discovery of the hyperfine-structure effect in the nuclear capture of muons, but the information yielded by this effect (the relative spin orientation of the muon and the nucleus) is also limited.

Therefore a method which could offer more complete information about the bound states of the muon and about the corresponding absolute energy values would be highly desirable.

As to muonic molecules in nuclear reactions it would be very important to know what happens to the muon after the nuclear reaction, i.e., if and in what system it remains bound before decaying.

It is the purpose of this note to suggest a method which might in the near future, with the muon beam intensities available from meson factories, contribute to the solution of the problems mentioned above.

In atomic physics absolute energy levels are measured in ionization processes. With muonic atoms, because of the instability of these systems and the low intensities of muonic beams, such experiments are at present, and in the foreseeable future, impossible. However, we will show that the radioactive character of the muon is just the feature which opens up new possibilities in this field.

Quantum mechanics tells us that the most complete information about a state is given by the wave function (w.f.), with the aid of which all the characteristics of the state, including its energy, can be calculated.

In the following we want to suggest a possible method for the direct experimental determination of the squared wave function of a bound muon. This method would consist in the measurement of the momentum distribution of the nuclei recoiling in the decay of bound muons. This distribution is closely related to the muon w.f. in the bound state from which it decays. For light nuclei the recoil distribution is practically identical with the momentum distribution of the muon in the Bohr orbit. Only muons could be studied in this way. For mesons like π and K mesons, the nuclear capture process predominates even at low Z.

[1] Y. Eisenberg and D. Kessler, Phys. Rev. **130**, 2349 (1963).

THE DECAY OF BOUND MUONS

The effect of binding on the electron spectrum and the decay rate of muons was predicted a long time ago by Corinaldesi[2] and Primakoff[3,4] and has been discussed in more detail recently[5–7] in connection with experimental work[8–10] on the decay rate. Although the average recoil energy has been evaluated in Ref. 3, since then, as far as we can gather, nobody has been concerned with this particular problem.

Consider a muonic atom in a stationary state Ψ^i. Since the muon is more than 200 times nearer the nucleus than the K electrons, we disregard the electrons in the following.

The w.f. of the initial state of the muon-plus-nucleus system can be written

$$\Psi_{FM}{}^i=\sum_{m_I} C(I,j,F;m_I,M-m_I)\psi_{I,m_I}\varphi_{j,M-m_I}{}^{n(l)}. \quad (1)$$

$C(I,j,F;m_I,M-m_I)$ are the Clebsch-Gordan coefficients, ψ_{I,m_I} the nuclear w.f., and $\varphi_{j,m_j}{}^{n(l)}$ the Schrödinger-Pauli w.f. of the muon in the field of the nucleus.[11] We have

$$\varphi_{j,m_j}{}^{n(l)}=\sum_{\mu} C(\tfrac{1}{2},l,j;\mu,m_j-\mu)\chi_\mu Y_{l,m_j-\mu}\varphi_{n,l}. \quad (2)$$

χ_μ are the components of the muon spinor w.f. and $\varphi_{n,l}$ is the radial w.f. Both the spherical functions $Y_{l,m-\mu}$ and $\varphi_{n,l}$ depend on the relative coordinate $\mathbf{r}_n-\mathbf{r}_\mu$, with $\mathbf{r}_n$ and $\mathbf{r}_\mu$ denoting, respectively, the coordinates of the nucleus (as a whole) and of the muon, in the laboratory system. In Eq. (1) it is assumed that the total linear momentum of the system in the initial state is zero.

After the muon decays there are four particles in the final state: the recoiling nucleus (atom), the electron and the two neutrinos. For low Z and the energies involved, one can neglect in a first approximation the final electromagnetic interaction between the nucleus and the emitted electron.[12] Applying sudden perturbation theory, the final w.f. of the system is then

$$\Psi^f=e^{i\mathbf{k}_n\cdot\mathbf{r}_n}e^{i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_e)\cdot\mathbf{r}_e}u_eu_1u_2\psi_{I',m_I'}, \quad (3)$$

where $\mathbf{k}_n$, $\mathbf{k}_e$, $\mathbf{k}_1$, $\mathbf{k}_2$ are the momenta of the nucleus, electron, and the two neutrinos, respectively, u_e, u_1, u_2 the leptonic spinors, and $\psi_{I',m_I'}$ the final nuclear w.f.

The matrix element of our process is

$$\begin{aligned}\mathfrak{M}&=\int\Psi^{*f}\mathcal{O}\Psi^i d\tau\\ &=\sum_{m_I,\mu} C(I,j,F;m_I,M-m_I)C(\tfrac{1}{2},l,j;\mu,M-m_I-\mu)\\ &\quad\times(\bar{u}_1\mathcal{O}_\alpha u_2)(\bar{u}_e\mathcal{O}_\alpha\chi_\mu)\int\varphi_{n,l}Y_{l,M-m_I-\mu}e^{-i\mathbf{k}_n\cdot\mathbf{r}_n}\\ &\quad\times e^{-i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_e)\cdot\mathbf{r}_e}d\mathbf{r}_nd\mathbf{r}_e\int\psi_{I',m_I'}{}^*\psi_{I,m_I}d\mathbf{R}. \quad (4)\end{aligned}$$

$\mathcal{O}$ is the weak-decay operator, $\mathbf{R}$ denotes the coordinates which define the nuclear w.f., and $\mathbf{r}_e=\mathbf{r}_\mu$. By putting $\mathbf{r}_n=\mathbf{r}+\mathbf{r}_e$ and integrating over $\mathbf{R}$ and $\mathbf{r}_e$, Eq. (4) becomes

$$\begin{aligned}\mathfrak{M}=\sum_{m_I,\mu} &C(I,j,F;m_I,M-m_I)C(\tfrac{1}{2},l,j;\mu,M-m_I-\mu)\\ &\times(\bar{u}_1\mathcal{O}_\alpha u_2)(\bar{u}_e\mathcal{O}_\alpha\chi_\mu)\delta(\mathbf{K})\bar{c}_{n,l,\ldots}(\mathbf{k}_n)\delta_{I,I'}\delta_{m_I,m_I'}, \quad (5)\end{aligned}$$

where

$$\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_e+\mathbf{k}_n \quad (6)$$

and

$$\bar{c}_{n,l\ldots}=\int\varphi_{n,l}(\mathbf{r})Y_{l,M-m_I'-\mu}e^{-i\mathbf{k}_n\cdot\mathbf{r}}d\mathbf{r}. \quad (7)$$

Expanding $e^{-i\mathbf{k}_n\cdot\mathbf{r}}$ into a series of spherical harmonics and integrating over the angular variables of $\mathbf{r}$, we get

$$\bar{c}_{n,l}=Y_{l,M-m_I'-\mu}(\mathbf{k}_n/k_n)c_{n,l}(k_n), \quad (8)$$

where

$$c_{n,l}(k_n)=\int\varphi_{n,l}(r)g_l(k_nr)r^2dr, \quad (9)$$

with g_l the lth spherical Bessel function.

The matrix element can now be written

$$\mathfrak{M}=\sum_\mu A_\mu Y_{l,M-m_I'-\mu}(\mathbf{k}_n/k_n)c_{n,l}(k_n), \quad (10)$$

where A_μ do not depend on $\mathbf{k}_n$. The corresponding probability is

$$P\sim\prod_{i=1}^{4}\sum\int|\mathfrak{M}|^2d\mathbf{k}_i\delta(\text{energy}). \quad (11)$$

From Eqs. (10) and (11) all the correlation formulas of the decay of bound muons, taking into account the nuclear recoil, can be obtained. Thus, e.g., by summing over $m_{I'}$ and the electron polarization and integrating over $\mathbf{k}_n$ ($\mathbf{k}_e$), one can get the angular distribution of

---

2 E. Corinaldesi, Nuovo Cimento **4**, 132 (1947).
3 H. Primakoff, Phys. Rev. **74**, 878 (1948).
4 C. E. Porter and H. Primakoff, Phys. Rev. **83**, 849 (1951).
5 H. Überall, Phys. Rev. **119**, 365 (1960).
6 V. Gilinsky and J. Mathews, Phys. Rev. **120**, 1450 (1960).
7 R. Huff, Ann. Phys. (N.Y.) **16**, 288 (1961).
8 R. A. Lundy, J. C. Sens, R. A. Swanson, V. L. Telegdi, and D. D. Yovanovitch, Phys. Rev. Letters **1**, 102 (1958).
9 W. A. Barret, F. E. Holmstrom, and J. W. Keuffel, Phys. Rev. **113**, 661 (1959).
10 A. Astbury, M. Hussain, M. A. Kemp, N. H. Lipman, H. Muirhead, R. G. P. Voss, and A. Kirk, Proc. Phys. Soc. (London) **73**, 314 (1959).
11 This nonrelativistic form for $\varphi_{j,m_j}{}^{n(l)}$ is a very good approximation at low and medium Z.
12 An evaluation of the error involved in this approximation can be obtained by applying an analogous argument as in the theory of β decay. There it is considered that the final Coulomb interaction affects the electron w.f. mainly at the origin where the influence of the nuclear Coulomb field is described by the Fermi function $F(Z,E)=|\varphi_{\text{coul}}(0)/\varphi_{\text{free}}(0)|^2$; φ_{coul} is the correct electron w.f. and φ_{free} a plane wave electron w.f. For low Z and high energies E, F differs only by a few percent from unity.

$\mathbf{k}_e$ ($\mathbf{k}_n$) at a given muon polarization μ; by averaging over M and summing over $m_{I'}$ and the electron polarization one obtains the angular correlation $\mathbf{k}_n \cdot \mathbf{k}_e$, and so on. In all cases of physical interest one has to sum over the neutrino spins and to integrate with respect to the corresponding momenta.

A simplification arises if we compare the average recoil momentum $\langle k_n \rangle_{\text{av}}$ with the leptonic momenta. From Eq. (7) we see that $\langle k_n \rangle_{\text{av}} \simeq k_\mu$, where k_μ is the average muon momentum in the Bohr orbit. Thus $\langle k_n \rangle_{\text{av}}$ is at least four orders of magnitude smaller than the average momenta of the leptons and we can neglect k_n in δ(energy) in Eq. (11). In this case, the emission of leptons and the recoil are incoherent and *the distribution of k_n is simply given by $|c_{n,l}|^2$, the square of the Bessel-Hankel transform of the bound w.f.*

For a bound muon state with $l=0$ (this is the most important case from the experimental point of view, because it is believed that at least 99% of the bound muons, which are not captured by the nucleus, decay from the ground state $1s_{1/2}$),

$$\bar{c}_{n,l=0} = c_{n,0} = \int e^{-i\mathbf{k}_n \cdot \mathbf{r}} \varphi_{n,0}(r) d\mathbf{r}\,, \qquad (12)$$

and $\mathfrak{M}$ is proportional to $\bar{c}_n$. In this case the recoil distribution is isotropic and identical with the momentum distribution of the bound muons in the Bohr orbit.

THE AVERAGE RECOIL AND THE ISOTOPE EFFECT—EXPERIMENTAL POSSIBILITIES

The average recoil energy is given by

$$\langle E_{\text{rec}} \rangle_{\text{av}} = k_\mu^2 / 2\text{M}; \quad k_\mu = 1/r_B\,, \qquad (13)$$

where r_B is the muon Bohr radius in the orbit from which it decays, and M the mass of the recoiling nucleus. From Eq. (13) we see that, for low Z, two isotopes of the same element will have substantially different recoil energies. We are thus faced with an analog of the mass effect in the isotope shift of atomic spectroscopy.

Another isotope effect in the recoil spectrum is due to the dependence of the nuclear charge distribution on the nuclear radius. This is the analog of the volume effect in the isotope shift. These effects may also provide information about the formation of molecules just before the muon decays.

To get an order of magnitude of the average recoil energy as given by Eq. (13), we will assume that the decay takes place in the K orbit (the stopped muon reaches this orbit in $\sim 10^{-10}$ sec). Furthermore, we will consider $\varphi_{n,0}$ in Eq. (12) to be the solution of the Schrödinger equation with a Coulomb potential. This leads to the following values of $\langle E_{\text{rec}} \rangle_{\text{av}}$: For H, $E_{\text{rec}} \sim 0.3$ keV, for Ne^{20} $E_{\text{rec}} \sim 1.6$ keV, and for Hg^{200}, $E_{\text{rec}} \sim 10$ keV.

To measure such small energies the recoil techniques of nuclear spectroscopy could be applied. A major handicap in the use of these methods would be the low pressure of the gas target in the spectrometer, which diminishes seriously the number of muonic atoms formed. However, with 10^6 negative muons per second stopped in liquid hydrogen, as one may hope to realize in the near future with a meson factory,[15] a reasonable number of events per second is to be expected even at low gas pressures.

FINAL DISCUSSION

Besides the principal reasons given in the introduction for which the detection of the recoil spectrum in muonic atoms is desirable, we should like to mention the following:

(a) It would be of great importance in connection with the x-ray puzzle[1] to determine the percentage of muons which reach metastable levels from which they decay or are captured by the nucleus. The recoil distribution may bring new insight to this problem. In the same order of ideas, the recoil spectrum may provide information on degenerate states, with the same energy but different wave functions.

(b) The muon beams are usually contaminated with π mesons which may confuse the x-ray spectrum but not the recoil spectrum (see the introduction).

ACKNOWLEDGMENTS

I should like to express my gratitude to Dr. E. M. Friedländer for instructive discussions on the experimental aspects of these problems and for reading the manuscript.

The author wishes also to thank Professor J. Ausländer, Dr. A. Gelberg, and V. Rittenberg for helpful comments on the manuscript.

A discussion with Professor E. H. S. Burhop is gratefully acknowledged.

[13] D. Yovanovitch, Phys. Rev. **117**, 1580 (1960).

[14] J. M. Blair, H. Muirhead, T. Woodhead, and J. N. Woulds, Proc. Phys. Soc. (London) 80, 938 (1962).

[15] E. R. Beringer, W. A. Blanpied, R. L. Gluckstern, V. W. Hughes, H. B. Knowles, S. Ohruma, and G. W. Wheeler, International Conference on Sector Focused Cyclotrons and Meson Factories, CERN, 1963 (unpublished) [quoted by G. I. Opat, Phys. Rev. **134**, B428 (1964)].