This is because there is a one-to-one correspondence between the octet bases and the operators H_i and E_α, and we normally consider that the charge conjugate of ζ_α is $\epsilon_\alpha\zeta_{-\alpha}$ (see Appendix). Then it is obligatory that the charge conjugate of H_i be $-H_i$ (see Sec. V).

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APPENDIX

Let us suppose ζ_α to be a self-conjugate multiplet such as the singlet or an octet, etc. Therefore, $\zeta_\alpha^* = \epsilon_\alpha\zeta_{-\alpha}$ with a certain numerical constant ϵ_α. Then if

$$\mathcal{R}\zeta_\alpha = a\zeta_\alpha\,, \tag{A1}$$

it follows that

$$\mathcal{R}\zeta_\alpha^* = a\zeta_\alpha^*. \tag{A2}$$

Then we have the two relations,

$$C\mathcal{R}\zeta_\alpha = Ca\zeta_\alpha = a\zeta_\alpha^* \tag{A3}$$

and

$$C\mathcal{R}\zeta_\alpha = C\mathcal{R}C^{-1}C\zeta_\alpha = -\mathcal{R}\zeta_\alpha^* = -a\zeta_\alpha^*. \tag{A4}$$

Apparently these two relations contradict each other unless $a=0$. Thus the invariant operator $\mathcal{R}$ which satisfies the relation (21) has zero eigenvalue for any self-conjugate multiplet.

Indeed the eigenvalue of the operator $I(3)$ vanishes for the self-conjugate multiplet, as is easily seen in the following. The eigenvalue of $I(3)$ is given in Ref. 3 as

$$\sum_i (l_i^3 - r_i^3)\,, \tag{A5}$$

where

$$l_i = m_i + r_i \quad \text{and} \quad r_i = \tfrac{1}{2}\sum_{\alpha>0} C_{i\alpha}{}^\alpha.$$

The m_i represent the highest weight and $r_1=1$, $r_2=0$, and $r_3=-1$. In the case of the octet, $m_1=1$, $m_2=0$ and $m_3=-1$, so that $\sum_i(l_1^3-r_1^3)=0$. More generally, for the self-conjugate multiplet, the highest weight m_i satisfies the relations,

$$m_1 = -m_3 \quad \text{and} \quad m_2 = 0.$$

Thus, it is evident that $\sum_i(l_i^3-r_i^3)=0$.

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Spin Determination for Boson Resonances*

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A method of spin and parity determination has been worked out for boson resonances of spin S which decay into a spin-1 particle and a spin-0 particle. It is shown that the quantity $S(S+1)$ can be given in terms of experimentally measurable averages. This affords a straightforward way of determining the spin uniquely. For the parity assignment, one finds that certain experimental averages are identically zero for one parity case and not the other. In addition, for the parity case in which two orbital angular momenta are allowed, decay parameters as well as multipole parameters for the spin-S particle can be determined.

I. INTRODUCTION

THE purpose of this paper is to present a method of determining spin and parity of boson resonances which decay into a spin-1 particle and a spin-0 particle. This method can be applied to $B\to\pi+\omega$, $A_{1,2}\to\pi+\rho$, and $K_C\to K+\rho$.[1] Several authors have discussed these problems before.[2–4]

In this paper we adopt the approach of Byers and Fenster, which they used to describe fermion resonances decaying into spin-$\frac{1}{2}$ and spinless particles.[5–8] We also use the helicity formalism of Jacob and Wick[9] for the decay particle of spin 1, which makes it possible to bring out certain salient features of the problem as well as to reduce the amount of algebra required. Perhaps the most interesting result would be the relation (23), with which one could determine, with enough statistics, the spin of the resonant particle unambiguously.[10]

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[1] For data on these particles, see the Summary by A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

[2] C. Zemach, Nuovo Cimento **32**, 1605 (1964).

[3] M. Ademollo, R. Gatto, and G. Preparata, Phys. Rev. Letters **12**, 462 (1964).

[4] W. R. Frazer, J. R. Fulco, and F. R. Halpern, Phys. Rev. **136**, B1207 (1964).

[5] N. Byers and S. Fenster, Phys. Rev. Letters **11**, 52 (1963).

[6] A similar approach has been applied to boson resonances decaying into two spinless particles; see P. E. Schlein, Phys. Rev. **135**, B1453 (1964).

[7] A slightly different approach to fermion resonances has been made by M. Ademollo and R. Gatto, Phys. Rev. **133**, B531 (1964).

[8] For applications of tests proposed by Byers and Fenster, see P. E. Schlein *et al.*, Phys. Rev. Letters **11**, 167 (1963) and J. B. Shafer and D. O. Huwe, Phys. Rev. **134**, B1372 (1964).

[9] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[10] For similar relation for fermion resonances, see Ref. 7; also J. B. Shafer in Ref. 8.

II. MULTIPOLE PARAMETERS

Consider a reaction of the type

$$A+B\to S\ (\text{spin } S)+C\,,\qquad S\to X\ (\text{spin } 1)+Y\ (\text{spin } 0). \tag{1}$$

The spin state of S is described in its own rest frame (S RF), where, for convenience, the quantization axis (z axis) is chosen to be along the production normal. One may define the density matrix of S in this frame by

$$\rho^{(s)}=\sum_{\mu\mu'}\rho_{\mu\mu'}{}^{(s)}|S\mu\rangle\langle S\mu'|\,, \tag{2}$$

where $|S\mu\rangle$ is the familiar spin state in the S RF.

Let us define an operator $T_L{}^M$ by[11]

$$T_L{}^M=\sum_{\mu\mu'}(S\mu'LM|S\mu)|S\mu\rangle\langle S\mu'|\,, \tag{3}$$

where $(S\mu'LM|S\mu)$ is the usual Clebsch-Gordan coefficient. In terms of this, one now defines the "multipole parameter" $t_L{}^M$ by[5]

$$\rho^{(s)}=\sum_{LM}(\hat{L}/\hat{S})^2 t_L{}^{M*}T_L{}^M\,, \tag{4}$$

where $\hat{L}=(2L+1)^{1/2}$ and $\hat{S}=(2S+1)^{1/2}$. Comparing (2) and (4), one finds

$$\rho_{\mu\mu'}{}^{(s)}=(\hat{L}/\hat{S})^2\sum_{LM}t_L{}^{M*}(S\mu'LM|S\mu)\,; \tag{5}$$

then, by inverting,

$$t_L{}^{M*}=\sum_{\mu\mu'}\rho_{\mu\mu'}{}^{(s)}(S\mu'LM|S\mu)\,. \tag{6}$$

Note that $L\leqslant 2S$ and that $t_0{}^0=1$ since $\mathrm{tr}\rho^{(s)}=1$. Because the Hermiticity of the density matrix, one has

$$t_L{}^{M*}=(-)^M t_L{}^{-M}. \tag{7}$$

If A and B are unpolarized and one sums over the spin states of C, one has the condition,[12] for parity-conserving reactions,

$$\rho_{\mu\mu'}{}^{(s)}=e^{i(\mu-\mu')\pi}\rho_{\mu\mu'}{}^{(s)}. \tag{8}$$

Substituting (8) into (6), one gets[5]

$$t_L{}^M=(-)^M t_L{}^M. \tag{9}$$

This means that $t_L{}^M=0$ for odd M.

We now turn to the description of the decay of S. Let us define $|\lambda\rangle$ to be the helicity state for the decay particle X and $\mathbf{k}$ to be its momentum in the direction $\Omega\equiv(\theta,\phi)$. If $\mathfrak{M}$ is the transition matrix for S decaying into X and Y, where X has momentum $\mathbf{k}$ and helicity λ, then $\rho^{(s)}$ becomes (see Appendix I):

$$\begin{aligned}\mathfrak{M}\rho^{(s)}\mathfrak{M}^\dagger &=\sum_{LM}(\hat{S}\hat{L}/4\pi)t_L{}^{M*}\sum_{\lambda\lambda'}(-)^{s-\lambda'}(S\lambda S-\lambda'|L\lambda-\lambda')\\ &\quad\times\mathfrak{D}_{M,\lambda-\lambda'}{}^{(L)*}(\phi,\theta,-\phi)A_\lambda A_{\lambda'}{}^*|\lambda\rangle\langle\lambda'|\,,\end{aligned} \tag{10}$$

where A_λ is the "helicity amplitude" for the decay.[13] It depends on the orbital angular momentum l_S. Assuming parity is conserved, one has two different sets for helicity amplitudes:

$$\begin{array}{ll} l_s=S\pm1 & l_s=S\\ A_{+1}=(1/\sqrt{2}\hat{S})[a_+S^{1/2}+a_-(S+1)^{1/2}] & A_{+1}=1/\sqrt{2}\\ A_{-1}=A_{+1} & A_{-1}=-1/\sqrt{2}\\ A_0=(1/\hat{S})[-a_+(S+1)^{1/2}+a_-S^{1/2}] & A_0=0\,, \end{array} \tag{11}$$

where $a_\pm=$ amplitudes for $l_s=S\pm1$ and $|a_+|^2+|a_-|^2=1$. If $S=0$, only one angular momentum state is possible, i.e., $l_s=1$. In this case, one has $A_{+1}=A_{-1}=0$ and $A_0=-1$.

The decay angular distribution can be calculated from Eq. (10) by taking the trace. Using the relation

$$\mathfrak{D}_{M0}{}^{(L)*}(\phi,\theta,-\phi)=[(4\pi)^{1/2}/\hat{L}]Y_L{}^M(\Omega)\,, \tag{12}$$

one gets, for angular distribution,

$$\begin{aligned}I(\Omega)&=\sum_{LM}[\hat{S}/(4\pi)^{1/2}]t_L{}^{M*}Y_L{}^M(\Omega)\\ &\quad\times\{\sum_\lambda(-)^{s-\lambda}(S\lambda S-\lambda|L0)|A_\lambda|^2\}.\end{aligned} \tag{13}$$

Note that the expression inside the bracket vanishes for odd L or for $L>2S$.

One may describe the density matrix of X in the same way as $\rho^{(s)}$. For the purpose, one defines the rest frame of X as follows: First, one rotates the S RF by Euler angles[14] $(\phi,\theta,-\phi)$ and then goes to the rest frame of X by pure time-like Lorentz transformation.[15] In this frame one may define the density-matrix operator $\rho^{(1)}$ for X by [in analogy to (2) and (3)]

$$\rho^{(1)}=\sum_{lm}(\hat{l}^2/3)r_l{}^{m*}\sum_{\lambda\lambda'}(1\lambda'lm|1\lambda)|\lambda\rangle\langle\lambda'|\,, \tag{14}$$

where $r_l{}^m$ stands for the multipole parameter of X. Note that $l\leqslant 2$ and that $r_0{}^0=1$ since $\mathrm{tr}\rho^{(1)}=1$. Note also that states $|\lambda\rangle$ are just the helicity states for X.

III. RELATION FOR SPIN DETERMINATION

Since the trace of $\rho^{(1)}$ is equal to 1, one may write $\mathfrak{M}\rho^{(s)}\mathfrak{M}^\dagger=I(\Omega)\rho^{(1)}$. One then obtains, by comparing (10) and (14),

$$\begin{aligned}I(\Omega)&\sum_l(\hat{l}/\sqrt{3})r_l{}^{(\lambda-\lambda')*}(-)^{1-\lambda'}(1\lambda1-\lambda'|l\lambda-\lambda')\\ &=\sum_{LM}(\hat{S}\hat{L}/4\pi)t_L{}^{M*}(-)^{s-\lambda'}(S\lambda S-\lambda'|L\lambda-\lambda')\\ &\quad\times A_\lambda A_{\lambda'}{}^*\mathfrak{D}_{M,\lambda-\lambda'}{}^{(L)*}(\phi,\theta,-\phi)\,,\end{aligned} \tag{15}$$

after using the relation (A10). Using the formula[14]

$$\int d\Omega\ \mathfrak{D}_{m\lambda}{}^{(l)*}\mathfrak{D}_{m'\lambda}{}^{(l')}=(4\pi/\hat{l}^2)\delta_{ll'}\delta_{mm'}\,, \tag{16}$$

[11] See Eq. (20), Ref. 5.
[12] R. H. Capps, Phys. Rev. **122**, 929 (1961).
[13] For the definition of $\mathfrak{D}_{MM'}{}^{(L)}$, see Ref. 9.
[14] See A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).
[15] See H. P. Stapp, Phys. Rev. **103**, 425 (1956).

one gets, from (15)

$$\sum_l(\hat{l}/\sqrt{3})(1\lambda 1-\lambda'|l\lambda-\lambda')$$
$$\times\int d\Omega\, I(\Omega) r_l^{(\lambda-\lambda')*}\mathfrak{D}_{M,\lambda-\lambda'}{}^{(L)}(\phi,\theta,-\phi)$$
$$=(-)^{s+1}(\hat{S}/\hat{L})t_L{}^{M*}(S\lambda S-\lambda'|L\lambda-\lambda')A_\lambda A_{\lambda'}{}^*.$$

For a fixed $m\equiv\lambda-\lambda'$, multiplying both sides by $(1\lambda 1-\lambda'|lm)$ and summing over λ and λ', one obtains

$$(\hat{l}\hat{L}/\sqrt{3})\langle r_l{}^m\mathfrak{D}_{Mm}{}^{(L)*}(\phi,\theta,-\phi)\rangle$$
$$=(-)^{s+1}\hat{S}t_L{}^M\sum_{\lambda\lambda'}(1\lambda 1-\lambda'|lm)(S\lambda S-\lambda'|Lm)A_\lambda{}^*A_{\lambda'},\quad(17)$$

where one has taken the complex conjugate.

Let us denote the left-hand side of (17) by

$$G(lm;LM)\equiv(\hat{l}\hat{L}/\sqrt{3})\langle r_l{}^m\mathfrak{D}_{Mm}{}^{(L)*}(\phi,\theta,-\phi)\rangle.\quad(18)$$

Then, we have

$$G(lm;LM)$$
$$=(-)^{s+1}\hat{S}t_L{}^M\sum_{\lambda\lambda'}(1\lambda 1-\lambda'|lm)(S\lambda S-\lambda'|Lm)A_\lambda{}^*A_{\lambda'}.\quad(19)$$

It is shown in Appendix II that $G(lm;LM)$ can be evaluated from experiment for $l=0$ or 2.

Using (7), (11), and (19), one has the condition

$$G(lm;LM)=(-)^LG(l-m;LM)\quad(\text{even } l)\quad(20\text{a})$$

and

$$G^*(lm;LM)=(-)^{L+M}G(lm;L-M)\quad(\text{even } l).\quad(20\text{b})$$

These relations show that it is not necessary to consider negative values of m and M when evaluating $G(lm;LM)$.

It is convenient to write down the relation (19) explicitly for different values of l and m:

$$G(22;LM)=(-)^{s+1}\hat{S}t_L{}^M(S1S1|L2)A_{+1}{}^*A_{-1}\quad(\text{even } L),\quad(21\text{a})$$
$$G(21;LM)=(-)^{s+1}\hat{S}t_L{}^M(S1S0|L1)\times(1/\sqrt{2})\{A_{+1}{}^*A_0+(-)^LA_0{}^*A_{-1}\},\quad(21\text{b})$$
$$G(20;LM)=(-1)^{s+1}\hat{S}t_L{}^M(2/3)^{1/2}\{(S1S-1|L0)|A_{+1}|^2+(S0S0|L0)|A_0|^2\}\quad(\text{even } L),\quad(21\text{c})$$
$$G(00;LM)=(-)^{s+1}\hat{S}t_L{}^M(1/3)^{1/2}\{2(S1S-1|L0)|A_{+1}|^2-(S0S0|L0)|A_0|^2\}\quad(\text{even } L).\quad(21\text{d})$$

From (21c) and (21d), one has

$$\sqrt{2}G(00;LM)+G(20;LM)$$
$$=(-)^{s+1}\hat{S}t_L{}^M(6)^{1/2}(S1S-1|L0)|A_{+1}|^2\quad(\text{even } L),\quad(22\text{a})$$
$$-G(00;LM)+\sqrt{2}G(20;LM)$$
$$=(-)^{s+1}\hat{S}t_L{}^M\sqrt{3}(S0S0|L0)|A_0|^2\quad(\text{even } L).\quad(22\text{b})$$

The Clebsch-Gordan coefficients in (21a) and (22a) can be expressed in terms of $(S0S0|L0)$ if L is even:

$$(S1S-1|L0)=\{[L(L+1)/2S(S+1)]-1\}(S0S0|L0),$$
$$(S1S1|L2)=[L(L+1)/(L-1)(L+2)]^{1/2}(S0S0|L0).$$

Taking the ratio of (21a) and (22a) and using the above two relations, one gets, for the spin $S(\geqslant 1)$,

$$S(S+1)=\frac{L(L+1)G(22;LM)}{\epsilon[2L(L+1)/3(L-1)(L+2)]^{1/2}[\sqrt{2}G(00;LM)+G(20;LM)]+2G(22;LM)},\quad(23)$$

where L is even $(\geqslant 2)$ and $\epsilon=\pm1$. $\epsilon=+1$ corresponds to $l_s=S\pm1$, and $\epsilon=-1$ to $l_s=S$.

It is to be understood that the relation (23) is true both for real and imaginary parts of $G(lm;LM)$ separately for all allowed values of L and M. Using (9) and (20b), one can show that there are $(L+1)$ independent tests for a given L. Note that formula (23) can be applied only after the parity is determined.

IV. TESTS FOR SPIN AND PARITY

Suppose $S=0$. Then only one angular momentum state is possible, i.e., $l_s=1$. For this case, all $G(lm;LM)$ should vanish except $G(00;00)$ and $G(20;00)$. Furthermore, we must have $G(00;00)=1/\sqrt{3}$ and $G(20;00)=-(2/3)^{1/2}$.

Now, consider the case $S\geqslant 1$. If the parity of S is such that $l_s=S$, we must have $A_0=0$ from (11). Therefore, for this parity assignment, we have

$$G(21;LM)=0,\quad(24\text{a})$$

where L can be either even or odd. Also, from (22b),

$$G(00;LM)=\sqrt{2}G(20;LM)\quad(\text{even } L).\quad(24\text{b})$$

One does not expect in general that these conditions hold for the other parity case $(l_s=S\pm1)$, so that conditions (24a) and (24b) afford a means of determining the parity of S. However, for the latter parity case $(l_s=S\pm1)$, one may have $G(21;LM)\approx0$ for odd L, if $l_s=S-1$ dominates over $l_s=S+1$. So (24a) is a strong test only for even L.

In order to determine the spin itself, one applies the condition that $G(lm;LM)=0$, if $L>2S$. This gives the minimum value of S consistent with the experimental data. For direct determination of the spin, one uses the relation (23). If $L_{\max}$ is the largest *even* value of L for which $G(lm;LM)$ is nonzero, one has

$$(1/4)L_{\max}(L_{\max}+4)$$

independent tests available for (23).

We refer to Ref. 8 for the statistical treatment involved in evaluating $G(lm;LM)$ from experimental data. One notes that some care needs to be taken when using the formula (23), for the statistical distribution of $S(S+1)$ as evaluated by the relation is not of Gaussian form. However, the distribution *can* be calculated for various hypotheses of S,[16] from which one can assess confidence levels on the experimental value of $S(S+1)$.

Once spin and parity are determined from experiment, one can evaluate decay parameters if the decay proceeds by two orbital angular momentum states $l_s=S\pm1$ (we consider the case $S\geqslant 1$). Decay parameters are defined in the usual way [see (11)]:

$$\begin{cases}\alpha=2\,\mathrm{Re}a_-a_+^*,\\ \beta=2\,\mathrm{Im}a_-a_+^*,\\ \gamma=|a_-|^2-|a_+|^2,\end{cases}\tag{25}$$

where $\alpha^2+\beta^2+\gamma^2=1$. In terms of these,

$$|A_{+1}|^2=[4(2S+1)]^{-1}\{2S+1+\gamma+2\alpha[S(S+1)]^{1/2}\},\tag{26a}$$

$$|A_{-1}|^2=|A_{+1}|^2,\tag{26b}$$

$$|A_0|^2=[2(2S+1)]^{-1}\{2S+1-\gamma-2\alpha[S(S+1)]^{1/2}\},\tag{26c}$$

$$A_{+1}{}^*A_0=[2\sqrt{2}(2S+1)]^{-1}\{-\alpha+2\gamma[S(S+1)]^{1/2}+i\beta(2S+1)\}.\tag{26d}$$

Using (21), (22), and (26), one evaluates various ratios from $G(lm;LM)$ for given L and M (both even) but with different l and m. This gives two (or more) independent equations involving α and γ, so that one can solve for them. Note that the sign of β cannot be determined.

Once these parameters are obtained, one can determine $t_L{}^M$ for all allowed values of L and M by using (21) and (22). However, one cannot determine the over-all sign of $t_L{}^M$ if L is odd. For a consistency check, one may apply inequality relationships which exist for absolute values[17] of $t_L{}^M$. If the parity of S is such that $l_s=S$, it is not possible to determine $t_L{}^M$ for odd L.

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[16] J. B. Shafer and D. W. Merrill, Lawrence Radiation Laboratory Report No. UCRL-11884, 1965 (unpublished).

[17] See Eqs. (23) and (24), Ref. 5.

APPENDIX I

Here we derive Eq. (10). We may define the "helicity amplitude" for the decay of S by

$$\mathfrak{M}|S\mu\rangle\equiv(-)^{s+1}\sum_\lambda A_\lambda\langle\Omega|S\mu;\lambda\rangle,\tag{A1}$$

where $\langle\Omega|S\mu;\lambda\rangle$ stands for a two-particle state with the relative momentum in the direction Ω and eigenvalues S and μ and helicity λ. In terms of orbital angular momentum l_s, one may write

$$\begin{aligned}\mathfrak{M}|S\mu\rangle&=\sum_{l_s}(-)^{l_s}a_{(l_s-S)}\langle\Omega|l_s1S\mu\rangle\\&=\sum_\nu(-)^{s+\nu}a_\nu\langle\Omega|S+\nu,1;S\mu\rangle,\end{aligned}\tag{A2}$$

where one has set $l_s\equiv S+\nu$ and $\nu=-1,0,+1$, and where a_ν is the amplitude for the orbital angular momentum $l_s=S+\nu$.

There is a prescription[18] which connects the helicity state with a state of definite orbital angular momentum:

$$(S\mu;\lambda|l_s1S\mu)=(\hat{l}_s/\hat{S})(l_s01\lambda|S\lambda).\tag{A3}$$

Using this relation, one has

$$\langle\Omega|l_s1S\mu\rangle=\sum_\lambda\langle\Omega|S\mu;\lambda\rangle(S\mu;\lambda|l_s1S\mu).$$

Then, from (A2),

$$\mathfrak{M}|S\mu\rangle=(-)^s\sum_\lambda\langle\Omega|S\mu;\lambda\rangle\times\sum_\nu(-)^\nu a_\nu[(2S+2\nu+1)^{1/2}/\hat{S}](S+\nu,0;1\lambda|S\lambda).$$

One may compare this with (A1) to get:

$$A_\lambda=\sum_\nu(-)^{\nu+1}a_\nu[(2S+2\nu+1)^{1/2}/\hat{S}]\times(S+\nu,0;1\lambda|S\lambda).\tag{A4}$$

The Clebsch-Gordan coefficients in (A4) can be calculated in general[13]

$$\begin{aligned}&(S-1,0;1,\pm1|S,\pm1)=[(S+1)/2(2S-1)]^{1/2},\\&(S+1,0;1,\pm1|S,\pm1)=[S/2(2S+3)]^{1/2},\\&(S,0;1,\pm1|S,\pm1)=\mp1/\sqrt{2},\\&(S-1,0;1,0|S,0)=[S/(2S-1)]^{1/2},\\&(S+1,0;1,0|S,0)=-[(S+1)/(2S+3)]^{1/2},\\&(S,0;1,0|S,0)=0,\end{aligned}\tag{A5}$$

where the first four relations are valid for $S\geqslant1$. Using these, one gets, from (A4),

$$\begin{aligned}A_{+1}&=(1/\sqrt{2})\{a_+[S/(2S+1)]^{1/2}+a_-[(S+1)/(2S+1)]^{1/2}+a_0\},\\A_{-1}&=(1/\sqrt{2})\{a_+[S/(2S+1)]^{1/2}+a_-[(S+1)/(2S+1)]^{1/2}-a_0\},\\A_0&=-a_+[(S+1)/(2S+1)]^{1/2}+a_-[S/(2S+1)]^{1/2}.\end{aligned}\tag{A6}$$

Note that $\sum_\lambda|A_\lambda|^2=1$ if $\sum_\nu|a_\nu|^2=1$. Since parity is conserved, (A6) breaks up into two different sets, which are given in (11).

States $\langle\Omega|S\mu;\lambda\rangle$ can be expressed[19] in terms of

[18] See Eq. (B5), Ref. 9.

[19] See Eq. (18), Ref. 9.

$\mathfrak{D}_{\mu\lambda}^{(s)*}(\phi,\theta,-\phi)$ and the helicity state $|\lambda\rangle$ of X:

$$\langle\Omega|S\mu;\lambda\rangle=[\hat{S}/(4\pi)^{1/2}]\mathfrak{D}_{\mu\lambda}^{(s)*}(\phi,\theta,-\phi)|\lambda\rangle. \quad \text{(A7)}$$

Substituting (A1) and (A7) into (4), one gets

$$\mathfrak{M}\rho^{(s)}\mathfrak{M}^{\dagger}=\sum_{LM}(\hat{L}^2/4\pi)t_L{}^{M*}\sum_{\mu\mu'}(S\mu'LM|S\mu) \times\sum_{\lambda\lambda'}\mathfrak{D}_{\mu\lambda}^{(s)*}(\phi,\theta,-\phi)\mathfrak{D}_{\mu'\lambda'}^{(s)}(\phi,\theta,-\phi) \times A_\lambda A_{\lambda'}^*|\lambda\rangle\langle\lambda'|. \quad \text{(A8)}$$

Using the relation[14]

$$\mathfrak{D}_{m_1m_2}^{(l)*}\mathfrak{D}_{m_1'm_2'}^{(l')} =\sum_j(lm_1;l'-m_1'|jm_1-m_1')(lm_2;l'-m_2'|jm_2-m_2') \times(-)^{m_1'-m_2'}\mathfrak{D}_{m_1-m_1',m_2-m_2'}^{(j)*}, \quad \text{(A9)}$$

and

$$(l_1m_1l_2m_2|l_3m_3) =(-)^{l_1-m_1}(\hat{l}_3/\hat{l}_2)(l_3m_3;l_1-m_1|l_2m_2) \quad \text{(A10)}$$

one can reduce (A8) into Eq. (10).

APPENDIX II

In order to evaluate $G(lm;LM)$ from experiment, one first needs to relate $r_l{}^m$ to experimentally measurable quantities.

For convenience, the rest frame of X as defined in Sec. II may be referred to as the X RF$_1$. In this frame, one defines a unit vector (or a pseudovector) $\hat{n}$ to describe the decay of X. Thus, in the case of ω, $\hat{n}$ is the unit vector normal to the decay plane of ω, whereas for the ρ decay $\hat{n}$ stands for the momentum direction for one of the decay pions.

As X decays, its helicity state $|\lambda\rangle$ transforms into $Y_1{}^\lambda(\hat{n})$. Using (14), one may then calculate the following average in the X RF$_1$:

$$\langle Y_l{}^{m*}(\hat{n})\rangle=\sum_{l'm'}\frac{\hat{l}^2}{3}r_l{}^{m*}\sum_{\lambda\lambda'}(1\lambda'l'm'|1\lambda) \times\int d\Omega(\hat{n})Y_1{}^\lambda(\hat{n})Y_1{}^{\lambda'*}(\hat{n})Y_l{}^{m*}(\hat{n}),$$

where $d\Omega(\hat{n})$ is the element of solid angle in the $\hat{n}$ space. The integral on the right-hand side is equal to

$$[3/(4\pi)^{1/2}\hat{l}](1010|l0)(-)^{\lambda'}(1\lambda1-\lambda'|lm).$$

Using (A10), one gets finally

$$r_l{}^m=-(4\pi/3)^{1/2}\langle Y_l{}^m(\hat{n})\rangle_\Omega/(1010|l0), \quad \text{(A11)}$$

where the average is to be performed for a fixed Ω, so that $r_l{}^m$ is now a function of Ω. Note that $r_l{}^m$ can be evaluated in this way only for even l, i.e., $l=0$ or $l=2$.

Now, one may evaluate $G(lm;LM)$ from experiment by using (A11) and (18):

$$G(00;LM)=(4\pi/3)^{1/2}\frac{1}{N}\sum_{i=1}^{N}Y_L{}^M(\Omega_i),$$
$$G(2m;LM)=-(10\pi/3)^{1/2}\frac{\hat{L}}{N}\sum_{i=1}^{N}Y_2{}^m(\hat{n}_i) \times\mathfrak{D}_{Mm}^{(L)*}(\phi_i,\theta_i,-\phi_i), \quad \text{(A12)}$$

where the sum is over all events in which the decay of the particle S is observed and N is the total number of these events. Note that $Y_L{}^M(\Omega_i)$ and $\mathfrak{D}_{Mm}^{(L)*}(\phi_i,\theta_i,-\phi_i)$ are evaluated in the S RF, whereas $Y_2{}^m(\hat{n}_i)$ is evaluated in the X RF$_1$.

Since $m\leq2$, $\mathfrak{D}_{Mm}^{(L)*}$ can be easily related to simpler functions. For convenience, we list a few useful formulas involving $\mathfrak{D}_{Mm}^{(L)}$[20]:

$$\mathfrak{D}_{Mm}^{(L)}(\phi,\theta,-\phi)=e^{-i(M-m)\phi}d_{Mm}^{(L)}(\theta), \quad \text{(A13)}$$

$$2[(L+2)(L+1)]^{1/2}d_{M2}^{(L)}(\theta) =[(L+M)(L+M-1)]^{1/2}(1+\cos\theta)d_{M-1,1}^{(L-1)}(\theta) +2(L^2-M^2)^{1/2}\sin\theta\, d_{M1}^{(L-1)}(\theta) +[(L-M)(L-M-1)]^{1/2}(1-\cos\theta) \times d_{M+1,1}^{(L-1)}(\theta) \quad \text{(A14)}$$

$$d_{M1}^{(L)}(\theta)=-[L(L+1)]^{-1/2}\{M(\csc\theta+\cot\theta)d_{M0}^{(L)}(\theta) +[(L-M)(L+M+1)]^{1/2}d_{M+1,0}^{(L)}\}, \quad \text{(A15)}$$

$$d_{M0}^{(L)}(\theta)=(-)^M\left[\frac{(L-M)!}{(L+M)!}\right]^{1/2}P_L{}^M(\cos\theta), \quad \text{(A16)}$$

where $P_L{}^M(\cos\theta)$ is the associated Legendre polynomial.

There exists an alternative method of evaluating $G(lm;LM)$. It involves defining multipole parameters of X in a different coordinate system. Let X RF$_2$ be the rest frame of X obtained by pure time-like Lorentz transformation directly from the S RF (no intermediate spatial rotation). Then the X RF$_1$ and the X RF$_2$ are related by Euler angles $(\phi,\theta,-\phi)$ with respect to each other. Since $r_l{}^m$ transforms in the same way as spherical harmonics $Y_l{}^m$ under spatial rotations,[5] one has

$$r_l{}^m=\sum_{m'}\tilde{r}_l{}^{m'}\mathfrak{D}_{m'm}^{(l)}(\phi,\theta,-\phi), \quad \text{(A17)}$$

where $\tilde{r}_l{}^m$ is the multipole parameter of X in the X RF$_2$. In this frame, $\tilde{r}_l{}^m$ is again given by (A11).

Now, one substitutes (A17) into (18) and then uses formulas (A9), (A10), and (12) to get[21]:

$$G(lm;LM) =(4\pi/3)^{1/2}(\hat{l}/\hat{L})\sum_{l'm'}\hat{l}'(l'0lm|Lm) \times(l'm'lM-m'|LM)\langle\tilde{r}_l{}^{(M-m')}Y_{l'}{}^{m'}\rangle, \quad \text{(A18)}$$

where $l=0$ or 2. Using (20a), one sees that l' has only *even* values. The average appearing in (A18) can be determined from experiment [in analogy to (A12)] by

$$\langle\tilde{r}_0{}^0Y_{l'}{}^{m'}\rangle=\frac{1}{N}\sum_{i=1}^{N}Y_{l'}{}^{m'}(\Omega_i)$$
$$\langle\tilde{r}_2{}^mY_{l'}{}^{m'}\rangle=\frac{-(2\pi)^{1/2}}{N}\sum_{i=1}^{N}Y_2{}^m(\hat{n}_i)Y_{l'}{}^{m'}(\Omega_i). \quad \text{(A19)}$$

Note that $Y_{l'}{}^{m'}(\Omega_i)$ is evaluated in the S RF, while $Y_2{}^m(\hat{n}_i)$ is evaluated in the X RF$_2$.

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[20] See, for instance, Eq. (A4) and Table I, Ref. 9.

[21] This is related to the "test functions" $A(l'l;LM)$ proposed in Ref. 3: $G(lm;LM)=(1/\sqrt{3}\hat{L})\sum_{l'}\hat{l}'(l'0lm|Lm)A(l'l;LM)$.