Unitary Symmetry Viewed as a Broken Rotational Invariance*

Y. NE'EMAN†
California Institute of Technology, Pasadena, California

AND

I. OZSVATH
Southwest Center for Advanced Studies, Dallas, Texas

(Received 18 January 1965)

The incorporation of $\Delta Q=-\Delta S$ currents into $SU(3)$ theory implies its extension into an $R(8)$ symmetry. Universality requires that these "skew" currents have a much weaker coupling than the $\Delta S=-\Delta Q$ transitions. The $R(8)$ currents may cause the breakdown of CP invariance as suggested by Wolfenstein. An $R(8)$ model would not allow the existence of $SU(3)$ triplets.

I. INTRODUCTION

UNITARY symmetry having been experimentally validated, it is only natural that the next item on the Particle Physics agenda should consist in unveiling any further physical superstructure the symmetry approach may yield. The most intriguing prospects reside in those extended symmetries that straddle internal and external transformations—chiral symmetries, spin-unitary spin ones and their suspected covariant parentage. Yet it also seems necessary to inquire whether unitary symmetry itself is the largest allowed endosymmetry ("internal" symmetry), i.e., whether there is no larger group whose algebra commutes with spin and parity.

One approach to this question has materialized in the work of Schwinger,[1] of Gürsey, Lee, and Nauenberg,[2] and of Bacry, Nuyts, and Van Hove.[3] The motivation in these studies has sprung from the fact that known particles span zero-triality[4] representations of $SU(3)$ only; triplets and other nonzero-triality multiplets, if they do exist, would have fractional baryon, hyper-, and electric charges[5] (the quark-ace[6,7] hypothesis). To allow for the hypothetical existence of integer-charge triplets, one has to add new quantum numbers to the basic algebra, i.e., to introduce new terms in the Gell-Mann–Nishijima relation. The minimum such scheme would be the $SU(4)$ theory proposed by Tarjanne and Teplitz[8] as interpreted by Maki[9] or Hara.[10] Other suggestions have included $SU(3)\times SU(3)^{1-3}$ and $Sp(6)$. The implications of this approach have been discussed by numerous workers in the field. Since triplets have yet to be discovered, most of these ideas do not readily yield to experimental checking. Some indications in favor of $SU(3)\times SU(3)$ have been extracted from the ϕ-ω degeneracy, with its suggestive nonet structure; yet this has been shown to be equally derivable from $SU(3)$ itself in its triplet version under certain dynamical assumptions.

The weak hadron vector current should reflect the structure of the endosymmetry's algebra. The eightfold A_2 algebra[11,12] of $SU(3)$ now seems to represent at least a good approximation, as shown by Cabibbo's work.[13] Earlier, when $\Delta Q=-\Delta S$ transitions had appeared to occur at the same rate as $\Delta Q=\Delta S$ ones,[14,15] it had seemed one might have to abandon the $SU(3)$ description which does not possess $\Delta Q=-\Delta S$ generators.[16] Alternatively, Ne'eman had suggested[17] using an extension of $SU(3)$ into eight-dimensional rotations, thereby incorporating the missing transitions. Some work on $R(8)$ was indeed initiated,[18] soon coming to a stop when further experiments seemed to invalidate

* Work supported in part by the U. S. Atomic Energy Commission, and the National Aeronautics and Space Administration. Prepared under Contract AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission, and Contract NsG 269-62 for NASA and AFOSR-Grant AF-AFOSR-903-65.

† On leave of absence from Tel Aviv University, Tel Aviv, Israel and the Israel Atomic Energy Commission, Tel Aviv, Israel.

[1] J. Schwinger, Phys. Rev. Letters **12**, 237 (1964).
[2] F. Gürsey, T. D. Lee, and M. Nauenberg, Phys. Rev. **135**, B467 (1964).
[3] H. Bacry, J. Nuyts, and L. Van Hove, Phys. Letters **9**, 279 (1964).
[4] G. E. Baird and L. C. Biedenharn, *Proceedings of the 1964 Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman, Publishers, San Francisco, 1964), p. 58.
[5] H. Goldberg and Y. Ne'eman, Nuovo Cimento **27**, 1 (1963).
[6] M. Gell-Mann, Phys. Letters **8**, 214 (1964).
[7] G. Zweig (unpublished).
[8] P. Tarjanne and V. L. Teplitz, Phys. Rev. Letters **11**, 447 (1963).
[9] Z. Maki, Progr. Theoret. Phys. (Kyoto) **31**, 331 (1964).
[10] Y. Hara, Phys. Rev. **134**, B701 (1964).
[11] W. Killing, Math. Ann. (Berlin) **31**, 252 (1888).
[12] E. Cartan, *Sur la Structure des Groupes de Transformations Finis et Continus*, thesis (Nony, Paris, 1894), 1st ed. (Vuibert, Paris, 1933), 2nd ed.
[13] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).
[14] R. P. Ely, W. M. Powell, H. White, M. Baldo-Ceolin, E. Calimani, S. Ciampolillo, O. Fabbri, F. Farini, C. Filippi, H. Huzita, G. Miari, U. Camerini, W. F. Fry, and S. Natali, Phys. Rev. Letters **8**, 132 (1962); G. Alexander, S. P. Almeida, and F. S. Crawford, Jr., Phys. Rev. Letters **9**, 69 (1962).
[15] A. Barbaro-Galtieri, W. H. Barkas, H. H. Heckman, J. W. Patrick, and F. M. Smith, Phys. Rev. Letters **9**, 26 (1962).
[16] B. d'Espagnat, in *Proceedings of the 1962 International Conference on High Energy Physics at CERN* (CERN, Geneva, 1962), p. 918.
[17] Y. Ne'eman, Phys. Letters **4**, 81, 312 (1963).
[18] D. Horn and Y. Ne'eman, Nuovo Cimento **29**, 760 (1963); **31**, 879 (1964); M. Gourdin, *ibid.* **30**, 587 (1963).

the previous observations, and put an upper limit of some 12% on the ratio $\Delta Q=-\Delta S/\Delta Q=+\Delta S$.[19]

Two experimental observations, however, seem to call for another look at the $R(8)$ theory, provided we include certain modifications connected with the rephrasing of weak-coupling universality in a two-neutrino picture.[13,20] It is our contention that this endosymmetry is preferred, in the context of conventional current-current formalism,[21] provided either or both of the following two experimental facts hold:

(a) The actual observation of the decay

$$\Sigma^- \rightarrow n+\mu^+ + \nu \qquad (1)$$

by Barbaro-Galtieri *et al.*[15];

(b) the recent observation of the mode

$$K_2^0 \rightarrow 2\pi \qquad (2)$$

and the related CP invariance breakdown.[22,23]

Neither of these decays is allowed to occur if $SU(3)$ is indeed the largest parity-retaining symmetry (or "restricted" endosymmetry, in the same sense as in "restricted Lorentz group"). A V-A current-current Hamiltonian with octet currents will yield no appropriate matrix elements (in the case of the CP experiment, one could perhaps introduce major alterations, but the consequences would be felt in the entire picture to a degree which does not seem to us justified at present).

In the following pages we introduce the mathematical physics of $R(8)$ symmetry; we then impose weak-current universality and see that it does indeed predict an extremely low rate of $\Delta Q=-\Delta S$ decays. We then study the possible emergence of $CP=-1$ terms in the current-current Hamiltonian and the observational implications of such an origin for the K_2^0 anomalous decay mode. Finally, we review other aspects of the $R(8)$ hypothesis, perhaps the most interesting one relating to the search for hadron triplets, quarks, or others: Their existence is not allowed at all by this theory.

II. THE D_4 ALGEBRA AND ITS A_2 CONTENT

We are using the rank-4 algebra D_4 of 8-dimensional rotations,[12] which found some applications in global symmetry times, yet in an entirely different physical picture. Our definition of the generators is picked so as to make D_4 appear as an extension of A_2, thus enriching the original physical $SU(3)$ algebra by adding additional operators. This fact is embodied in what we call the "A_2 content" of D_4. In the following exposition, we try to make this meaning precise. Throughout, we use tensor language, hoping to gain some didactic clarity from the geometric picture.

Let us start with some considerations concerning representations of D_4 and A_2 which are going to be of importance below. It is well known that in an 8-dimensional linear vector space X, those linear transformations A, B, which have skew-symmetric matrices, form a linear representation of D_4. We note that the regular representation of A_2 is contained in this representation of D_4, since these are the **F** matrices, Hermitian and imaginary,[24] i.e., eight skew-symmetric matrices spanning an 8-dimensional subspace of the 28-dimensional linear vector space of the D_4 algebra.

The Kronecker product of this **8** representation of D_4 with itself induces in the space T of second-rank tensors (in X space) T_{ab} $(a, b\cdots=1, \cdots, 8)$ the linear transformations

$$T_{ab}'=\sum_{c=1}^{8}\sum_{d=1}^{8} A_{ac}A_{bd}T_{cd}, \qquad (3)$$

the $(A_{ac}A_{bd})$ forming a 64-dimensional representation of D_4. Note that this representation of D_4 then contains a 64-dimensional representation of A_2, since it includes the Kronecker self-product of the A_2 **8** regular representations. To visualize the transformations (3) in this light, denote (ab) by A, (cd) by B, etc., (A, B thus go from 1 to 64), and $A_{AB}=A_{(ab)(cd)}=A_{ac}A_{bd}$, so that (3) is mapped into

$$T_A'=\sum_{B=1}^{64} A_{AB}T_B. \qquad (4)$$

Let us now return to the former picture of a tensor space T_{ab} (in an 8-space X), i.e., a set of 64 (8 by 8) matrices. This is a reducible representation space with respect to D_4, decomposing into the direct sum

$$(D_4)=\mathbf{8}\times\mathbf{8}=\mathbf{1}+\mathbf{35}+\mathbf{28}. \qquad (5)$$

The corresponding invariant subspaces

$$T=I+S+K$$

are the trace T_{aa}, the symmetric tensors $T_{(ab)}$ and the skew-symmetric tensors $T_{[ab]}$, respectively.[25]

The representation matrices A_{ac} of (3) contain the F_i of A_2. The tensor space T_{ab} thus decomposes further with respect to this subalgebra into

$$(A_2)\quad \mathbf{8}\times\mathbf{8}=\mathbf{1}+(\mathbf{27}+\mathbf{8})+(\mathbf{8}+\mathbf{10}+\overline{\mathbf{10}}), \qquad (6)$$

[19] L. Kirsch, R. J. Plane, J. Steinberger, and P. Franzini, Phys. Rev. Letters **13**, 35 (1964); R. W. Birge *et al.*, *ibid.* **11**, 35 (1963); W. Willis *et al.*, *ibid.* **13**, 291 (1964).

[20] M. Gell-Mann, in *Proceedings of the 1960 International Conference on High Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), pp. 508–513.

[21] E. C. G. Sudarshan and R. E. Marshak, in *Proceedings of the Padua-Venice Conference on Mesons and Recently Discovered Particles, 1957*, and Phys. Rev. **109**, 1860 (1958); R. P. Feynman and Gell-Mann, Phys. Rev. **109**, 193 (1958).

[22] J. W. Christensen, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Letters **13**, 138 (1964).

[23] A. Abashian, R. J. Abrams, D. W. Carpenter, G. P. Fisher, B. M. K. Nefkens, and J. H. Smith, Phys. Rev. Letters **13**, 286 (1964).

[24] M. Gell-Mann, California Institute of Technology Synchrotron Laboratory Report CTSL-20, 1961, published in M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin, Inc., New York, 1964), p. 11.

[25] We use repeated indices to denote summation.

the parentheses reproducing the connection with (5). The K piece thus reduces into the subspaces

$$K=Z+V+U\,, \tag{7}$$

where we have adopted the notation[26]

$$Z=\mathbf{8}\,,\quad V=\mathbf{10}\,,\quad U=\overline{\mathbf{10}}.$$

Z, V, and U are then irreducible invariant subspaces in terms of this 28-dimensional representation of A_2. If we choose in K space a base such that the first eight vectors form a basis of Z, the next ten a basis of V, and the last ten of U, then the twenty-eight (28 by 28) matrices representing D_4 when operating on K (the irreducible $T_{[ab]}$ subspace) contain the eight A_2 matrices in their decomposed form, i.e., filling up three disconnected boxes (8 by 8, 10 by 10, $\overline{10}$ by $\overline{10}$) along the main diagonal:

$$\left|\begin{array}{c|c|c} \mathbf{8} & & \\ \hline & \mathbf{10} & \\ \hline & & \overline{\mathbf{10}} \end{array}\right| \quad \left|\begin{array}{c} z \\ v \\ u \end{array}\right| \Big\} K. \tag{8}$$

Now D_4 has only one irreducible representation **28** (up to similarity transformations, corresponding to different choices of the basis in the underlying vector space): This is the regular representation, operating by definition upon the 28-dimensional linear vector space of D_4 itself. The picture is the same as in the traditional view of the Pauli σ set as a basis whose three unit vectors σ_1, σ_2, σ_3 span a three-dimensional vector space (e.g., the expression $\boldsymbol{\sigma}\cdot\boldsymbol{\phi}$); or of the $SU(3)$ $\lambda_1\cdots\lambda_8$ spanning an 8-dimensional space,[27] for example, in coupling the meson octet to a quark bilinear $\bar{q}\gamma_5\boldsymbol{\lambda}\cdot\boldsymbol{\phi}q$. Any 28-dimensional vector space supporting an irreducible linear representation of D_4 can therefore be regarded as the vector space spanned by the D_4 algebra matrices themselves, in some particular basis.

Our K space thus supports the regular representation **28** of D_4; and K itself is identifiable with the D_4 algebra matrices, each unit vector in the K basis representing a certain matrix. The basis is now chosen so that the A_2 subset of the D_4 representation (28 by 28 matrices) is in the above reduced form. The three boxes contain the corresponding generators of A_2 in their **8**, **10**, $\overline{\mathbf{10}}$ representations. The **8** box operating on the Z subspace is identifiable with the regular representation of A_2, since this is the only existing 8-dimensional representation of this algebra, up to equivalence (i.e., it can be brought by a unitary transformation into any specific form we prefer).

To reproduce the correspondence between the K basis and the D_4 basic set of (8 by 8) generators, we follow the following line of reasoning. Any z_k is some linear combination[28] of the λ_i matrices of A_2, viewed as vectors; we transfer to a λ_i basis, where we can now use the usual mapping through commutation relations[24]

$$[\lambda_j,\lambda_k]=if_{jkl}\lambda_l \quad (j,k,l=1,\cdots 8) \tag{9}$$

into the form

$$[F_j,F_k]=if_{jkl}F_l\,, \tag{10}$$

through the definition

$$F_j{}^{(kl)}=-if_{jkl}\,, \tag{11}$$

thus getting the explicit (kl) matrix element for all (8 by 8) F_j matrices; we can regard the z_k itself from now on as the appropriate 8 by 8 F_k matrix. We can make the further identification of this z_k with a certain Z_k, a (28 by 28) matrix of the decomposed form (8). This Z_k operates upon the V or U subspaces, yielding

$$\begin{aligned} Z_kv_r&=i\sum_{s=1}^{10} f_{krs}{}^{V}v_s\,,\\ Z_ku_r&=i\sum_{s=1}^{10} f_{krs}{}^{U}u_s\,,\quad (k=1\cdots 8;\, r,s=1\cdots 10)\,, \end{aligned} \tag{12}$$

where the constants $f_{krs}{}^V$ and $f_{krs}{}^U$ are the (rs) matrix elements of **10** and $\overline{\mathbf{10}}$ representations of A_2, and now appear in the Z_k matrix

$$F_k{}^{V(rs)}=-if_{krs}{}^V\,,\quad F_k{}^{U(rs)}=-if_{krs}{}^U\,. \tag{13}$$

Considering that v and u span the regular representation of D_4, the equations (12) can be rephrased as commutation relations of the D_4 algebra in our particular A_2-reduced definition,

$$\begin{aligned} [z_k,v_r]&=if_{krs}{}^Vv_s\,,\\ [z_k,u_r]&=if_{krs}{}^Uu_s\,, \end{aligned} \tag{14}$$

thus identifying the (8 by 8) matrices v_r and u_r. We can now complete our identifications by reading (14) backwards, to define their (28 by 28) representatives,

$$V_rz_k=-if_{krs}{}^Vv_s\,,\quad U_rz_k=-if_{krs}{}^Uu_s \tag{15}$$

which implies a form

$$\left|\begin{array}{c|c|c} 0 & & \\ \hline f^V & & \\ \hline 0 & & \end{array}\right| \qquad \left|\begin{array}{c|c|c} & & \\ \hline & & \\ \hline f^U & & \end{array}\right| \tag{16}$$

for these matrices operating on K space.

The entire K-space basis is now identifiable with a thus defined base of the D_4 generators set. The commutation relations are such that

$$[Z,Z]\subset Z\,,\quad [Z,V]\subset V\,,\quad [Z,U]\subset U\,. \tag{17}$$

From our knowledge of the $SU(3)$ transformation

---

[26] H. Goldberg, Israel Atomic Energy Commission Report IA-834, Table 3 (unpublished).

[27] This is the picture in which $SU(3)$ was introduced in Ne'eman's version, where these matrices form the **V** basis: Nucl. Phys. **26**, 222 (1961).

[28] The correspondence between Goldberg's set and the Gell-Mann λ_i is $Z(1)\sim\frac{1}{2}(\lambda_4+i\lambda_5)$, $Z(2)\sim\frac{1}{2}(\lambda_6+i\lambda_7)$, $Z(3)\sim\frac{1}{2}(\lambda_1+i\lambda_2)$, $Z(4)\sim 2^{-1/2}\lambda_8$, $Z(5)\sim 2^{-1/2}\lambda_3$, $Z(6)\sim\frac{1}{2}(\lambda_1-i\lambda_2)$, $Z(7)\sim\frac{1}{2}(\lambda_6-i\lambda_7)$, $Z(8)\sim\frac{1}{2}(\lambda_4-i\lambda_5)$.

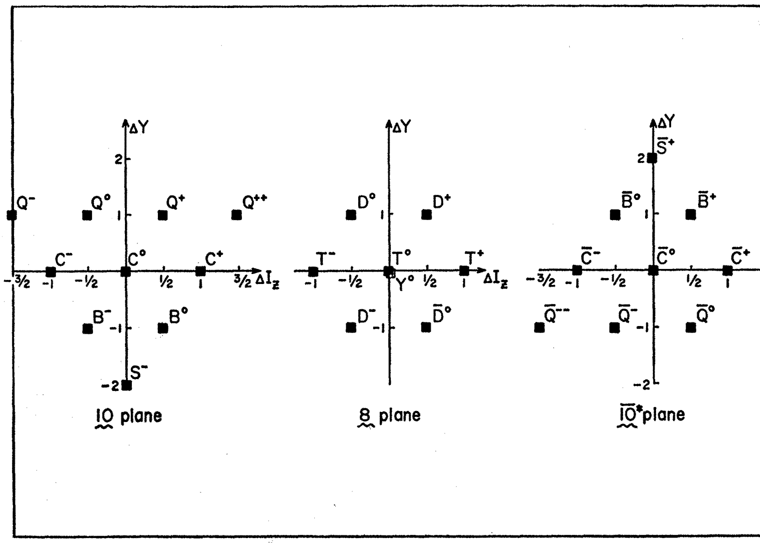

FIG. 1. $SU(3)$ reduction of the generators of $R(8)$.

properties of the Z, V, and U, we can complete this with

$$[V,V]\subset U,\quad [U,U]\subset V,\quad [V,U]\subset Z \tag{18}$$

which is yielded unambiguously by the Kronecker products[29]

$$\begin{aligned} \mathbf{10}\times\mathbf{10} &= \mathbf{35}+\mathbf{28}+\mathbf{27}+\overline{\mathbf{10}}, \\ \overline{\mathbf{10}}\times\overline{\mathbf{10}} &= \overline{\mathbf{35}}+\overline{\mathbf{28}}+\mathbf{27}+\mathbf{10}, \\ \mathbf{10}\times\overline{\mathbf{10}} &= \mathbf{64}+\mathbf{27}+\mathbf{8}+\mathbf{1}. \end{aligned} \tag{19}$$

We now make use of our above definition of the D_4 set in its reduced A_2 form to suggest an easy way to construct the matrices explicitly. We remind ourselves that the z_k, v_r, and u_s vectors span the K subspace of $T_{[ab]}$ skew-symmetric tensors of the second rank in the original 8-dimensional space X, as exhibited in (3). The basis of X is also defined by A_2, and the reduction (6) is indeed generally given by a table relating the Z, V, U to the original vectors x,

$$v_r = \mathfrak{v}_r{}^{jk} x_j x_k = \tilde{x}_j \mathfrak{v}_r x_k, \tag{20}$$

where the second expression displays $\mathfrak{v}_r$ as a matrix whose (jk) component is the corresponding component of the reduction tensor. We can now insert these expressions in (13)

$$\begin{aligned} if_{lrs}{}^V v_s = -F_l{}^{V(rs)}\mathrm{v} &= -\tilde{x}_j \mathfrak{v} F_l{}^{(rs)} x_k \\ &\quad -\tilde{x}_j F_l{}^{(rs)} \mathfrak{v} x_k = \tilde{x}_j [F_l, \mathfrak{v}_r] x_k, \end{aligned} \tag{21}$$

where we have expressed the l variation of v_s, first in terms of the operation of the A_2 generator's representative directly in **10** space, and then in terms of the A_2 generators in the X space itself. We have also used the fact that the F_l are skew-symmetric.

This can now be compared with (14),

$$if_{lrs}{}^V v_s = [z_l, v_r]. \tag{14'}$$

[29] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).

We have already identified the z_l with the F_l set in (11); we now learn that *the reduction matrices* $\mathfrak{v}_r$ *are a representation of the* V_r. Our proof can be extended to the u_r and to the z_k themselves, and we have gained a *construction-procedure* for our D_4 algebra basic set in their A_2 reduced form.

If we perform in the original X space a unitary transformation diagonalizing F_3 and F_8, and keep the same diagonalization in the reduction of K space, the base vectors z_k, v_r, u_s will be eigenstates of λ, μ, I_z, and y [λ and μ are one way[30] of characterizing an irreducible representation of $SU(3)$, I_z is the third component of isospin and y is the hypercharge]. The corresponding D_4 generators will produce transformations with just these quantum numbers; inserted into a Dirac bilinear, they will produce a boson operator destroying these quantum numbers. If this is a vector bilinear, we have a destruction operator for a current.

In Appendix A, we have produced an explicit set of generators, based upon a matrix construction from the Goldberg reduction tables[26] of 8×8 in $SU(3)$. The weight diagram of these operators (Fig. 1) is useful for quick identification and suggests algebraic shortcuts (e.g., commutators are given by vector addition, provided the three A_2 representations are regarded as three levels 1, 0, -1 of a third dimension).

To allow us to exponentiate, we define a Hermitian set. In K space, this reduces to a transition from a metric of the transposing type (like the g_{ij} of Ref. 27) to a Euclidean metric; the Hermitian matrices are represented by real vectors in the basis, replacing the former complex set of I_z, y eigenvectors. Eight of these Hermitian matrices are the F_i of Ref. 24; we have named the others G_9 to G_{28}, attempting throughout to make the numbering suggestive of the A_2 content.

D_4 has four diagonalizable commuting generators,

[30] See, for example, H. Goldberg, in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), p. 219.

i.e., states have 4 weights or quantum numbers. Two of these are I_z and y, given by

$$I_z = T^0 = F_3, \tag{22}$$

$$y = 2(3)^{-1/2} Y^0 = 2(3)^{-1/2} F_8, \tag{23}$$

the other two are

$$u = 3^{-1/2} i (C^0 - \bar{C}^0) = (\tfrac{2}{3})^{1/2} G_{23}, \tag{24}$$

$$v = (C^0 + \bar{C}^0) = \sqrt{2} G_{13}. \tag{25}$$

The (u,w) eigenvalues of the baryon octet are

$$p(0,1),\quad n(0,-1),\quad \Sigma^+(0,-1),\quad \Sigma^0(1,0),\quad \Sigma^-(0,1),\quad \Lambda(-1,0),\quad \Xi^0(0,1),\quad \Xi^-(0,-1). \tag{26}$$

Although $SU(3)$ is a subgroup of $R(8)$, u and v do not commute with $SU(3)$; this is because Schur's lemma does not allow any operator except for the identity to commute with $SU(3)$, since its **8** representation is irreducible.[17] Similarly, u and v do not commute with isospin.

The introduction of a higher symmetry always restricts couplings allowed by the lower symmetry. In the original search for a good physical endosymmetry,[31] it was important to restrict the rank to two, in order to allow all reactions of the basic multiplets allowed by I_z and y. At the present stage, we know *a priori* that the higher endosymmetry is going to be badly broken [the situation is the same in $SU(3)\times SU(3)$, etc.].

III. UNIVERSALITY

Weak-interaction universality has gradually been generalized to include all types of weak coupling. The basic formalism consists in requiring weak currents to satisfy a set of commutation relations, thus defining a Lie algebra. The algebra generates a symmetry group; any interaction which is invariant with respect to the group will thereby conserve the "charges" carried by these weak currents, i.e., it will keep their couplings at some fixed value throughout. Usually, one makes use of the fact that for the hadrons, the weak-symmetry algebra coincides with a subalgebra of the strong interactions; the result is then a lack of renormalization of the weak coupling by the strong interactions. The best-studied example of this type is found in the neutron beta-decay vector coupling, which retains approximately the same value as in μ decay. That the lepton currents also satisfy some such algebraic condition can be learned from the equivalence of muon and lepton currents. Indeed, these seem to satisfy universality in their axial-vector part almost as well as in the vector part; considering that the partial conservation of the beta-decay axial-vector current is required by the successful Goldberger-Treiman relation, one is finally led to the concept of *total universality*, which we adopt here.

Gell-Mann noted[20] that in a two-neutrino theory, the lepton currents can be regarded as the partially conserved currents of an algebra of $SU(2)$ generators, the **W** spin; the leptons transform as 2-spinors[32] (μ,ν_μ) and (e,ν_e). The same group **W** now operates on the hadrons, yielding a hadron current where both n and Λ decay into p; the effective spinor is then $(an+b\Lambda, p)$ with $a^2+b^2=1$. Note that this parallelism is required by the fact that the couplings of hadrons and leptons are equal; had the hadrons appeared in a different representation of W, we would have gotten factors of $\sqrt{2}$ etc. Since n-beta decay is similar in strength to μ decay, $a\sim 1$ and $b\ll 1$. This is Cabibbo's weak hadron current[13] in $SU(3)$ theory. The entire formulation is made with respect to the complete weak current, i.e., the left-handed spinor representations

$$H_W = (G/\sqrt{2}) W^\mu W_\mu{}^+, \tag{27}$$

$$W^\mu = W^\mu{}_{\text{hadrons}} + W^\mu{}_{\text{muons}} + W^\mu{}_{\text{electrons}}, \tag{28}$$

$$W^\mu{}_{\text{hadrons}} = (L/\sqrt{2}) \bar{\psi}_h (1+\gamma_5)\gamma^\mu \{aT^- + bD^-\} \psi_h, \tag{29}$$

where $\Delta Q = -1$, T^- and D^- are the $SU(3)$ generators $2^{-1/2}(F_1+iF_2)$ and $2^{-1/2}(F_4+iF_5)$, respectively.

$$W^\mu{}_{\text{muons}} = \bar{\psi}_{\text{mu}} (1+\gamma_5)\gamma^\mu (\tau_1 + i\tau_2) \psi_{\text{mu}}, \tag{30}$$

$$W^\mu{}_{\text{electrons}} = \bar{\psi}_{\text{el}} (1+\gamma_5)\gamma^\mu (\tau_1 + i\tau_2) \psi_{\text{el}}. \tag{31}$$

Cabibbo's formulation seems to be at least a good approximation. We now note that if this is to hold, any new term we would like to add to the current will display an *extremely weak* coupling, since the a and b terms in (30) are experimentally known to yield

$$(a^2+b^2)\sim 1. \tag{32}$$

It would be difficult to assign an exact limit to the summed squares of all additional terms, but it seems we can exclude anything stronger than

$$0.01 > c^2 + d^2 + \cdots. \tag{33}$$

This is in fact a theoretical prediction: Any hadron weak currents besides T^-, D^-, and their Hermitian conjugates produce extremely weak interactions. This runs counter to phenomenological studies that have assumed the existence of large $\Delta Q = -\Delta S$ currents.[33]

Such considerations seem superfluous in $SU(3)$ theory, where $T^\mp$ and $D^\mp$ have anyhow used up all $|\Delta Q| = 1$ currents, and no neutral currents have yet been observed. On the other hand, the observed Σ^+ beta decay[20] in (1), if confirmed, would imply the existence

[31] Y. Ne'eman, in *Proceedings of the International Conference on Nucleon Structure*, edited by R. Hofstadter and L. I. Schiff (Stanford University Press, Stanford, 1964), pp. 172–187.

[32] An alternative description of the leptons in terms of the Konopinski-Mahmoud definition of lepton number (μ^+, ν, e^- are the leptons, with both ν_e and ν_μ redefined appropriately and included in a four-component ν) leads to a different structure of **W**, which was studied by Y. Ne'eman [Nuovo Cimento **27**, 922 (1963)]. The two descriptions are equivalent with respect to the leptons as long as no neutral currents are involved, and if the neutrinos are both massless. The connection with the hadrons can be different; it has recently been used by A. Salam and J. C. Ward, Phys. Letters **13**, 168 (1964).

[33] R. G. Sachs, Phys. Rev. Letters **13**, 286 (1964).

of a very weak (only this one event and another "possible" one[19] have been seen to date) current of the $\Delta Q=-\Delta S$ type, with $|\Delta \mathbf{I}|=\frac{3}{2}$. This leads us out of $SU(3)$ proper, and we now view the $R(8)$ extension described in the previous section. Clearly, our particular current is described by the operator

$$Q^{-}=2^{-1/2}(G_{19}+iG_{20}) \quad (\text{in } \mathbf{10}) \tag{34}$$

and its conjugate $\bar{Q}^{+}$ (in $\mathbf{10}$).

IV. CONSTRUCTION OF THE WEAK CURRENT

We are faced with the problem of determining the contents of a new $\mathbf{W}$ set of currents, including

$$W^{+}\sim aT^{+}+bD^{+}+c\bar{Q}^{+}+\cdots, \tag{35}$$

such that it will again yield for

$$W_1=2^{-1/2}(W^{+}+W^{-}), \tag{36}$$

$$W_2=2^{-1/2}i(W^{+}-W^{-}), \tag{37}$$

$$[W_i,W_j]=ie_{ijk}W_k \quad (i,\, j,\, k=1,\, 2,\, 3). \tag{38}$$

Finding exact solutions of the most general type for $\mathbf{W}$ is a laborious problem which we shall tackle in Appendix B. In fact, we study the vector part V_i; the problem of $W_i=V_i+A_i$ is left to our discussion of chiral extensions. For the physics we are interested in at this stage, we shall adopt an approximate solution, discarding terms in all but first order in b or c in the amended current (35). Following Cabibbo's transformation

$$e^{-2i\theta F_7}T^{+}e^{2i\theta F_7}=(\cos\theta)T^{+}+(\sin\theta)D^{+}, \tag{39}$$

we shall look for a transformation of T^{+} into $\bar{Q}^{+}$ or T^{-} into Q^{-}. A look at the weight diagram will show that Q^0 in $\mathbf{10}$ will give the appropriate transformation; since Q^0 is not Hermitian, we shall have to combine it with $\bar{Q}^0$ in $\overline{\mathbf{10}}$. There is a choice we can make in the phases, and we shall use it to produce an answer to both the Σ^{+} beta decay and the recent $CP=-1$ decay,[22,23] Eq. (2).

To understand the emergence of a 90° phase, let us replace the exponential by its first term, in Cabibbo's transformation

$$(1-2i\theta F_7)T^{-}(1+2i\theta F_7)=T^{-}-2i\theta[F_7,T^{-}] \tag{40}$$

we adjoin a transformation

$$(1-2i\theta F_6)T^{-}(1+2i\theta F_6)=T^{-}-2i\theta[F_6,T^{-}]. \tag{41}$$

Since

$$2^{-1/2}(F_6-iF_7)=D^0, \tag{42}$$

$$2^{-1/2}(F_6+iF_7)=\bar{D}^0, \tag{43}$$

we find from vector addition in the weight diagram (or from the commutation tables in Appendix A)

$$[D^0,T^{-}]=0, \tag{44}$$

$$[\bar{D}^0,T^{-}]=-(1/\sqrt{2})D^{-}, \tag{45}$$

which leads to

$$-2i\theta[F_6,T^{-}]=i\theta D^{-}, \tag{46}$$

$$-2i\theta[F_7,T^{-}]=\theta D^{-}. \tag{47}$$

We see that the F_7 variation used by Cabibbo produces an in-phase D^{-} current whereas an F_6 variation would have yielded a 90° out-of-phase D^{-}. Wolfenstein[34] has shown that the inclusion of an out-of-phase small current with

$$10^{-7}<c<10^{-8} \tag{48}$$

could reproduce the $K_2{}^0$ anomalous 2π decay.

To achieve such a result, in the context of current universality, we use

$$e^{-i\alpha G_{26}}(\cos\theta T^{-}+\sin\theta D^{-})e^{i\alpha G_{26}} \tag{49}$$

noting that

$$G_{26}=2^{-1/2}(Q^0+\bar{Q}^0), \tag{50}$$

$$[Q^0,T^{-}]=(\tfrac{3}{2})^{1/2}Q^{-}, \tag{51}$$

$$[\bar{Q}^0,T^{-}]=-\sqrt{2}\bar{Q}^{-}. \tag{52}$$

We get in first order in α or θ a current $[X_\mu$ implies $\bar{\psi}(1+\gamma_5)\gamma_\mu X\psi]$

$$\begin{aligned}W_\mu{}^{-}=\cos\theta\,\cos\alpha T_\mu{}^{-}+\sin\theta\,\cos\alpha D_\mu{}^{-}+\sqrt{2}i\,\cos\theta\,\sin\alpha\bar{Q}^{-}\\ -2^{-1/2}\sqrt{3}i\,\cos\theta\,\sin\alpha Q^{-}+\text{lepton currents } L^{-}\\ +\text{terms in }\sin\alpha\,\sin\theta,\end{aligned} \tag{53}$$

where the $T_\mu{}^{-}$ and $D_\mu{}^{-}$ couplings are lowered only very slightly through multiplication by $\cos\alpha$, $\bar{Q}^{-}$ has $|\Delta T|=\frac{3}{2}$, $\Delta Q=\Delta S$, and Q^{-} is our $\Delta Q=-\Delta S$ operator. Note that both new currents are out of phase and will produce $CP=-1$ terms in the Hamiltonian,

$$\begin{aligned}H'=-i\,\cos\theta\,\sin\alpha G\{(L^{-\mu}Q_\mu{}^{+}-\bar{Q}^{-\mu}L_\mu{}^{+})\\ +\cos\theta\,\cos\alpha(T^{-\mu}Q_\mu{}^{+}-\bar{Q}^{-\mu}T_\mu{}^{+})\\ +\sin\theta\,\cos\alpha(D^{-\mu}Q_\mu{}^{+}-\bar{Q}^{-\mu}D_\mu{}^{+})\}\\ +i\sqrt{3}\,\cos\theta\,\sin\alpha G\{(L^{\mu-}\bar{Q}^{+\mu}-Q^{-\mu}L_\mu{}^{+})\\ +\cos\theta\,\cos\alpha(T^{\mu-}\bar{Q}_\mu{}^{+}-Q^{-\mu}T_\mu{}^{+})\\ +\sin\theta\,\cos\alpha(D^{-\mu}\bar{Q}_\mu{}^{+}-Q^{-\mu}D_\mu{}^{+})\}.\end{aligned} \tag{54}$$

V. *CP* VIOLATION

The various terms in H' all contain the factor $\sin\alpha$' which makes them generally weaker than a CP-invariant weak interaction. However, it may occur that some reaction is allowed to proceed via H' in first order, whereas it would appear in higher order only in H.

We first check the effect of H' upon the neutral K decays. Now that a $CP=-1$ Hamiltonian allows some $K_2{}^0\to 2\pi$ and $K_1{}^0\to 3\pi$, we shall have to redefine the two actual components of the K^0 system in weak interactions. This is done by computing the squared-mass matrix in the $(K^0,\bar{K}^0)$ basis and diagonalizing it to reproduce the physically distinguishable states. We keep the names $K_1{}^0$ and $K_2{}^0$ for the CP eigenstates, since these are still useful for computational purposes:

[34] L. Wolfenstein, Phys. Rev. Letters **13**, 562 (1964).

The various parts of the Hamiltonian have definite CP eigenvalues. The new physical states will be denoted $K_S{}^0$ (short-lived) and $K_L{}^0$ (long-lived). We use the convention:

$$C|K^0\rangle=+|\bar{K}^0\rangle,\quad P|K^0\rangle=-|K^0\rangle,$$
$$C|\bar{K}^0\rangle=+|K^0\rangle,\quad P|\bar{K}^0\rangle=-|\bar{K}^0\rangle, \tag{55}$$

$$|K^0\rangle=\frac{1}{\sqrt{2}}[|K_2{}^0\rangle+|K_1{}^0\rangle],$$
$$|\bar{K}^0\rangle=\frac{1}{\sqrt{2}}[|H_2{}^0\rangle-|K_1{}^0\rangle], \tag{56}$$

$$|K_1{}^0\rangle=\frac{1}{\sqrt{2}}[|K^0\rangle-|\bar{K}^0\rangle],$$
$$|K_2{}^0\rangle=\frac{1}{\sqrt{2}}[|K^0\rangle+|\bar{K}^0\rangle], \tag{57}$$

$$CP|K_1{}^0\rangle=1,\quad CP|K_2{}^0\rangle=-1. \tag{58}$$

The two diagonal matrix elements $\langle K^0|M^2|K^0\rangle$ and $\langle\bar{K}^0|M^2|\bar{K}^0\rangle$ will have only time-reversal-symmetric contributions,

$$\langle K^0|M^2|K^0\rangle=\langle K_1{}^0|M^2|2\pi\rangle\langle 2\pi|M^2|K_1{}^0\rangle+\langle K_2{}^0|M^2|3\pi\rangle\langle 3\pi|M^2|K_2{}^0\rangle+\cdots+\langle K_2{}^0|M^2|2\pi\rangle\langle 2\pi|M^2|K_2{}^0\rangle+\cdots. \tag{59}$$

Odd CP terms like the last one here are allowed to contribute, provided they multiply conjugates. The result is thus a sum A of positive-definite quantities, which will reappear identically in $\langle\bar{K}^0|M^2|\bar{K}^0\rangle$.

The off-diagonal matrix elements are each other's Hermitian conjugates. There are two kinds of terms: real and symmetric, which we denote by B, and an imaginary and antisymmetric part $-iC$ (for $\langle K^0|M^2|\bar{K}^0\rangle$).

In the pre-CP, violation age, only B existed and was given by

$$\langle K^0|M^2|\bar{K}^0\rangle=-|\langle K_1{}^0|M^2|2\pi\rangle|^2+|\langle K_2{}^0|M^2|3\pi\rangle|^2. \tag{60}$$

It will now acquire terms like $B'=-|\langle K_1{}^0|M^2|3\pi\rangle|^2+|\langle K_2{}^0|M^2|2\pi\rangle|^2$.

The resulting matrix is now

$$M^2=A\mathbf{1}+B\tau_1+C\tau_2. \tag{61}$$

For $C=0$, diagonalization had been achieved through the unitary transformation

$$UM^2U^\dagger=M_{\text{diag}}{}^2,$$
$$U=\frac{1}{\sqrt{2}}\begin{vmatrix}1 & 1\\ i & -i\end{vmatrix}, \tag{62}$$

thus defining the diagonal states as

$$(1/\sqrt{2})[|K_0\rangle+|\bar{K}^0\rangle]=K_2{}^0,$$
$$(i/\sqrt{2})[|K_0\rangle-|\bar{K}_0\rangle]=iK_1{}^0. \tag{63}$$

In the new system, we have to use

$$U'=\frac{1}{\sqrt{2}}\begin{vmatrix}\left(\dfrac{B+iC}{B-iC}\right)^{1/2} & 1\\ i & -i\left(\dfrac{B+iC}{B-iC}\right)^{-1/2}\end{vmatrix}. \tag{64}$$

The physical states are thus

$$|K_L{}^0\rangle=(1/\sqrt{2})[t|K^0\rangle+|\bar{K}^0\rangle],$$
$$|K_S{}^0\rangle=(i/\sqrt{2})[|K^0\rangle-\bar{t}|\bar{K}^0\rangle], \tag{65}$$

where

$$t=((B+iC)/(B-iC))^{1/2}=\bar{t}^{-1}. \tag{66}$$

The mass difference is

$$M^2(K_L{}^0)-M^2(K_S{}^0)=2(B^2+C^2)^{1/2}. \tag{67}$$

We can now study the 2π decay mode, following roughly the analysis in Refs. 33 and 34, and using the experimental result of Ref. 22:

$$\xi=2.6\times10^{-3}=\left|\frac{\langle 2\pi|H+H'|K_L{}^0\rangle}{\langle 2\pi|H+H'|K_S{}^0\rangle}\right|\sim\frac{1+\zeta|t|}{1-\zeta|t|}, \tag{68}$$

$$\zeta=\frac{\langle 2\pi|H+H'|\bar{K}^0\rangle}{\langle 2\pi|H+H'|K^0\rangle}=\frac{\langle 2\pi|H'|K_2{}^0\rangle-\langle 2\pi|H|K_1{}^0\rangle}{\langle 2\pi|H'|K_2{}^0\rangle+\langle 2\pi|H|K_1{}^0\rangle}\sim-\frac{i\sin\alpha G\Lambda^2-GL^2}{i\sin\alpha G\Lambda^2+GL^2},$$

$$\zeta\sim-(1-i\alpha)/(1+i\alpha). \tag{69}$$

$\zeta=-1$ for a CP-invariant theory. We get an upper bound for α from the off-diagonal mass matrix element of (67); experimentally, this was shown to fit a second-order process in the weak interaction,[35,36] i.e., it is of order $G^2\Lambda^4$ (Λ is a cutoff). For the first-order process in C, we would get $\alpha G\Lambda^2$, so that

$$\alpha<G\Lambda^2.$$

With α thus of the order of the weak coupling or less, we may use $\zeta\sim-1$ and get from (68)

$$t=1+2\xi. \tag{70}$$

Reinserting this value for t^2 into Eq. (66), and utilizing again our knowledge of the couplings involved in B and

[35] See, for example, R. H. Good, R. P. Matsen, F. Muller, O. Piccioni, W. M. Powell, H. S. White, W. B. Fowler, and R. W. Bridge, Phys. Rev. **124**, 1223 (1961).

[36] For the theoretical argument, see L. Okun and B. Pontecorvo, Zh. Eksperim. i Teor. Fiz. **32**, 1587 (1957) [English transl.: Soviet Phys.—JETP **5**, 1297 (1947)].

C, we find

$$\alpha/G\Lambda^2 \sim C/B \sim \xi, \qquad \alpha \sim 10^{-7} \text{ to } 10^{-8}. \tag{71}$$

This estimate is Wolfenstein's.[34] On the other hand, it seems we should also consider the fact that the $\Delta S=2$ Hamiltonian is in an $SU(3)$ 27, or in some higher $R(8)$ representation. The dynamical octet enhancement results[37] would account for a factor of 1/700 in α_{eff}, since we are comparing one "non-octet" to a double "octet" amplitude, i.e., a squared enhancement equal to the rate-enhancement in $(K^+/K_1{}^0) \to 2\pi$ decays; thus, $\alpha \sim 10^{-5} \sim G\Lambda^2$.

VI. OTHER WEAK EFFECTS

Considering that any amplitude containing a non-eightfold current will be reduced by the factor α, it is doubtful whether such reactions can be observed in the near future. The Σ^+ beta decay (1) if repeated, would imply a larger value of α. Contrary to what was expected in an $R(8)$ theory without a Gell-Mann–Cabibbo type of universality,[18] we do not expect to find decays of the type

$$\Xi^- \to n + e^- + \bar{\nu} \tag{72}$$

as we have included only two additional currents (to first order) out of the 20 allowed by $R(8)$ outside the octet.

It is in the K^0 system that we may expect to find easier verifications of these ideas. Wolfenstein has noted that $K_2{}^0$ leptonic decays would be 5% asymmetric between the two charge states:

$$\frac{K_L{}^0 \to \pi^+ + e^- + \bar{\nu}}{K_L{}^0 \to \pi^- + e^+ + \nu} = \frac{1}{t^2}. \tag{73}$$

Again, there is a nonvanishing amplitude for

$$K_S{}^0 \to 3\pi \tag{74}$$

experimentally complicated by phase-space factors.

In cases where the new currents provide for a weak reaction in first order, as against second-order processes in the former currents, we can distinguish the effects only to the extent that we would have reached these interactions observationally anyhow. For the $K_S{}^0 - K_L{}^0$ mass difference, we would expect to find a larger δm, i.e., an increase of 25–100% over the expectation for a second-order weak interaction.

VII. CHIRAL EXTENSION AND MULTIPLET ASSIGNMENTS

There are various ways in which $R(8)$ could be defined as the vector-current algebra within a larger chiral symmetry. One is tempted to use $R(9)$, with only an octet of axial vectors added to the vector currents; $SU(8)$, with 35 axial vectors currents [reducing into $SU(3)$ **8**+**27**] has some advantages (the axial octet couples through D). However, the nonlinear equations corresponding to the universality idea do not allow these choices.[38] One is thus led to postulate

$$R(8) \times R(8)$$

as the chiral symmetry. The proof of universality of $SU(3) \times SU(3)$ in Ref. 38 can be generalized to this symmetry.

In this picture, the baryon octet (separated into left- and right-handed components) forms an **(8,1)**+**(1,8)**. Note that we have at this stage no clear knowledge as to which **8** in $R(8)$ this is, since all three representations have the same $SU(3)$ content. It is conceivable that by studying the reactions occurring between baryons in terms of the two new additive quantum numbers supplied by $R(8)$, X, and Z, we may be able to determine whether the baryons use a spinor or a vector representation.

For vector and axial vector mesons, the assignment should be (28,1)+(1,28). This predicts the existence of an $SU(3)$ icosuplet of vector mesons and an octet and icosuplet of axial vector mesons. Lee, Okubo, and Schecter[39] have conjectured the existence of a boson icosuplet including the B (1220-MeV π-ω resonance)[40] and the A (1200-MeV π-ρ resonance)[41] as diagonalized unitary parity eigenstates of the two isotriplets. They also included the 1175-MeV $K\pi\pi$ resonance[42] in this representation, with $T=\frac{1}{2}$ or $\frac{3}{2}$ and conjectured that another such resonance with the complementary isospin should be found at 1265 MeV.

It seems probable that an icosuplet will indeed be the right answer. A $K\pi\pi$ resonance with $T=\frac{1}{2}$,[43] $\mu=1215$ MeV (it is not clear that this is what was observed in Ref. 42) has been observed and another $K\pi\pi$ resonance has been found[43,44] at 1270 MeV and has $T=\frac{3}{2}$. There seems to be some difficulty with the masses, which have moved away from the mass formula level: The A is now considered to have a mass[45] of 1090 MeV and since the lower $K\pi\pi$ is put at 1215 MeV, both strange resonances have risen above the center of the triplets. However, these moving peaks are broad

[37] R. Dashen and S. Frautschi, Phys. Rev. Letters **13**, 497 (1964).

[38] M. Gell-Mann and Y. Ne'eman, Ann. Phys. (N. Y.) **30**, 360 (1964).

[39] B. W. Lee, S. Okubo, and J. Schecter, Phys. Rev. **135**, B219 (1964).

[40] M. Abolins, R. L. Lander, W. W. Mehlhop, Ng.-H. Xuong, and P. M. Yager, Phys. Rev. Letters **11**, 381 (1963).

[41] G. Goldhaber, J. L. Brown, S. Goldhaber, J. A. Kadyk, B. C. Shen, and G. H. Trilling, Phys. Rev. Letters **12**, 336 (1964).

[42] T. P. Wangler, W. D. Walker, and A. R. Erwin, Phys. Letters **9**, 71 (1964).

[43] R. H. Dalitz, Chicago APS Review, October 1964.

[44] R. Amenteros, D. N. Edwards, T. Jacobsen, L. Montanet, A. Shapira, J. Vandermeulen, CH. D'anlau, A. Astier, P. Baillon, J. Cohen-Ganouna, C. Defoix, J. Siaud, C. Ghesquiere, and P. Rivet, Phys. Letters **9**, 207 (1964).

[45] S. U. Chung, O. I. Dahn, L. M. Hardy, R. I. Hess, G. R. Kalbfleisch, J. Kirz, D. H. Miller, and G. A. Smith, Phys. Rev. Letters **12**, 621 (1964).

and they could well all be thought to lie at nearly 1200 MeV. The spin-parity 1^+ is favored in most cases, considering that these resonances do not seem to decay into two pseudoscalar mesons (except for some $\pi\eta$ traces reported by one group). In the case of the 1270-MeV "C" meson, the decay

$$C^0 \to K_1^0 + \rho^0 \searrow \pi^+ + \pi^-$$

shows no deviations for isotropy[44] in the first step, which would imply an S wave and give $j=1^+$ indeed.

It should be noted that a baryon-antibaryon-vector meson coupling is $R(8)$ invariant only if we put the baryons in the vector **8**. This assignment would require 28 pseudoscalars too; alternatively, we could put the pseudoscalars in an **8** and have the Yukawa pseudoscalar interaction break $R(8)$, leaving $SU(3)$ invariant. (This seems preferable to having an **8** for the vector mesons,[46] considering that they should appear in the adjoint representation and couple to quasiconserved currents.) In this case, the symmetry-breaking Hamiltonian could be in either **56** (vector) or **112** (vector) [these representations include an $SU(3)$ scalar and appear in the product of $\mathbf{8}\times\mathbf{28}$ and $\mathbf{8}\times\mathbf{35}$, with $8\times 8 = \mathbf{1}+\mathbf{28}+\mathbf{35}$].

VIII. DYNAMICS OF THE SYMMETRY BREAKING

In the light of bootstrap theories, we would prefer getting the $R(8)$ breakdown from the same model we suggested for $SU(3)$.[47] The basic idea is that although the bootstrap may settle on a nonsymmetric solution, there is no visible way in which it could pick an asymmetry which does not correspond to the existing imbalance—electromagnetism. We therefore introduce an interaction similar to electromagnetism and to the weak interactions, in that it is defined by a given coupling and is not generated by the bootstrap; it is mediated by a vector meson with definite elementary properties, i.e., it is a (C.D.D.) Castillejo-Dalitz-Dyson pole and exhibits the appropriate behavior with respect to the Levinson theorem count.[48]

In our model, this vector meson is coupled to an $SU(3)$ octet eighth component, mixed with a singlet. For this current to be approximately conserved, we have to assign it in $R(8)$ to a **28**. This determines the algebraic features of the symmetry breaking, which should be given by the self-product of our current,

$$\mathbf{28}\times\mathbf{28}=\mathbf{1}+\mathbf{300}+\mathbf{350}+\cdots. \tag{75}$$

(we list only representations containing the $SU(3)$ scalar) where the figures in parentheses are the $SU(3)$ representation dimensionality and the index of the appropriate component. The product contains three $SU(3)$ scalars; one of them is identical with the $R(8)$ scalar, the other appears in the representation **300**; the third scalar, in **350**, is antisymmetric and does not contain a $(\lambda_8)^2$ contribution. The **300** breaks $R(8)$ and leaves $SU(3)$ as a good symmetry. Note that its eigenvalues split 28 into 20+8, which fits in with the experimental picture in which we have, to date, perhaps an $SU(3)$ **8** for vector mesons at $\sim$0.9 BeV and a **20** for the axial vectors at 1.2 BeV: The missing 20 vectors and 8 axial vectors should be clustered around different energy levels.

IX. NONEXISTENCE OF $SU(3)$ TRIPLETS

If our $R(8)$ model should be true, there would be one additional important implication with respect to $SU(3)$: The eightfold symmetry is then only $SU(3)/Z(3)$, and $SU(3)$ itself never arises. $R(8)$ yields only zero-triality representations, starting at **8**, and will never generate a triplet, either independently or inside some larger representation. We could then understand more readily the nonappearance of such representations to date; if any triplets be found, this model would not hold.

ACKNOWLEDGMENTS

This work was begun while one of us (YN) was enjoying the hospitality of the Southwest Center for Advanced Studies (Dallas), which it gives him great pleasure to thank. He is also indebted to Professor F. Zachariasen for some enlightening comments.

APPENDIX A: THE LADDER OPERATORS

The following set of generators of the D_4 algebra (i.e., the infinitesimal algebras of the group of rotations in 8-dimensional space) in an $SU(3)$-oriented classification is based upon the Goldberg 8×8 reduction table.[26] Our generators operate on the same space as Gell-Mann's[24] F_i. The correspondence between the bases is given by comparing the meson octet in Table 2 of Goldberg's report with Table 4.2 in Gell-Mann's. This is done explicitly in our Table I.

TABLE I. Connections between notations.

Goldberg	Gell-Mann
ϕ_1	$(\pi_4-i\pi_5)/\sqrt{2}$
ϕ_2	$(\pi_6-i\pi_7)/\sqrt{2}$
ϕ_3	$(\pi_1-i\pi_2)/\sqrt{2}$
ϕ_4	π_8
ϕ_5	π_3
ϕ_6	$(\pi_1+i\pi_2)/\sqrt{2}$
ϕ_7	$(\pi_6+i\pi_7)/\sqrt{2}$
ϕ_8	$(\pi_4+i\pi_5)/\sqrt{2}$

[46] G. Cocho, Phys. Rev. **137**, B1255 (1965), has dealt with $R(8)$ and suggested using the spinor **8** for the baryons and the vector **8** for the pseudoscalar mesons. This would imply that the vector meson coupling to the baryons breaks $R(8)$ though leaving $SU(3)$ invariant.

[47] Y. Ne'eman, Phys. Rev. **134**, B1355 (1964).

[48] See, for example, S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963), p. 34.

The notation for the "ladder" operators is given in Table II. Here, $\Delta(\lambda,\mu)$ is taken in the sense of the commutation relations (17) and (18) of our text. All Goldberg tensors are multiplied by $\sqrt{3}$ to bring about the same normalization as for the F_i. The Hermitian generators are listed in Table III.

TABLE II. Notation for the "ladder" operators.

	$\Delta(\lambda,\mu)$	$\lvert\Delta I\rvert$	ΔI_z	ΔY		$\Delta(\lambda,\mu)$	$\lvert\Delta I\rvert$	ΔI_z	ΔY
T^+	8	1	1	0	C^-	10	1	-1	0
T^0	8	1	0	0	B^0	10	$\frac{1}{2}$	$\frac{1}{2}$	-1
T^-	8	1	-1	0	B^-	10	$\frac{1}{2}$	$-\frac{1}{2}$	-1
D^+	8	$\frac{1}{2}$	$\frac{1}{2}$	1	S^-	10	0	0	-2
D^0	8	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\bar{Q}^{--}$	$\overline{10}$	$\frac{3}{2}$	$-\frac{3}{2}$	-1
$\bar{D}^0$	8	$\frac{1}{2}$	$\frac{1}{2}$	-1	$\bar{Q}^-$	$\overline{10}$	$\frac{3}{2}$	$-\frac{1}{2}$	-1
D^-	8	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\bar{Q}^0$	$\overline{10}$	$\frac{3}{2}$	$\frac{1}{2}$	-1
Y^0	8	0	0	0	$\bar{Q}^+$	$\overline{10}$	$\frac{3}{2}$	$\frac{3}{2}$	-1
Q^{++}	10	$\frac{3}{2}$	$\frac{3}{2}$	1	$\bar{C}^-$	$\overline{10}$	1	-1	0
Q^+	10	$\frac{3}{2}$	$\frac{1}{2}$	1	$\bar{C}^0$	$\overline{10}$	1	0	0
Q^0	10	$\frac{3}{2}$	$-\frac{1}{2}$	1	$\bar{C}^+$	$\overline{10}$	1	1	0
Q^-	10	$\frac{3}{2}$	$-\frac{3}{2}$	1	$\bar{B}^0$	$\overline{10}$	$\frac{1}{2}$	$-\frac{1}{2}$	1
C^+	10	1	1	0	$\bar{B}^+$	$\overline{10}$	$\frac{1}{2}$	$\frac{1}{2}$	1
C^0	10	1	0	0	$\bar{S}^+$	$\overline{10}$	0	0	2

TABLE III. The Hermitian generators.

$$T^+ = (1/\sqrt{2})(F_1 - iF_2) \qquad D^+ = (1/\sqrt{2})(F_4 - iF_5) \qquad D^- = (1/\sqrt{2})(F_4 + iF_5)$$
$$T^0 = F_3 \qquad D^0 = (1/\sqrt{2})(F_6 - iF_7) \qquad \bar{D}^0 = (1/\sqrt{2})(F_6 + iF_7)$$
$$T^- = (1/\sqrt{2})(F_1 + iF_2)$$
$$Y^0 = F_8$$
$$Q^{++} = (1/\sqrt{2})(G_9 - iG_{10}) \qquad Q^+ = (1/\sqrt{2})(G_{24} - iG_{25}) \qquad Q^0 = (1/\sqrt{2})(G_{26} - iG_{27})$$
$$Q^- = (1/\sqrt{2})(G_{19} + iG_{20}) \qquad \bar{Q}^{--} = (1/\sqrt{2})(G_9 + iG_{10}) \qquad \bar{Q}^- = (1/\sqrt{2})(G_{24} + iG_{25})$$
$$\bar{Q}^0 = (1/\sqrt{2})(G_{26} + iG_{27}) \qquad \bar{Q}^+ = (1/\sqrt{2})(G_{19} - iG_{20})$$
$$C^+ = (1/\sqrt{2})(G_{21} - iG_{22}) \qquad C_0 = (1/\sqrt{2})(G_{13} - iG_{23}) \qquad C^- = (1/\sqrt{2})(G_{11} + iG_{12})$$
$$\bar{C}^- = (1/\sqrt{2})(G_{21} + iG_{22}) \qquad \bar{C}^0 = (1/\sqrt{2})(G_{13} + iG_{23}) \qquad \bar{C}^+ = (1/\sqrt{2})(G_{11} - iG_{12})$$
$$B^0 = (1/\sqrt{2})(G_{16} + iG_{17}) \qquad B^- = (1/\sqrt{2})(G_{14} + iG_{15})$$
$$\bar{B}^0 = (1/\sqrt{2})(G_{16} - iG_{17}) \qquad \bar{B}^+ = (1/\sqrt{2})(G_{14} - iG_{15})$$
$$S^- = (1/\sqrt{2})(G_{13} + iG_{28})$$
$$\bar{S}^+ = (1/\sqrt{2})(G_{18} - iG_{28})$$

In Table IV we provide an explicit matrix representation. In Tables V (a)–(f) we give the commutation relations explicitly.

TABLE IV. The $R(8)$ generators.

$$T^+ = \frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot \\ -2 & 2i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & i & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$T^0 = \frac{1}{2}\begin{vmatrix} \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$T^- = \frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 2i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & i & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

TABLE IV (*continued*)

$$D^{+}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & i & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \sqrt{3} \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ 1 & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\sqrt{3} & i\sqrt{3} & \cdot & \cdot & \cdot \end{vmatrix} \qquad D^{0}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & -1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i & \cdot \\ 1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \sqrt{3} \\ \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} & i\sqrt{3} & \cdot \end{vmatrix}$$

$$\bar{D}^{-}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & i & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ -1 & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{3} & i\sqrt{3} & \cdot & \cdot & \cdot \end{vmatrix} \qquad \bar{D}^{0}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & 1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -i & \cdot \\ -1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} \\ \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} & i\sqrt{3} & \cdot \end{vmatrix}$$

$$Y^{0}=\frac{\sqrt{3}}{2}\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$Q^{++}=\frac{\sqrt{6}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & -1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad Q^{+}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & -1 & \cdot \\ \cdot & \cdot & \cdot & -2 & 2i & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ i & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$Q^{0}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & -1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -2 & 2i & \cdot \\ 1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad Q^{-}=\frac{\sqrt{6}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$C^{+}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \sqrt{3} \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ -1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & -1 & \cdot & \cdot & \cdot \\ -\sqrt{3} & i\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad C^{0}=\frac{1}{2}\begin{vmatrix} \cdot & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot \\ \cdot & \cdot & \sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$C^{-}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \sqrt{3} \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & i\sqrt{3} \\ 1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & 1 & \cdot & \cdot & \cdot \\ -\sqrt{3} & -i\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad B^{0}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & -1 & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & i & \cdot \\ 1 & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ i & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} & i\sqrt{3} & \cdot \end{vmatrix}$$

$$B^{-}=\frac{\sqrt{2}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} \\ \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ 1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{3} & i\sqrt{3} & \cdot & \cdot & \cdot \end{vmatrix} \qquad S^{-}=\frac{\sqrt{6}}{4}\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

TABLE IV (*continued*).

$$\bar{Q}^{--}=\frac{\sqrt{6}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & \cdot & 1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right| \qquad \bar{Q}^{-}=\frac{\sqrt{2}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & 1 & \cdot \\ \cdot & \cdot & \cdot & 2 & 2i & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right|$$

$$\bar{Q}^{0}=\frac{\sqrt{2}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & \cdot & 1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 2i & \cdot \\ -1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2i & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right| \qquad \bar{Q}^{+}=\frac{\sqrt{6}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right|$$

$$\bar{C}^{-}=\frac{\sqrt{2}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ 1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & 1 & \cdot & \cdot & \cdot \\ \sqrt{3} & i\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right| \qquad \bar{C}^{0}=\frac{1}{2}\left|\begin{array}{cccccccc} \cdot & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & \cdot & \cdot \\ \cdot & \cdot & -\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right|$$

$$\bar{C}^{+}=\frac{\sqrt{2}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & i\sqrt{3} \\ -1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i & -1 & \cdot & \cdot & \cdot \\ \sqrt{3} & -i\sqrt{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right| \qquad \bar{B}^{0}=\frac{\sqrt{2}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & \cdot & 1 & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & i & \cdot \\ -1 & -i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ i & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \sqrt{3} \\ \cdot & \cdot & -i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{3} & i\sqrt{3} & \cdot \end{array}\right|$$

$$\bar{B}^{+}=\frac{\sqrt{2}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \sqrt{3} \\ \cdot & \cdot & i & \cdot & \cdot & \cdot & \cdot & -i\sqrt{3} \\ -1 & i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ i & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\sqrt{3} & i\sqrt{3} & \cdot & \cdot & \cdot \end{array}\right| \qquad \bar{S}^{+}=\frac{\sqrt{6}}{4}\left|\begin{array}{cccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -i & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right|$$

TABLE V. Commutation relations.

	T^+	T^0	T^-	D^+	D^0	$\bar{D}^0$	D^-	Y^0
T^+	0	T^+	$-T^0$	0	$-(1/\sqrt{2})D^+$	0	$(1/\sqrt{2})\bar{D}^0$	0
T^0		0	T^-	$-\frac{1}{2}D^+$	$\frac{1}{2}D^0$	$-\frac{1}{2}\bar{D}^0$	$\frac{1}{2}D^-$	0
T^-			0	$-(1/\sqrt{2})D^0$	0	$(1/\sqrt{2})D^-$	0	0
D^+				0	0	$-(1/\sqrt{2})T^+$	$-\frac{1}{2}T^0 - \frac{1}{2}\sqrt{3}Y^0$	$\frac{1}{2}\sqrt{3}D^+$
D^0					0	$\frac{1}{2}T^0 - \frac{1}{2}\sqrt{3}Y^0$	$-(1/\sqrt{2})T^-$	$\frac{1}{2}\sqrt{3}D^0$
$\bar{D}^0$						0	0	$-\frac{1}{2}\sqrt{3}\bar{D}^0$
D^-							0	$-\frac{1}{2}\sqrt{3}D^-$
Y^0								0

	Q^{++}	Q^+	Q^0	Q^-	C^+	C^0	C^-	B^0	B^-	S^-
T^+	0	$(\frac{3}{2})^{1/2}Q^{++}$	$-\sqrt{2}Q^+$	$-(\frac{3}{2})^{1/2}Q^0$	0	$-C^+$	C^0	0	$(1/\sqrt{2})B^0$	0
T^0	$-\frac{3}{2}Q^{++}$	$-\frac{1}{2}Q^+$	$\frac{1}{2}Q^0$	$\frac{3}{2}Q^-$	$-C^+$	0	C^-	$-\frac{1}{2}B^0$	$\frac{1}{2}B^-$	0
T^-	$(\frac{3}{2})^{1/2}Q^+$	$-\sqrt{2}Q^0$	$-(\frac{3}{2})^{1/2}Q^-$	0	$-C^0$	C^-	0	$(1/\sqrt{2})B^-$	0	0
D^+	0	0	0	0	$-(\frac{3}{2})^{1/2}Q^{++}$	Q^+	$-(1/\sqrt{2})Q^0$	$\sqrt{2}C^+$	$-C^0$	$(\frac{3}{2})^{1/2}B^0$
D^0	0	0	0	0	$(1/\sqrt{2})Q^+$	Q^0	$-(\frac{3}{2})^{1/2}Q^-$	C^0	$\sqrt{2}C^-$	$-(\frac{3}{2})^{1/2}B^-$
$\bar{D}^0$	0	$(1/\sqrt{2})C^+$	C^0	$-(\frac{3}{2})^{1/2}C^-$	0	B^0	$\sqrt{2}B^-$	0	$-(\frac{3}{2})^{1/2}S^-$	0
D^-	$-(\frac{3}{2})^{1/2}C^+$	C^0	$-(1/\sqrt{2})C^-$	0	$\sqrt{2}B^0$	$-B^-$	0	$(\frac{3}{2})^{1/2}S^-$	0	0
Y^0	$-\frac{1}{2}\sqrt{3}Q^{++}$	$-\frac{1}{2}\sqrt{3}Q^+$	$-\frac{1}{2}\sqrt{3}Q^0$	$-\frac{1}{2}\sqrt{3}Q^-$	0	0	0	$\frac{1}{2}\sqrt{3}B^0$	$\frac{1}{2}\sqrt{3}B^-$	$\sqrt{3}S^-$

TABLE V (*continued*)

	$\bar{Q}^{--}$	$\bar{Q}^{-}$	$\bar{Q}^{0}$	$\bar{Q}^{+}$	$\bar{C}^{-}$	$\bar{C}^{0}$	$\bar{C}^{+}$	$\bar{B}^{0}$	$\bar{B}^{+}$	$\bar{S}^{+}$
T^+	$-(\frac{3}{2})^{1/2}\bar{Q}^{-}$	$\sqrt{2}\bar{Q}^{0}$	$(\frac{3}{2})^{1/2}\bar{Q}^{+}$	0	$\bar{C}^{0}$	$-\bar{C}^{+}$	0	$-(1/\sqrt{2})\bar{B}^{+}$	0	0
T^0	$\frac{3}{2}Q^{--}$	$\frac{1}{2}\bar{Q}^{-}$	$-\frac{1}{2}\bar{Q}^{0}$	$-\frac{3}{2}\bar{Q}^{+}$	$\bar{C}^{-}$	0	$-\bar{C}^{+}$	$\frac{1}{2}\bar{B}^{0}$	$-\frac{1}{2}\bar{B}^{+}$	0
T^-	0	$-(\frac{3}{2})^{1/2}\bar{Q}^{--}$	$\sqrt{2}\bar{Q}^{-}$	$(\frac{3}{2})^{1/2}\bar{Q}^{0}$	0	$\bar{C}^{-}$	$-\bar{C}^{0}$	0	$-(1/\sqrt{2})\bar{B}^{0}$	0
D^+	$(\frac{3}{2})^{1/2}\bar{C}^{-}$	$-\bar{C}^{0}$	$(1/\sqrt{2})\bar{C}^{+}$	0	$-\sqrt{2}\bar{B}^{0}$	$\bar{B}^{+}$	0	$-(\frac{3}{2})^{1/2}\bar{S}^{+}$	0	0
D^0	0	$-(1/\sqrt{2})\bar{C}^{-}$	$-\bar{C}^{0}$	$(\frac{3}{2})^{1/2}\bar{C}^{+}$	0	$-\bar{B}^{0}$	$-\sqrt{2}\bar{B}^{+}$	0	$(\frac{3}{2})^{1/2}\bar{S}^{+}$	0
$\bar{D}^0$	0	0	0	0	$-(1/\sqrt{2})\bar{Q}^{-}$	$-\bar{Q}^{0}$	$(\frac{3}{2})^{1/2}\bar{Q}^{+}$	$-\bar{C}^{0}$	$-\sqrt{2}\bar{C}^{+}$	$(\frac{3}{2})^{1/2}\bar{B}^{+}$
D^-	0	0	0	0	$(\frac{3}{2})^{1/2}\bar{Q}^{--}$	$-\bar{Q}^{-}$	$(1/\sqrt{2})\bar{Q}^{0}$	$-\sqrt{2}\bar{C}^{-}$	$\bar{C}^{0}$	$-(\frac{3}{2})^{1/2}\bar{B}^{0}$
Y^0	$\frac{1}{2}\sqrt{3}\bar{Q}^{--}$	$\frac{1}{2}\sqrt{3}\bar{Q}^{-}$	$\frac{1}{2}\sqrt{3}\bar{Q}^{0}$	$\frac{1}{2}\sqrt{3}\bar{Q}^{+}$	0	0	0	$-\frac{1}{2}\sqrt{3}\bar{B}^{0}$	$-\frac{1}{2}\sqrt{3}\bar{B}^{+}$	$-\sqrt{3}\bar{S}^{+}$

	Q^{++}	Q^{+}	Q^{0}	Q^{-}	C^{+}	C^{0}	C^{-}	B^{0}	B^{-}	S^{-}
Q^{++}	0	0	0	$-(\frac{3}{2})^{1/2}\bar{S}^{+}$	0	0	$(\frac{3}{2})^{1/2}\bar{B}^{+}$	0	$-(\frac{3}{2})^{1/2}\bar{C}^{+}$	$(\frac{3}{2})^{1/2}\bar{Q}^{+}$
Q^{+}		0	$-(\frac{3}{2})^{1/2}\bar{S}^{+}$	0	0	$-\bar{B}^{+}$	$-(1/\sqrt{2})\bar{B}^{0}$	$(1/\sqrt{2})\bar{C}^{+}$	$\bar{C}^{0}$	$(\frac{3}{2})^{1/2}\bar{Q}^{0}$
Q^{0}			0	0	$(1/\sqrt{2})\bar{B}^{+}$	$-\bar{B}^{0}$	0	$\bar{C}^{0}$	$-(1/\sqrt{2})\bar{C}^{-}$	$-(\frac{3}{2})^{1/2}\bar{Q}^{-}$
Q^{-}				0	$(\frac{3}{2})^{1/2}\bar{B}^{0}$	0	0	$-(\frac{3}{2})^{1/2}\bar{C}^{-}$	0	$-(\frac{3}{2})^{1/2}\bar{Q}^{--}$
C^{+}					0	$\bar{C}^{+}$	$-\bar{C}^{0}$	$(\frac{3}{2})^{1/2}\bar{Q}^{+}$	$(1/\sqrt{2})\bar{Q}^{0}$	0
C^{0}						0	$\bar{C}^{-}$	$-\bar{Q}^{0}$	$-\bar{Q}^{-}$	0
C^{-}							0	$-(1/\sqrt{2})\bar{Q}^{-}$	$(\frac{3}{2})^{1/2}\bar{Q}^{--}$	0
B^{0}								0	0	0
B^{-}									0	0
S^{-}										0

	$\bar{Q}^{--}$	$\bar{Q}^{-}$	$\bar{Q}^{0}$	$\bar{Q}^{+}$	$\bar{C}^{-}$	$\bar{C}^{0}$	$\bar{C}^{+}$	$\bar{B}^{0}$	$\bar{B}^{+}$	$\bar{S}^{+}$
Q^{++}	$-\frac{3}{2}T^0$ $-\frac{1}{2}\sqrt{3}Y^0$	$(\frac{3}{2})^{1/2}T^+$	0	0	$-(\frac{3}{2})^{1/2}D^+$	0	0	0	0	0
Q^{+}	$(\frac{3}{2})^{1/2}T^-$	$-\frac{1}{2}T^0$ $-\frac{1}{2}\sqrt{3}Y^0$	$-\sqrt{2}T^+$	0	$(1/\sqrt{2})D^0$	D^+	0	0	0	0
Q^{0}	0	$-\sqrt{2}T^-$	$\frac{1}{2}T^0$ $-\frac{1}{2}\sqrt{3}Y^0$	$-(\frac{3}{2})^{1/2}T^+$	0	D^0	$-(1/\sqrt{2})D^+$	0	0	0
Q^{-}	0	0	$-(\frac{3}{2})^{1/2}T^-$	$\frac{3}{2}T^0$ $-\frac{1}{2}\sqrt{3}Y^0$	0	0	$-(\frac{3}{2})^{1/2}D^0$	0	0	0
C^{+}	$-(\frac{3}{2})^{1/2}D^-$	$(1/\sqrt{2})\bar{D}^0$	0	0	$-T^0$	$-T^+$	0	$\sqrt{2}D^+$	0	0
C^{0}	0	D^-	$\bar{D}^0$	0	$-T^-$	0	T^+	D^0	$-D^+$	0
C^{-}	0	0	$-(1/\sqrt{2})D^-$	$-(\frac{3}{2})^{1/2}\bar{D}^0$	0	T^-	T^0	0	$\sqrt{2}D^0$	0
B^{0}	0	0	0	0	$\sqrt{2}D^-$	$\bar{D}^0$	0	$-\frac{1}{2}T^0$ $+\frac{1}{2}\sqrt{3}Y^0$	$(1/\sqrt{2})T^+$	$(\frac{3}{2})^{1/2}D^+$
B^{-}	0	0	0	0	0	$-D^-$	$\sqrt{2}\bar{D}^0$	$(1/\sqrt{2})T^-$	$\frac{1}{2}T^0$ $\frac{1}{2}\sqrt{3}Y^0$	$-(\frac{3}{2})^{1/2}D^0$
S^{-}	0	0	0	0	0	0	0	$(\frac{3}{2})^{1/2}D^-$	$-(\frac{3}{2})^{1/2}\bar{D}^0$	$\sqrt{3}Y^0$

	$\bar{Q}^{--}$	$\bar{Q}^{-}$	$\bar{Q}^{0}$	$\bar{Q}^{+}$	$\bar{C}^{-}$	$\bar{C}^{0}$	$\bar{C}^{+}$	$\bar{B}^{0}$	$\bar{B}^{+}$	$\bar{S}^{+}$
$\bar{Q}^{--}$	0	0	0	$(\frac{3}{2})^{1/2}S^-$	0	0	$-(\frac{3}{2})^{1/2}B^-$	0	$(\frac{3}{2})^{1/2}C^-$	$-(1/\sqrt{2})Q^-$
$\bar{Q}^{-}$		0	$(\frac{3}{2})^{1/2}S^-$	0	0	B^-	$(1/\sqrt{2})B^0$	$-(1/\sqrt{2})C^-$	$-C^0$	$-(\frac{3}{2})^{1/2}Q^0$
$\bar{Q}^{0}$			0	0	$-(1/\sqrt{2})B^-$	B^0	0	$-C^0$	$(1/\sqrt{2})C^+$	$(\frac{3}{2})^{1/2}Q^+$
$\bar{Q}^{+}$				0	$-(\frac{3}{2})^{1/2}B^0$	0	0	$(\frac{3}{2})^{1/2}C^+$	0	$(\frac{3}{2})^{1/2}Q^{++}$
$\bar{C}^{-}$					0	$-C^-$	C^0	$-(\frac{3}{2})^{1/2}Q^-$	$-(1/\sqrt{2})Q^0$	0
$\bar{C}^{0}$						0	$-C^+$	Q^0	Q^+	0
$\bar{C}^{+}$							0	$(1/\sqrt{2})Q^+$	$-(\frac{3}{2})^{1/2}Q^{++}$	0
$\bar{B}^{0}$								0	0	0
$\bar{B}^{+}$									0	0
$\bar{S}^{+}$										0

APPENDIX B. THE FORMAL PART OF THE UNIVERSALITY PROBLEM

We consider here the university problem formally. We consider two operators of $R(8)$ defined by

$$V^+=aT^++bD^++cQ^++dC^++e\bar{B}^++f\bar{S}^++g\bar{Q}^++h\bar{C}^+$$

and

$$V^-=\bar{a}T^-+\bar{b}D^-+\bar{c}\bar{Q}^-+\bar{d}\bar{C}^-+\bar{e}B^-+\bar{f}S^-+\bar{g}Q^-+\bar{h}C^-, \quad \text{(B1)}$$

where $a, b, \cdots, h$ are complex numbers $\bar{a}, \bar{b}, \cdots \bar{h}$ their complex conjugates satisfying the equation

$$a\bar{a}+\cdots+h\bar{h}=1. \quad \text{(B2)}$$

We then define the operators V_1 and V_2 by the equations

$$V_1=(1/\sqrt{2})(V^++V^-), \quad V_2=(i/\sqrt{2})(V^+-V^-) \quad \text{(B3)}$$

and a third operator V_3 by

$$iV_3=[V_1V_2]. \quad \text{(B4)}$$

The problem is the following: Can we find $a, b, \cdots, h$ such that V_1, V_2, V_3 satisfy the further commutation relations

$$iV_1=[V_2V_3] \quad\text{and}\quad iV_2=[V_3V_1]. \tag{B5}$$

One sees immediately from (B3) and (B4) that

$$V_3=[V^-V^+]. \tag{B6}$$

Using (B1) and the commutator relations given in Table II of Appendix A, we find that

$$V_3=\tau T^0+\delta D^0+\bar{\delta}\bar{D}^0+\eta Y^0+\kappa Q^0+\bar{\kappa}\bar{Q}^0 +\gamma C^0+\bar{\gamma}\bar{C}^0+\beta B^0+\bar{\beta}\bar{B}^0, \tag{B7}$$

where we introduced the notations

$$\tau=a\bar{a}+\tfrac{1}{2}b\bar{b}+\tfrac{1}{2}c\bar{c}+d\bar{d}+\tfrac{1}{2}e\bar{e}+\tfrac{3}{2}g\bar{g}+h\bar{h}, \tag{B8a}$$

$$\delta=-(\bar{a}b/\sqrt{2})-(c\bar{d}/\sqrt{2})-(\tfrac{3}{2})^{1/2}\bar{e}f -(\tfrac{3}{2})^{1/2}\bar{g}h+\sqrt{2}e\bar{h}, \tag{B8b}$$

$$\eta=\tfrac{1}{2}\sqrt{3}b\bar{b}+\tfrac{1}{2}\sqrt{3}c\bar{c}+\tfrac{1}{2}\sqrt{3}e\bar{e}+\sqrt{3}f\bar{f}-\tfrac{1}{2}\sqrt{3}g\bar{g}, \tag{B8c}$$

$$\kappa=-\sqrt{2}\bar{a}c-(\tfrac{3}{2})^{1/2}\bar{c}f-(\bar{d}e/\sqrt{2}) +(\tfrac{3}{2})^{1/2}a\bar{g}+(b\bar{h}/\sqrt{2}), \tag{B8d}$$

$$\gamma=-\bar{a}d+\bar{b}c-\bar{c}e+\bar{d}h+b\bar{e}-a\bar{h}, \tag{B8e}$$

$$\beta=-(a\bar{e}/\sqrt{2})+\sqrt{2}\bar{b}d+(\bar{c}h/\sqrt{2}) +(\tfrac{3}{2})^{1/2}\bar{d}g-(\tfrac{3}{2})^{1/2}b\bar{f}, \tag{B8f}$$

$$a\bar{a}+\cdots+h\bar{h}=1. \tag{B8g}$$

We note that

$$\tau+(1/\sqrt{3})\eta=1 \tag{B9}$$

which is a consequence of (B8a), (B8c), and (B8g) and may be useful by later calculations.

The requirements (B5) impose the conditions

$$a=a\tau-(b/\sqrt{2})\bar{\delta}+(\tfrac{3}{2})^{1/2}g\kappa-\sqrt{2}c\bar{\kappa} -h\gamma-d\bar{\gamma}-(e/\sqrt{2})\beta, \tag{B10a}$$

$$b=\tfrac{1}{2}b\tau-(a/\sqrt{2})\delta+\tfrac{1}{2}\sqrt{3}b\eta+(h/\sqrt{2})\kappa +e\gamma+c\bar{\gamma}-(\tfrac{3}{2})^{1/2}f\beta+\sqrt{2}d\bar{\beta}, \tag{B10b}$$

$$c=\tfrac{1}{2}c\tau-(d/\sqrt{2})\delta+\tfrac{1}{2}\sqrt{3}c\eta-\sqrt{2}a\kappa -(\tfrac{3}{2})^{1/2}f\bar{\kappa}+b\gamma-e\bar{\gamma}+(h/\sqrt{2})\bar{\beta}, \tag{B10c}$$

$$d=d\tau-(c/\sqrt{2})\bar{\delta}-(e/\sqrt{2})\bar{\kappa}-a\gamma +h\bar{\gamma}+\sqrt{2}b\beta+(\tfrac{3}{2})^{1/2}g\bar{\beta}, \tag{B10d}$$

$$e=\tfrac{1}{2}e\tau+\sqrt{2}h\delta-(\tfrac{3}{2})^{1/2}f\bar{\delta}+\tfrac{1}{2}\sqrt{3}e\eta-(d/\sqrt{2})\kappa -c\gamma+b\bar{\gamma}-(a/\sqrt{2})\bar{\beta}, \tag{B10e}$$

$$f=-(\tfrac{3}{2})^{1/2}h\delta+\sqrt{3}f\eta-(\tfrac{3}{2})^{1/2}c\kappa-(\tfrac{3}{2})^{1/2}b\bar{\beta}, \tag{B10f}$$

$$g=\tfrac{3}{2}g\tau-(\tfrac{3}{2})^{1/2}h\bar{\delta}-\tfrac{1}{2}\sqrt{3}g\eta+(\tfrac{3}{2})^{1/2}a\bar{\kappa}+(\tfrac{3}{2})^{1/2}d\beta, \tag{B10g}$$

$$h=h\tau-(\tfrac{3}{2})^{1/2}g\delta+\sqrt{2}e\bar{\delta}+(b/\sqrt{2})\bar{\kappa} +d\gamma-a\bar{\gamma}+(c/\sqrt{2})\beta. \tag{B10h}$$

Multiplying the Eqs. (B10a) through (B10h) with $\bar{a}, \cdots \bar{h}$, respectively, and adding up, we get the equation

$$\tau^2+2\delta\bar{\delta}+\eta^2+2\kappa\bar{\kappa}+2\gamma\bar{\gamma}+2\beta\bar{\beta}=1 \tag{B11}$$

as a consequence of (B8) and (B10).

The formal part of the universality problem would now be to find $a, b, \cdots, h$ satisfying these equations. If one would substitute (B8) into (B10), one would get a system of cubic equations whose solution would give all physical currents allowed by the universality condition. One would hope that one such solution would have some special features, providing a possible reason for its appearance in the weak current. We have preferred not to attack the exact problem and use instead a linearization procedure.

One imagines that one has a solution as a function of some real parameter p, $a=a(p), \cdots, b=b(p)$, which one can expand into a power series according to p:

$$a=a_0+\sum_{k=1}^{\infty}a_kp^k, \cdots, h=h_0+\sum_{k=1}^{\infty}h_kp^k. \tag{B12}$$

The leading terms $a_0, \cdots, h_0$ give an exact solution of Eqs. (B8) to (B10) and $a_k, \cdots, h_k$ are unknown complex numbers. Substituting (B12) into (B8) to (B10) and rearranging it according to the powers of p, we get systems of linear equations with known coefficients for our unknowns, which we could solve successively. Since we know that exact solutions exist (the T system, or the Cabibbo set), one can make the assumption that for small values of p our procedure gives a good approximation even if we break at small powers of p.

It is easy to see that the Cabibbo set

$$a=\cos\phi,\quad b=\sin\phi,\quad c=d=e=f=g=h=0 \tag{B13}$$

is an exact solution of (B8) to (B10). Choosing this as the leading term in (B12) which we then use in the simple form

$$a=\cos\phi+Ap,\quad b=\sin\phi+Bp, \quad c=Cp,\quad \cdots,\quad h=Hp, \tag{B14}$$

where

$$A, B, \cdots, H \tag{B15}$$

are unknown complex numbers, and substituting into (B8) to (B10) and considering only zero and first-order terms in p, we get the equations

$$(A+\bar{A})\cos\phi+(B+\bar{B})\sin\phi=0 \tag{B16}$$

and a system of homogeneous linear equations for $C, \cdots H$, which has the solution

$$C\text{ arbitrary},\quad D=E=F=0,\quad G=\tfrac{1}{2}\sqrt{3}\bar{C},\quad H=\tfrac{1}{2}. \tag{B17}$$

Therefore, (B14) is a solution of (B8) to (B10) up to higher than first-order terms in p, if (B15) satisfies (B16) and (B17). Therefore, taking the special choice

$$A=B=0,\quad C=1,$$

we see that the set

$$\begin{aligned} V^+&=\cos\phi T^+ +\sin\phi D^+ +pQ^+ +\tfrac{1}{2}\sqrt{3}p\bar{Q}^+ +\tfrac{1}{2}P, \\ V^-&=\cos\phi T^- +\sin\phi D^- +p\bar{Q}^- +\tfrac{1}{2}\sqrt{3}pQ^- +\tfrac{1}{2}P, \end{aligned} \tag{B18}$$

is a solution of the universality problem up to first-order terms in the parameter p.