# Multimeson Resonances and Nucleon-Nucleon Interaction*

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Relativistic partial-wave dispersion relations are formulated for elastic nucleon-nucleon scattering. These dispersion relations are integral equations with an inhomogeneous term taken from single-particle exchange contributions. The particles under consideration are $\pi(I=1$, pseudoscalar), $\eta(I=0$, pseudoscalar), $\rho(I=1$, vector), $\omega(I=0$, vector), $\varphi(I=0$, vector), and $\sigma(I=0$, scalar). The existence of a $\sigma$ meson is not well established. Two possibilities are considered: (i) The $\sigma$ meson exists, in which case the mass and coupling constants are taken to be two parameters of the present problem. (ii) The $\sigma$ meson does not exist but the $I=0$, $J=0$ two-pion continuum is taken into account. This two-pion continuum can be treated as a superposition of scalar particles with a mass spectrum determined by pion-nucleon and pion-pion interactions. Information on the $\pi N$ interaction is obtained from $\pi N$ scattering data, while the $S$-wave $\pi\pi$ interaction is represented with a relativistic scattering-length approximation. In addition to the $\pi\pi$ scattering length, a cutoff on the two-pion spectrum is introduced. Thus two parameters are introduced in either (i) or (ii). Aside from the masses and coupling constants of the particles mentioned, a cutoff parameter is needed for each of the vector mesons $\rho$, $\omega$, and $\varphi$. These are taken to be coefficients in an exponentially decreasing factor suggested by the Regge-pole behavior of composite particles. A total of twelve adjustable parameters is used and a search program is formulated to fit 560 $pp$ and $np$ data collected by the Livermore group ranging from 9.68 to 388 MeV. In both cases (i) and (ii), a fit is obtained with a "goodness to fit" value of approximately 8%, meaning that the $\chi^2$ is $\sim$548 if the uncertainty inherent in the theory is assumed to be 8%.

## I. INTRODUCTION

ONE of the longstanding problems in the meson theory of nucleon-nucleon interaction is the treatment of multimeson exchange processes in nucleon-nucleon scattering. It is not until the discovery of the multimeson resonances that this problem becomes computationally feasible. Essentially, the existence of multimeson resonances makes it plausible that some of these systems can be treated as single-particle states, with masses and quantum numbers determined by production experiments.[1] The following well-established resonances (mesons) have quantum numbers which allow strong coupling to the nucleon-antinucleon system: (1) the $I=0$, $J=0$ $\eta$ meson with mass $m_\eta=548$ MeV; (2) the $I=1$, $J=1$ $\rho$ meson with mass $m_\rho\simeq750$ MeV; (3) the $I=0$, $J=1$ $\omega$ meson with mass $m_\omega=780$ MeV; (4) the $I=0$, $J=1$ $\varphi$ meson with mass $m_\varphi=1020$ MeV. All of these mesons as well as the pion have negative parity. Therefore, they couple to the $N\bar{N}$ system in the $S$ state (and partly in the $D$ state for the vector mesons). In addition, there may exist a $\sigma$ resonance of $I=0$, $J=0$ parity plus coupled strongly to the two-pion system in the $S$ state and to the $N\bar{N}$ system in the $^3P_0$ state. As we shall see later, whether this resonance exists or not, the $S$-wave $\pi\pi$ pair gives a significant contribution to the medium-range attractive force between two nucleons. If the resonance exists, one can use the single-particle approximation. Otherwise, the $J=0$ $\pi\pi$ system will have to be treated as a continuous spectrum. Both possibilities are considered in the text. In view of the fact that the new mesons have masses comparable to that of the nucleon, it is desirable that the scattering problem be formulated in a covariant manner in order to take proper account of the rather short-range forces. In the present work, we use relativistic partial-wave dispersion relations for the calculation of phase shifts and subsequently the calculation of direct experimental observables. Our procedure is as follows.

First, we calculate the pole terms (renormalized Born approximation) corresponding to the exchange of single mesons $\pi$, $\eta$, $\rho$, $\omega$, and $\varphi$. For all the masses except $m_\rho$ we use the experimental values given above. Since the width of the $\rho$ resonance is rather broad (between 75 and 130 MeV),[1] the effective mass that enters into the calculation of the nucleon-nucleon scattering amplitudes may be shifted substantially from the peak of $\approx750$ MeV observed in production processes. Therefore we take $m_\rho$ as an adjustable parameter. A sizeable shifting of the effective $\rho$ mass is in fact consistent with measurements of the isovector form factors of the nucleon.[2,3] As for the coupling constants, we take $g_\pi^2=14$ and $g_\eta^2$, $g_{\rho1}^2$, $g_{\rho2}^2$, $g_\omega^2$, and $g_\varphi^2$ as free parameters of the problem. There are two coupling constants for the $\rho$ meson, $g_{\rho1}$ and $g_{\rho2}$, proportional to the two-pion contribution to

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[1] See, for example, Proceedings of the 1964 Conference on High Energy Physics (to be published).

[2] *Nucleon Structure*, edited by R. Hofstadter and L. I. Schiff (Stanford University Press, Stanford, California, 1964), Chap. II.

[3] J. S. Ball and D. Y. Wong, Phys. Rev. **130**, 2112 (1963).

the isovector charge and anomalous magnetic moment of the nucleon. We introduce only one coupling constant for each of the $\omega$ and $\varphi$ mesons on the ground that the isoscalar anomalous magnetic moment of the nucleon is very small. In addition to the pole terms, we calculate the exchange of a pair of $\pi$ mesons in the $S$ state ($I=0$, $J=0$). In the case where the existence of the $\sigma$ is is assumed, we simply use the single-particle approximation for a scalar particle of mass $m_\sigma$ and coupling constant $g_\sigma^2$. In the case where we assume that the $\sigma$ is absent, we calculate the $\pi\pi$ spectrum in terms of the $N\bar{N}\rightarrow\pi\pi$ amplitude. For this calculation, the familiar "pair suppression" is accomplished by making use of the pion-nucleon forward-scattering dispersion relation to normalize the $N\bar{N}\rightarrow\pi\pi$ amplitude at zero total energy. A cutoff parameter $t_c$ and a $\pi$-$\pi$ scattering length $a_\pi$ are introduced in this calculation of the pair contribution.

Next, we formulate the dispersion relations in the form of a Fredholm integral equation of the second kind whose solution gives directly the partial-wave amplitudes. Both the inhomogeneous term and the kernel of the integral equation are constructed in terms of the pole terms together with the pair contribution. Our procedure is very similar to the formulation of the Bethe-Salpeter equation in that both start with the single-particle (and two-particle) terms and impose the elastic unitarity condition by solving an integral equation and that both neglect inelastic effects. The difference lies in the treatment of singularities in the unphysical region. It is not clear which method will yield more reliable results. On the other hand, the dispersion theory is certainly somewhat simpler. In the formulation of our integral equations, we introduce a cutoff parameter for each of the vector mesons $\rho$, $\omega$, and $\varphi$. We choose an exponential cutoff as suggested by the Regge-pole description of composite particles. The parameters are denoted by $c_\rho$, $c_\omega$, and $c_\varphi$. For the two $S$-wave amplitudes ($I=0$, ${}^3S_1$ and $I=1$, ${}^1S_0$), we introduce the scattering lengths as subtraction constants. This normalization of the $S$-wave amplitude at threshold presumably absorbs the over-all contribution of extreme short-range forces which has not been taken into account explicitly. Finally, one additional parameter $s_0$ is introduced in the formulation of the equations for $l\geq 2$ partial-wave amplitudes. Essentially, this parameter is introduced to compensate for the approximate treatment of the centrifugal barrier in the partial-wave dispersion relations. To summarize, we have eight predetermined constants $g_\pi^2=14$, $m_\pi=135$ MeV, $m_\eta=548$ MeV, $m_\omega=780$ MeV, $m_\varphi=1020$ MeV, $a_1^{pp}=-7.7\times10^{-13}$ cm, $a_1^{np}=-23.74\times10^{-13}$ cm, $a_3^{np}=5.4\times10^{-13}$ cm and twelve adjustable parameters $g_\mu^2$, $g_{\rho^1}^2$, $g_{\rho^2}^2$, $g_\omega^2$, $g_\varphi^2$, $m_\rho$, $c_\rho$, $c_\omega$, $c_\varphi$, $t_c$, $a_\pi$, and $s_0$ (or $g_\sigma^2$, $m_\sigma^2$ in the place of $t_c$ and $a_\pi$). For any given set of these parameters, all of the $np$ and $pp$ partial-wave amplitudes are calculated including the static Coulomb correction in the $pp$ case. Having obtained the partial-wave amplitudes, the observables $\sigma$, $P$, $D$, $R$, and $A$ are calculated and compared with a set of 560 pieces of $np$ and $pp$ data collected by the Livermore group.[4] A search on the twelve parameters is performed to obtain a good fit to the data. If one assumes that our present approximation in the theory is inherently good to only 8%, then our results can be considered good fits in both the case where the $\sigma$ is assumed and the case where we have only a continuous $\pi\pi$ spectrum. With our calculated phase shifts, an evaluation of the $\chi^2$ has been performed by the Livermore group.[4] They quoted a value of $\chi^2$ of 822 for our result using 377 pieces of $pp$ data.

In Sec. II, we illustrate the mathematical procedure by the fictitious example of the scattering of two spinless nucleons. In Sec. III, we treat the real nucleons and define scattering amplitudes taken between states of definite initial and final helicity and their partial-wave projections. We also express the observables $\sigma$, $P$, $D$, $R$, and $A$ in terms of the helicity amplitudes. The one-meson (resonance) exchange terms and the pair contribution are given in Sec. IV. In Sec. V, we formulate partial-wave dispersion relations and corresponding integral equations. Numerical results are discussed in Sec. VI. A detailed account of the Coulomb correction is given in an Appendix. Preliminary results of our work have been published elsewhere.[5] Some references to earlier works can also be found there.

## II. SPINLESS NUCLEONS

We first consider the scattering of two fictitious spinless nucleons. We introduce kinematical variables, invariant energy and momentum transfer squared, in the conventional way

$$\begin{aligned} s&=-(P_1+P_2)^2=4(p^2+m^2)\,,\\ t&=-(P_1-P_3)=-2p^2(1-z)\,,\\ u&=-(P_1-P_4)=-2p^2(1+z)\,,\end{aligned}$$

with the constraint $s+t+u=4m^2$, where $P_1$, $P_2$ and $P_3$, $P_4$ are the incoming and outgoing four momenta of the nucleons, $m$ is the nucleon mass, $p$ is the magnitude of the center-of-mass momentum, and $z$ the cosine of the scattering angle.

If there exists a scalar meson of mass $\mu_S$ that can be exchanged between the two nucleons, it will correspond to a pole in the nucleon-antinucleon scattering amplitude, that is to say, in our simple example, a pole in the nucleon-antinucleon $S$-wave amplitude. We will have then the following contribution:

$$\varphi_{NN}{}^E(s;t)=\varphi_{N\bar{N}}{}^E(t;s)=g_S{}^2(t)/(\mu_S{}^2-t)\,,\qquad(1)$$

[4] Some results of the Livermore phase shift analysis are published by M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nucl. Sci. 10, 291 (1960). The data selection we use has been communicated to us through H. P. Noyes. The $\chi^2$ of our calculated phase shifts are evaluated by the Livermore MIDPOP III program.

[5] A. Scotti and D. Y. Wong, Phys. Rev. Letters 10, 142 (1963), Proceedings of the Topical Conference on Newly Discovered Particles, Athens, Ohio, April 1963, Ref. 2, and Proceedings of the International Congress on Nuclear Physics, Paris, July 1964 (to be published).

where the first variable of $\varphi$ denotes the invariant energy squared in the system indicated by the subscript and the second variable is the corresponding momentum transfer squared. The vertex function $g_S(t)$ is independent of $s$ and $g_S(\mu_S^2)$ is defined as the coupling constant. The superscript $E$ indicates the (single-particle) exchange contribution.

In addition to the exchange of a scalar meson, let us consider now the exchange of a $J=1$ resonance system. If we treat the $J=1$ resonance as a vector meson, then the only formal difference from the scalar case is that we are now considering a $P$-wave amplitude in the nucleon-antinucleon channel. Hence

$$\varphi_{NN}{}^E(s;t)=\varphi_{N\bar{N}}{}^E(t;s)=[g_S{}^2(t)/(\mu_S{}^2-t)]+[g_V{}^2(t)/(\mu_V{}^2-t)]P_1(\bar{z})\,, \quad (2)$$

where

$$\bar{z}\equiv\cos\theta_{N\bar{N}}=1+[2s/(t-4m^2)]$$

and $P_1(\bar{z})$ is the Legendre polynomial of order 1,

$$P_1(\bar{z})=\bar{z}\,.$$

We obtain therefore a contribution which is linearly divergent in $s$. As we shall see later, the corresponding integral equations for the partial-wave amplitudes would not have, in general, any solution.

If we consider instead the $J=1$ resonance as a composite system, a natural cutoff is contained in its description as a Regge pole.[6] We will use instead of (2) the following expression for the $J=1$ resonance exchange contribution:

$$\begin{aligned}\varphi_{NN}{}^E(s;t)&=\varphi_{N\bar{N}}{}^E(s;t)\\&=[g_S{}^2(t)/(\mu_S{}^2-t)]+[g_V{}^2(t)/(\mu_V{}^2-t)]P_1(\bar{z})\\&\quad\times\exp\{c_V(t-\mu_V{}^2)\ln[(s/2m^2)-1]\}\,, \quad (3)\end{aligned}$$

where $c_V$ is a positive real parameter. This particular form is chosen in such a way that at threshold $s\to 4m^2$ it reduces exactly to the vector meson contribution (2) and for $s\to+\infty$ it becomes the asymptotic form of the exact Regge-pole expression. Needless to say, the above expression reduces to the form of a vector particle exchange term at $t=\mu_V{}^2$.

We shall refer to both terms in the representation (3) as one-meson exchange terms. Once $\varphi_{NN}{}^E$ is given (in the realistic case we will have contributions from $\pi$, $\eta$, $\omega$, $\rho$, and $\varphi$ "mesons" and the $S$-wave $\pi\pi$ pair), we use dispersion relations to construct scattering amplitudes which satisfy the elastic unitarity condition.

The unitarity condition implies that the partial-wave amplitudes have a branch point at the physical threshold of the nucleon-nucleon system. The discontinuity across the corresponding branch cut is proportional to the square of the amplitude. Since we are dealing with strong interactions in the low-energy region, such a branch cut certainly has a significant influence on the scattering amplitude. However, as we shall see in the realistic problem, the $l=6$ and higher partial waves phase shifts are very small in the elastic scattering region (<400 MeV) and the branch cut due to the unitarity condition can be neglected. For intermediate partial waves, from $F$ to $H$, the phase shifts are also rather small but such branch cuts are not completely negligible. We take into account the unitarity condition for these waves in an approximate way by considering the contribution to the imaginary part in the physical region coming only from the first iteration of the one-pion exchange. For $S$, $P$, and $D$ waves we impose the complete elastic unitarity condition.

Let us normalize the partial-wave amplitudes as follows:

$$h_l(s)=[(p^2+m^2)/p^2m^2]^{1/2}e^{i\delta_l}\sin\delta_l\,, \quad (4)$$

and denote by $h_l{}^E$ the $l$th partial-wave projection of $\varphi_{NN}{}^E(s;t)$. Our basic set of equations will then be

$$h_0(s)=-a+h_0{}^E(s)-h_0{}^E(4m^2)+\frac{s-4m^2}{\pi}\int_{4m^2}^{+\infty}ds'\left[\frac{m^2}{s'(s'-4m^2)}\right]^{1/2}\frac{|h_0(s')|^2}{(s'-s)}\,, \quad (5)$$

$$h_1(s)=h_1{}^E(s)+\frac{s-4m^2}{\pi}\int_{4m^2}^{+\infty}ds'\left[\frac{m^2}{s'(s'-4m^2)}\right]^{1/2}\frac{|h_1(s')|^2}{(s'-s)}\,, \quad (6)$$

$$h_2(s)=h_2{}^E(s)+\frac{(s-4m^2)^2}{\pi(s-s_0)}\int_{4m^2}^{+\infty}ds'\left[\frac{m^2}{s'(s'-4m^2)}\right]^{1/2}\left(\frac{s'-s_0}{s'-4m^2}\right)\frac{|h_2(s')|^2}{(s'-s)}\,, \quad (7)$$

$$h_l(s)=h_l{}^E(s)+\left(\frac{s-4m^2}{s-s_0}\right)^{l-1}\frac{s-4m^2}{\pi}\int_{4m^2}^{+\infty}ds'\left[\frac{m^2}{s'(s'-4m^2)}\right]^{1/2}\left(\frac{s'-s_0}{s'-4m^2}\right)^{l-1}\frac{|h_l{}^{(\pi)}(s')|^2}{(s'-s)};\quad 2<l\leq 5\,, \quad (8)$$

$$h_l(s)=h_l{}^E(s);\quad l>5\,. \quad (9)$$

[6] T. Regge, Nuovo Cimento 18, 947 (1960).

The $S$-wave scattering length is introduced as a subtraction constant. Equations (5), (6), and (7) are integral equations that can be solved by the well-known $N/D$ method[7,8] if $h_l^E(s)$ is a known function analytic and bounded to the right of $s=4m^2$. This is indeed the case for the partial-wave projection of our representation for $\varphi_{NN}^E(s;t)$. For $D$ and higher waves the threshold behavior $(s-4m^2)^l$ cannot be satisfied if one imposes the elastic umitarity condition without introducing an additional singularity to the left of $s=4m^2$ aside from singularities already present in the single-particle exchange terms. This is due to the fact that each single-particle exchange term satisfies the threshold behavior while the dispersion integral at $s=4m^2$ is positive definite. Hence the dispersion integral and the multiparticle exchange terms must cancel each other (to the $l$th order) at threshold.

For $l\geq 2$, we represent the effect of all multiparticle exchange terms by a pole at $s_0$ and obtain Eqs. (7) and (8) which satisfy the threshold behavior. Since the single-particle exchange terms are the dominant terms for high partial waves, the solutions are not sensitive to $s_0$. For simplicity, we choose a single $s_0$ for all $l\geq 2$ waves. After solving Eqs. (5), (6), and (7) and evaluating the integrals in (8), one obtains $h_l(s)$. The complete scattering amplitude $\varphi_{NN}(s;t)$ can now be evaluated as follows:

$$\varphi_{NN}(s;t)=\varphi_{NN}^E(s;t)+\sum_{l=0}^{5}(2l+1)h_l^R(s)P_l(z)\,, \quad (10)$$

where

$$h_l^R(s)=h_l(s)-h_l^E(s)\,. \quad (11)$$

This completes the evaluation of $\varphi_{NN}(s;t)$.

Before turning to the realistic nucleon-nucleon problem, we return briefly to the partial-wave projection of the one-meson exchange term corresponding to a $J=1$ resonance.

$$h_l^E(s)=\frac{1}{2}\int_{-1}^{+1}dz\,P_l(z)\frac{g_V^2(t)}{\mu_V^2-t}P_1(\bar{z})\exp\left[c_V(t-\mu_V^2)\ln\left(\frac{s}{2m^2}-1\right)\right]$$
$$=\frac{1}{(s-4m^2)}\int_{-(s-4m^2)}^{0}dt\,P_l\left(1+\frac{2t}{s-4m^2}\right)\frac{g_V^2(t)}{\mu_V^2-t}\left(1+\frac{2s}{t-4m^2}\right)\exp\left[c_V(t-\mu_V^2)\ln\left(\frac{s}{2m^2}-1\right)\right]. \quad (12)$$

It is clear first of all that $h_l^E(s)$ is not bounded for large $s$ if one takes the usual Born approximation for vector particle exchange: $c_V=0$ and $g_V^2(t)=g_V^2\times(t-4m^2)$, where $g_V^2$ is a constant. We shall retain this approximation for the vertex function but introduce the cutoff parameter $c_V>0$. As regards the crude approximation to the Regge-pole expression we remark that, as long as $c_V>0$, only small negative values of $t$ are relevant in the integration in (12): this is obvious when $(s-4m^2)$ small; however, even when $(s-4m^2)$ is large the dominant contribution to the integral still comes from small negative values of $t$ since the exponential factor is rapidly decreasing as $t$ becomes large negative.

### III. SCATTERING AMPLITUDES AND THE "OBSERVABLES"

If one assumes invariance under space reflections, time reversal, and charge independence, it is well known[9] that the scattering of two nucleons can be described by five amplitudes for each isotopic-spin state $I=0$ or $I=1$. One of the five amplitudes corresponds to transitions in a singlet-spin state and the other four to transitions in a triplet-spin state. The choice of the scattering amplitudes is, of course, not unique. We shall use the helicity amplitudes $\varphi_n^{(I)}$; $n=1, \cdots, 5$; $I=0, 1$, defined in GGMW.[10] The relation between helicity amplitudes and the scattering amplitudes of Stapp, Ypsilantis, and Metropolis[9] is given in Appendix A.

To avoid all possible confusion about normalization, we give here an explicit formula of the helicity amplitudes in terms of the conventional singlet and triplet phase shifts. Let the partial-wave amplitudes be defined as

$$h_J=(E/2imp)(e^{2i\delta_J}-1)\,, \quad (13a)$$

$$h_{JJ}=(E/2imp)(e^{2i\delta_{JJ}}-1)\,, \quad (13b)$$

$$h_{J-1,J}=(E/2imp)[(\cos 2\epsilon_J)\exp(2i\delta_{J-1,J})-1]\,, \quad (13c)$$

$$h_{J+1,J}=(E/2imp)[(\cos 2\epsilon_J)\exp(2i\delta_{J+1,J})-1]\,, \quad (13d)$$

$$h^J=(E/2mp)\sin 2\epsilon_J\exp[i(\delta_{J-1,J}+\delta_{J+1,J})]\,, \quad (13e)$$

where $E=\frac{1}{2}\sqrt{s}$.

The first one is the singlet amplitude, the second one is the uncoupled-triplet amplitude and the last three are the coupled-triplet amplitudes expressed in the nuclear-bar phase shifts and coupling parameters.[9] The

[7] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[8] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).
[9] See, for example, H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1958).

[10] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960); hereafter referred to as GGMW. The relativistic nucleon-nucleon problem has also been investigated by D. Amati, E. Leader, and B. Vitale, Nuovo Cimento **17**, 68 (1960); **18**, 409, 458 (1960); Phys. Rev. **130**, 750 (1963).

partial-wave expansion of the helicity amplitudes is given by[10]

$$(E/m)\varphi_1^{(0,1)}=\sum_{J\ \text{odd, even}}\{(2J+1)h_J^{(0,1)}+Jh_{J-1,J}^{(0,1)}+(J+1)h_{J+1,J}^{(0,1)}+2[J(J+1)]^{1/2}h^{J(0,1)}\}P_J\,,\tag{14a}$$

$$(E/m)\varphi_2^{(0,1)}=\sum_{J\ \text{odd, even}}\{-(2J+1)h_J^{(0,1)}+Jh_{J-1,J}^{(0,1)}+(J+1)h_{J+1,J}^{(0,1)}+2[J(J+1)]^{1/2}h^{J(0,1)}\}P_J\,,\tag{14b}$$

$$(E/m)\varphi_3^{(0,1)}=\sum_{J\ \text{odd, even}}\{(J+1)h_{J-1,J}^{(0,1)}+Jh_{J+1,J}^{(0,1)}-2[J(J+1)]^{1/2}h^{J(0,1)}\}d_{11}^J+\sum_{J\ \text{even, odd}}[(2J+1)h_{JJ}^{(0,1)}]d_{11}^J\,,\tag{14c}$$

$$(E/m)\varphi_4^{(0,1)}=\sum_{J\ \text{odd, even}}\{(J+1)h_{J-1,J}^{(0,1)}+Jh_{J+1,J}^{(0,1)}-2[J(J+1)]^{1/2}h^{J(0,1)}\}d_{-11}^J-\sum_{J\ \text{even, odd}}[(2J+1)h_{JJ}^{(0,1)}]d_{-11}^J\,,\tag{14d}$$

$$[1/(1-z^2)^{1/2}]\varphi_5^{(0,1)}=(m/E)\sum_{J\ \text{odd, even}}\{[J(J+1)]^{1/2}(h_{J-1,J}^{(0,1)}-h_{J+1,J}^{(0,1)})+h^{J(0,1)}\}d_{10}^J/(1-z^2)^{1/2}.\tag{14e}$$

$P_J$ is the usual Legendre polynomial and

$$d_{11}^J=\frac{1}{1+z}\left[P_J+\left(\frac{J+1}{2J+1}\right)P_{J-1}+\left(\frac{J}{2J+1}\right)P_{J+1}\right],\tag{15a}$$

$$d_{-11}^J=\frac{1}{1-z}\left[-P_J+\left(\frac{J+1}{2J+1}\right)P_{J-1}+\left(\frac{J}{2J+1}\right)P_{J+1}\right],\tag{15b}$$

$$d_{10}^J/(1-z^2)^{1/2}=-\left(\frac{J+1}{J}\right)^{1/2}\left[\frac{zP_J-P_{J+1}}{(1-z^2)}\right]=\frac{[J(J+1)]^{1/2}}{2J+1}\left[\frac{P_{J+1}-P_{J-1}}{1-z^2}\right].\tag{15c}$$

As shown in GGMW, the left-hand side of Eqs. (14a) to (14e) has no $\sqrt{s}$ kinematical singularity. The appearance of the $(m/E)$ factor on the right-hand side of (14e) implies that the coupled triplet partial-wave amplitudes must contain the kinematical singularity at $s=0$. However, since this singularity is far removed from the physical region, its effect on the unitarity integral should be quite small. In our present work, we keep this kinematical singularity in the partial-wave projection of the exchange terms and introduce no additional singularity of this kind in the process of imposing the unitarity condition.

By making use of the orthonormality properties of the $P$ and $d$ functions, one obtains

$$h_J^{(0,1)}=\frac{E}{4m}\int_{-1}^{+1}dz\,P_J[\varphi_1^{(0,1)}-\varphi_2^{(0,1)}],\tag{16a}$$

$$h_{JJ}^{(0,1)}=\frac{E}{4m}\int_{-1}^{+1}dz[d_{11}^J\varphi_3^{(0,1)}-d_{-11}^J\varphi_4^{(0,1)}],\tag{16b}$$

$$h_{J-1,J}^{(0,1)}=\frac{1}{2J+1}\frac{E}{4m}\int_{-1}^{+1}dz\{JP_J(\varphi_1^{(0,1)}+\varphi_2^{(0,1)})+(J+1)(d_{11}^J\varphi_3^{(0,1)}+d_{-11}^J\varphi_4^{(0,1)})+4[J(J+1)]^{1/2}d_{10}^J\varphi_5^{(0,1)}\},\tag{16c}$$

$$h_{J+1,J}^{(0,1)}=\frac{1}{2J+1}\frac{E}{4m}\int_{-1}^{+1}dz\{(J+1)P_J(\varphi_1^{(0,1)}+\varphi_2^{(0,1)})+J(d_{11}^J\varphi_3^{(0,1)}+d_{-11}^J\varphi_4^{(0,1)})-4[J(J+1)]^{1/2}d_{10}^J\varphi_5^{(0,1)}\},\tag{16d}$$

$$h^{J(0,1)}=\frac{[J(J+1)]^{1/2}}{2J+1}\frac{E}{4m}\int_{-1}^{+1}dz\big(P_J(\varphi_1^{(0,1)}+\varphi_2^{(0,1)})-(d_{11}^J\varphi_3^{(0,1)}+d_{-11}^J\varphi_4^{(0,1)})+\{2/[J(J+1)]^{1/2}\}d_{10}^J\varphi_5^{(0,1)}\big).\tag{16e}$$

Having defined our scattering amplitudes and their partial-wave projection, let us now turn to the expression of the experimental "observables": these are well known[11] and are quoted here for convenience and completeness.

We first give the expression of the Wolfenstein parameters[11] in terms of the helicity amplitudes

$$\begin{aligned}
a&=\tfrac{1}{4}\{\varphi_1-\varphi_2+\varphi_3+\varphi_4+z(\varphi_1+\varphi_2+\varphi_3-\varphi_4)-4y\varphi_5\}\,,\\
ic&=\tfrac{1}{4}\{\varphi_1+\varphi_2+\varphi_3-\varphi_4)+4z\varphi_5\}\,,\\
m&=\tfrac{1}{4}\{-(\varphi_1-\varphi_2+\varphi_3+\varphi_4)\\
&\qquad+z(\varphi_1+\varphi_2+\varphi_3-\varphi_4)-4y\varphi_5\}\,,\qquad(17)\\
g&=\tfrac{1}{4}\{\varphi_3+\varphi_4-\varphi_1+\varphi_2\}\,,\\
h&=\tfrac{1}{4}\{\varphi_3-\varphi_4-\varphi_1-\varphi_2\}\,,
\end{aligned}$$

and then the expression of $\sigma$, $P$, $D$, $R$, and $A$ in terms of the Wolfenstein parameters

$$\begin{aligned}
\sigma&=|a|^2+|m|^2+2|c|^2+2|g|^2+2|h|^2\,,\\
\sigma P&=2\,\mathrm{Re}[c^*(a+m)]\,,\\
\sigma(1-D)&=4|g|^2+4|h|^2\,,\\
\sigma R&=[|a|^2-|m|^2-4\,\mathrm{Re}(hg^*)]\cos(\theta/2)\qquad(18)\\
&\qquad+2\,\mathrm{Re}[ic(a^*-m^*)]\sin(\theta/2)\,,\\
\sigma A&=-[|a|^2-|m|^2-4\,\mathrm{Re}(hg^*)]\sin(\theta/2)\\
&\qquad+2\,\mathrm{Re}[ic(a^*-m^*)]\cos(\theta/2)\,,
\end{aligned}$$

where the isotopic-spin index has been suppressed, $y=\sin\theta$, $\sigma$ is the differential cross section, and $\theta$ is the scattering angle. For the definition of $P$, $D$, $R$, $A$, and of the Wolfenstein parameters we again refer the reader to Ref. 11.

## IV. ONE-MESON EXCHANGE TERMS

### A. The Single Pion

The one-pion exchange term is well known. Although it is easily derivable from conventional Lagrangian field theory we prefer to derive it, according to what we said in Sec. II, by considering a pole in the nucleon-antinucleon scattering amplitude which satisfies all the selection rules for a transition via the intermediate state of a pseudoscalar particle with isospin one. We will derive all the other one-meson exchange terms using this prescription. By making use of (6.7) and (4.23) of Ref. 10, we obtain

$$\begin{aligned}
(E/m)\varphi_{1,\pi}^{(0)}&=0\,,\\
\frac{E}{m}\varphi_{2,\pi}^{(0)}&=\frac{-3g_\pi^2}{4m}\left[\frac{u}{m_\pi^2-u}-\frac{t}{m_\pi^2-t}\right],\\
\frac{E}{m}\varphi_{3,\pi}^{(0)}&=\frac{3g_\pi^2}{4m}(1+z)\left[\frac{-2p^2}{m_\pi^2-u}\right],\qquad(19)\\
\frac{E}{m}\varphi_{4,\pi}^{(0)}&=\frac{3g_\pi^2}{4m}(1-z)\left[\frac{-2p^2}{m_\pi^2-t}\right],\\
[1/(1-z^2)^{1/2}]\varphi_{5,\pi}^{(0)}&=0\,.
\end{aligned}$$

The amplitudes for $I=1$ are obtained from the corresponding amplitudes for $I=0$ by supplying a factor of $\frac{1}{3}$ and changing the sign of the $t$ contributions.

### B. The $\eta$ Meson

For the $\eta$ meson, which differs from the $\pi$ meson only for the isospin, we obtain

$$\begin{aligned}
(E/m)\varphi_{1,\eta}^{(0)}&=0\,,\\
\frac{E}{m}\varphi_{2,\eta}^{(0)}&=\frac{g_\eta^2}{4m}\left[\frac{u}{m_\eta^2-u}-\frac{t}{m_\eta^2-t}\right],\\
\frac{E}{m}\varphi_{3,\eta}^{(0)}&=-\frac{g_\eta^2}{4m}(1+z)\left[\frac{-2p^2}{m_\eta^2-u}\right],\qquad(20)\\
\frac{E}{m}\varphi_{4,\eta}^{(0)}&=-\frac{g_\eta^2}{4m}(1-z)\left[\frac{-2p^2}{m_\eta^2-t}\right],\\
[1/(1-z^2)^{1/2}]\varphi_{5,\eta}^{(0)}&=0\,.
\end{aligned}$$

And the $I=1$ amplitudes are obtained by changing sign of the $u$ contributions.

### C. The $\omega$ and $\varphi$ Mesons

Taking into account the quantum numbers involved in the exchange of a $\omega$ or $\varphi$ meson we can write the corresponding spectral representation for the $N\bar{N}\to N\bar{N}$ amplitude in the following way:

$$V^2(t)=\frac{1}{\pi}\int_{9m_\pi^2}^{+\infty}dt'\frac{g^2(t')}{t'-t}\,.\qquad(21)$$

In principle $g^2(t')$ is related to the $N\bar{N}\to 3\pi$ and $N\bar{N}\to K\bar{K}$ amplitudes. However, since the widths of the $\omega$ and $\varphi$ resonances are rather narrow, we use the

[11] M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nucl. Sci. **10**, 291 (1960).

much simpler approximate representation

$$V^2(t)=\frac{g_\omega^2}{m_\omega^2-t}+\frac{g_\varphi^2}{m_\varphi^2-t}, \qquad (22)$$

where now $g_\omega^2$ and $g_\varphi^2$ are constants. Neglecting the coupling to the anomalous magnetic moment of the nucleon, as mentioned in the introduction, and following our prescription, we obtain for the $\omega$-meson exchange terms

$$\frac{E}{m}\varphi_{1,\omega}^{(0)}=\frac{g_\omega^2}{m}\left[\frac{\frac{1}{2}s-\frac{3}{2}m^2-\frac{1}{2}m^2z}{m_\omega^2-u}-\frac{\frac{1}{2}s-\frac{3}{2}m^2+\frac{1}{2}m^2z}{m_\omega^2-t}\right],$$

$$\frac{E}{m}\varphi_{2,\omega}^{(0)}=\frac{g_\omega^2}{m}\left[\frac{-\frac{1}{2}m^2-\frac{1}{2}m^2z}{m_\omega^2-u}+\frac{\frac{1}{2}m^2-\frac{1}{2}m^2z}{m_\omega^2-t}\right],$$

$$\frac{E}{m}\varphi_{3,\omega}^{(0)}=\frac{g_\omega^2(1+z)}{m}\left[\frac{-\frac{1}{2}m^2}{m_\omega^2-u}+\frac{-\frac{1}{4}s+\frac{1}{2}m^2}{m_\omega^2-t}\right], \qquad (23)$$

$$\frac{E}{m}\varphi_{4,\omega}^{(0)}=\frac{g_\omega^2(1-z)}{m}\left[\frac{-\frac{1}{4}s+\frac{1}{2}m^2}{m_\omega^2-u}+\frac{-\frac{1}{2}m^2}{m_\omega^2-t}\right],$$

$$\frac{1}{(1-z^2)^{1/2}}\varphi_{5,\omega}^{(0)}=\frac{g_\omega^2}{m}\left[\frac{\frac{1}{2}m^2}{m_\omega^2-u}+\frac{\frac{1}{2}m^2}{m_\omega^2-t}\right].$$

The $I=1$ amplitudes are obtained by changing sign to the $u$ contributions. Obviously the $\varphi$-meson contributions are simply obtained from (23) by substituting $g_\varphi^2$ for $g_\omega^2$ and $m_\varphi^2$ for $m_\omega^2$. As explained in Sec. II the fact that we are dealing here with $J=1$ resonances requires the introduction of a cutoff which we take to be a simple factor for each of the $t$ contributions $\exp[c_\omega(t-m_\omega^2)\times\ln(-1+s/2m^2)]$ and $\exp[c_\varphi(t-m_\varphi^2)\ln(-1+s/2m^2)]$ for the $\omega$ and $\varphi$ meson, respectively, and similarly for the $u$ channel.

### D. The ρ Meson

Although the contribution of the anomalous magnetic moment term is of order $(p^2/m^2)$ smaller than the charge term, this is not negligible in the case of the $\rho$ resonance since the isovector anomalous magnetic moment is quite large. Consequently, we will take it into account here. Let us begin, as before, by defining the following spectral representations

$$V_{\rho 1}^2(t)=\frac{3}{8\pi}\int_{4m_\pi^2}^{\infty}dt'\frac{(t'/4-m_\pi^2)^{3/2}|2\Gamma_1(t')|^2}{(\frac{1}{4}t')^{1/2}(t'-t)}, \qquad (24a)$$

$$V_{\rho 2}^2(t)=\frac{3}{8\pi}\int_{4m_\pi^2}^{+\infty}dt'\frac{(\frac{1}{4}t'-m_\pi^2)^{3/2}|4m\Gamma_2(t')|^2}{(\frac{1}{4}t')^{1/2}(t'-t)}, \qquad (24b)$$

$$V_{\rho 1\rho 2}(t)=\frac{3}{8\pi}\int_{4m_\pi^2}^{+\infty}dt'\frac{(\frac{1}{4}t'-m_\pi^2)^{3/2}|8m\Gamma_2(t')\Gamma_1(t')|}{(\frac{1}{4}t')^{1/2}(t'-t)}, \qquad (24c)$$

where $\Gamma_1$ snd $\Gamma_2$ are the Frazer-Fulco[12] amplitudes linearly related to the isovector form factors. Note that a factor of 4 has been introduced here in changing from the isospin (−) convention of Frazer and Fulco to the isospin 1 amplitudes.

From Eqs. (24a), (24b), and (24c) we see that in approximating the right-hand side by a simple pole,

$$V_{\rho 1}^2(t)\cong g_{\rho 1}^2/(m_\rho^2-t), \qquad (24a')$$

$$V_{\rho 2}^2(t)\cong g_{\rho 2}^2/(m_\rho^2-t), \qquad (24b')$$

$$V_{\rho 1\rho 2}^2\cong g_{\rho 1}g_{\rho 2}/(m_\rho^2-t). \qquad (24c')$$

The value $m_\rho$ of the "effective" mass of the $\rho$ meson is controlled by the behavior of the functions $|\Gamma_i|$ $(i=1,2)$. If $D_\pi(t)$ is the $\pi$-$\pi$ $P$-wave denominator function, $D_\pi(t)\Gamma_i(t)$ is real[2] and $|\Gamma_i|^2$ can be written in the following way

$$|\Gamma_i|^2=|D_\pi(t)\Gamma_i(t)/D_\pi(t)|^2=(D_\pi(t)\Gamma_i(t))^2|F_\pi(t)|^2, \qquad (25)$$

where $F_\pi(t)$ is the pion form factor.

From the work of Ball and Wong[3] on the nucleon form factors, one sees that $D_\pi(t)\Gamma_i(t)$ are smooth functions of $t$, but weigh heavily the spectral function below the resonance. Together with the fact that the $\rho$ resonance is rather broad, we see then that a substantial lowering of the "effective" mass $m_\rho$ from the experimental value found in production processes is not surprising. Experimental analysis of the form factors indeed gives strong support to such shifting. Consequently $m_\rho$ will be considered as a free parameter. We remark here that if the $\rho$ width is substantially below the 130-MeV value taken by Ball and Wong, then the effective $\rho$ mass will be only slightly below the resonance peak. However, in this case, the $\rho$-resonance contribution to the static anomalous magnetic moment (and probably the charge) will exceed the experimental value, thus requiring an opposite contribution from the higher energy spectral functions. The net result could still be represented by an average pole substantially *below* the $\rho$ resonance.

In the approximation corresponding to Eqs. (24a′), (24b′), and (24c′) we obtain now the following expressions for the $\rho$-meson exchange terms corresponding to

[12] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960); **119**, 1420 (1960).

the three types of couplings:

$$
\begin{aligned}
\frac{E}{m}\varphi_{1,\rho 1}{}^{(0)} &= \frac{3}{m} g_{\rho 1}{}^2 \left[\frac{-\frac{1}{2}s+\frac{3}{2}m^2+\frac{1}{2}m^2 z}{m_\rho{}^2-u}+\frac{\frac{1}{2}s-\frac{3}{2}m^2+\frac{1}{2}m^2 z}{m_\rho{}^2-t}\right],\\
\frac{E}{m}\varphi_{2,\rho 1}{}^{(0)} &= \frac{3}{m} g_{\rho 1}{}^2 \left[\frac{\frac{1}{2}m^2+\frac{1}{2}m^2 z}{m_\rho{}^2-u}+\frac{-\frac{1}{2}m^2+\frac{1}{2}m^2 z}{m_\rho{}^2-t}\right],\\
\frac{E}{m}\varphi_{3,\rho 1}{}^{(0)} &= \frac{3}{m} g_{\rho 1}{}^2 (1+z)\left[\frac{\frac{1}{2}m^2}{m_\rho{}^2-u}+\frac{\frac{1}{4}s-\frac{1}{2}m^2}{m_\rho{}^2-t}\right],\\
\frac{E}{m}\varphi_{4,\rho 1}{}^{(0)} &= \frac{3}{m} g_{\rho 1}{}^2 (1-z)\left[\frac{\frac{1}{4}s-\frac{1}{2}m^2}{m_\rho{}^2-u}+\frac{\frac{1}{2}m^2}{m_\rho{}^2-t}\right],\\
\frac{1}{(1-z^2)^{1/2}}\varphi_{5,\rho 1}{}^{(0)} &= \frac{3}{m} g_{\rho 1}{}^2 \left[\frac{-\frac{1}{2}m^2}{m_\rho{}^2-u}-\frac{\frac{1}{2}m^2}{m_\rho{}^2-t}\right],
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\frac{E}{m}\varphi_{1,\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 2}{}^2 \left\{\frac{\frac{1}{8}m_\rho{}^2(z+3)}{m_\rho{}^2-u}-\frac{\frac{1}{8}m_\rho{}^2(3-z)}{m_\rho{}^2-t}\right\},\\
\frac{E}{m}\varphi_{2,\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 2}{}^2 \Bigg(\frac{\{\frac{1}{8}u[(s/m^2)-2]+\frac{1}{8}m_\rho{}^2[-(7s/4m^2)+(sz/4m^2)+4]\}}{m_\rho{}^2-u}\\
&\qquad -\frac{\{\frac{1}{8}t[(s/m^2)-2]+\frac{1}{8}m_\rho{}^2[-(7s/4m^2)-(sz/4m^2)+4]\}}{m_\rho{}^2-t}\Bigg),\\
\frac{E}{m}\varphi_{3,\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 2}{}^2 (1+z)\left\{\frac{[(s^2/16m^2)+(sm_\rho{}^2/32m^2)-\frac{3}{8}s+\frac{1}{2}m^2]}{m_\rho{}^2-u}+\frac{\frac{1}{8}m_\rho{}^2}{m_\rho{}^2-t}\right\},\\
\frac{E}{m}\varphi_{4,\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 2}{}^2 (1-z)\left\{\frac{\frac{1}{8}m_\rho{}^2}{m_\rho{}^2-u}+\frac{[(s^2/16m^2)+(sm_\rho{}^2/32m^2)-\frac{3}{8}s+\frac{1}{2}m^2]}{m_\rho{}^2-t}\right\},\\
\frac{1}{(1-z^2)^{1/2}}\varphi_{5,\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 2}{}^2 \left\{-\frac{\frac{1}{8}m_\rho{}^2}{m_\rho{}^2-u}-\frac{\frac{1}{8}m_\rho{}^2}{m_\rho{}^2-t}\right\},
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\frac{E}{m}\varphi_{1,\rho 1\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 1} g_{\rho 2} \left\{\frac{\frac{1}{2}(u+m_\rho{}^2)}{m_\rho{}^2-u}-\frac{\frac{1}{2}(t+m_\rho{}^2)}{m_\rho{}^2-t}\right\},\\
\frac{E}{m}\varphi_{2,\rho 1\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 1} g_{\rho 2} \left\{\frac{[us/4m^2-m_\rho{}^2 s/4m^2+\frac{1}{2}m_\rho{}^2]}{m_\rho{}^2-u}-\frac{[ts/4m^2-m_\rho{}^2 s/4m^2+\frac{1}{2}m_\rho{}^2]}{m_\rho{}^2-t}\right\},\\
\frac{E}{m}\varphi_{3,\rho 1\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 1} g_{\rho 2} (1+z)\left\{\frac{-\frac{1}{4}s+m^2}{m_\rho{}^2-u}\right\},\\
\frac{E}{m}\varphi_{4,\rho 1\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 1} g_{\rho 2} (1-z)\left\{\frac{-\frac{1}{4}s+m^2}{m_\rho{}^2-t}\right\},\\
\frac{1}{(1-z^2)^{1/2}}\varphi_{5,\rho 1\rho 2}{}^{(0)} &= \frac{3}{m} g_{\rho 1} g_{\rho 2} \left\{\frac{\frac{1}{4}s-m^2}{m_\rho{}^2-u}+\frac{\frac{1}{4}s-m^2}{m_\rho{}^2-t}\right\}.
\end{aligned}
\tag{28}
$$

The $I=1$ terms are obtained by multiplying by $\frac{1}{3}$ and changing the sign of the $t$ contributions. Since the $\rho$ meson is a $J=1$ resonance like the $\omega$ and $\varphi$, in (26), (27), (28), each $t(u)$ contribution is to be multiplied by a factor $\exp[c_\rho(t-m_\rho^2)\ln(-1+s/2m^2)](\exp[c_\rho(u-m_\rho^2)\times\ln(-1+s/2m^2)])$.

Before leaving this section, we mention that our $g_{\rho 1}$ and $g_{\rho 2}$ are related to coupling constants in the Lagrangian formalism by

$$\mathcal{L}=i\pi^{1/2}(g_{\rho 1}+g_{\rho 2})\bar{\psi}(p')\boldsymbol{\tau}\gamma_\mu\psi(p)\mathbf{A}^\mu - \frac{\pi^{1/2}}{2m}g_{\rho 2}\bar{\psi}(p')\boldsymbol{\tau}\psi(p)\mathbf{A}^\mu(p+p')_\mu .$$

Similar expressions hold for the $\omega$ and $\varphi$ coupling constants except for omitting the isospin operator $\boldsymbol{\tau}$ and setting $g_{\omega 2}=g_{\varphi 2}=0$. In equations which involve $g_{\rho 2}$ [(27) and (28)] there is a difference between our expressions and those obtained by using the above Lagrangian in that the $m_\rho^2$ terms in the numerator should be replaced by $u$ (or $t$) in the Lagrangian calculation. This ambiguity is associated with the question of whether the spectral representation (24a), (24b), and (24c) requires a subtraction. Since we are only concerned with the contribution of the spectral function near the $\rho$-resonance region, it seems to us reasonable to use a no-subtraction form. Note that the difference between our expression and that of the Lagrangian calculation affects only the $S$- and the $P$-wave amplitudes for $NN$ scattering.

### E. The $I=0$ $S$-Wave Pion Pair Exchange Terms

Here again taking into account the appropriate quantum numbers we may write for the corresponding spectral representation of the $N\bar{N}$ amplitude

$$V_S^2(t)=\frac{1}{\pi}\int_{4m_\pi^2}^{+\infty}dt'\frac{g_S^2(t')}{t-t'}, \qquad (29)$$

where $g_S^2(t)$ is proportional to the absolute square of the $S$-wave $N\bar{N}\to\pi\pi$ amplitude $f_+^{(0)}(t')$, defined by Frazer and Fulco[12]

$$g_S^2(t')=\frac{3}{2}\left(\frac{t'-4m_\pi^2}{t'}\right)^{1/2}|f_+^{(0)}(t')/(\tfrac{1}{4}t'-m^2)|^2 \equiv \frac{3}{2}\left(\frac{t'-4m_\pi^2}{t'}\right)^{1/2}|\Gamma_0(t')|^2. \qquad (30)$$

Here we consider two alternative approximations of $g_s^2(t)$ depending on whther the $\sigma$ resonance exists or not. (i) Assuming that the $\sigma$ exists and is the dominant contributor to $g_s^2$, we write

$$g_s^2(t')\simeq 2\pi g_\sigma^2\delta(t'-m_\sigma^2),$$

thus

$$V_s^2(t)=2g_\sigma^2/(m_\sigma^2-t). \qquad (31)$$

This definition of $g_\sigma$ coincides with the conventional scalar coupling constant in a Lagrangian

$$\mathcal{L}=[g_\sigma/(4\pi)^{1/2}]\bar{\psi}\psi\varphi_\sigma .$$

(ii) Assuming that the $\sigma$ does not exist, we construct an approximate expression for $\Gamma_0$ by considering the contributions from the nucleon and the $N^*$ (3,3 resonance) and imposing the unitarity condition that $\Gamma_0$ should have the same phase as the $\pi$-$\pi$ $S$-wave amplitude $A_{\pi\pi}^S$. A simple scattering-length approximation is used for the latter amplitude since this approximation seems to be able to describe rather well the experimental data.[13,14] If we put

$$A_{\pi\pi}^S(t)=N_{\pi\pi}(t)/D_{\pi\pi}(t)$$

we have then

$$N_{\pi\pi}(t)=-a_\pi,$$

$$D_{\pi\pi}(t)=1-\frac{t}{\pi}\int_{4m^2}^{+\infty}dt'\left(\frac{t'-4m^2}{t'}\right)^{1/2}\frac{N_{\pi\pi}(t')}{t'(t'-t)}. \qquad (32)$$

The integral in (32) can be evaluated analytically and gives

$$D_{\pi\pi}(t)=1-\frac{2a_\pi}{\pi}\left[\left(\frac{t-4m_\pi^2}{t}\right)^{1/2}\ln[\tfrac{1}{2}(t-4m_\pi^2)^{1/2}+\tfrac{1}{2}t^{1/2}]-1\right]+ia_\pi\left(\frac{t-4m_\pi^2}{t}\right)^{1/2};\quad t>4m_\pi^2. \qquad (33)$$

Note that the conventional scattering length is $[a_\pi/(1+2a_\pi/\pi)]$.

Denoting by $\Gamma_B(t)$ the contribution to $\Gamma_0(t)$ from the nucleon and the (3,3) resonance,[15] we have

$$\Gamma_B(t)=g_\pi^2\frac{m}{p^2}Q_1\left(\frac{t-2m_\pi^2}{4pq}\right)+\frac{2\gamma_{33}}{pq}\left\{[\tfrac{3}{8}(E_{33}+m)^2(m_{33}-m)+(m_{33}+m)(m^2+m_\pi^2-\tfrac{1}{2}m_{33}^2-t)]Q_0\left(\frac{m_{33}^2+p^2+q^2}{2pq}\right)\right.$$
$$\left.+[\tfrac{2}{3}(E_{33}+m)^2-(m^2+m_\pi^2-\tfrac{1}{2}m_{33}^2-t)]\frac{mq}{p}Q_1\left(\frac{m_{33}^2+p^2+q^2}{2pq}\right)\right\}, \qquad (34)$$

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[13] K. M. Crowe, Phys. Rev. Letters **5**, 258 (1960); **7**, 35 (1961); A. Abashian, N. Booth, K. Crowe, R. Hill, and E. Rogers, Phys. Rev. **132**, 2296 (1963); M. E. Booth, A. Abashian, and K. Crowe, *ibid.* **132**, 2309, 2314 (1963).

[14] T. N. Truong, Phys. Rev. Letters **5**, 308 (1961).

[15] J. S. Ball and D. Y. Wong, Phys. Rev. **133**, B179 (1964).

where $Q_0$ and $Q_1$ are Legendre functions of the second kind, $q$ is the pion c.m. momentum, $\gamma_{33}=0.06$ is the $N^*\pi N$ coupling,[2] $m_{33}=8.7m_\pi$ is the mass of the (3,3) resonance and $E_{33}=(m_{33}^2+m^2-m_\pi^2)/(2m_{33})$.

Our expression for $\Gamma_0(t)$ is then given by

$$\frac{1}{D_{\pi\pi}(t)}\left\{\bar{\Gamma}_B(t)+\frac{t}{\pi}\int_{4m_\pi^2}^{t_c}dt'\frac{\bar{\Gamma}_B(t')-\bar{\Gamma}_B(t)}{t'(t'-t)}\left(\frac{t'-4m_\pi^2}{t'}\right)^{1/2}N_{\pi\pi}(t')\right\}, \tag{35}$$

where $t_c$ is a cutoff parameter introduced on account of the bad asymptotic behavior of the nucleon and the $N^*$ contributions. $\bar{\Gamma}_B(t)$ is obtained from $\Gamma_B(t)$ after two modifications. One consists in subtracting from the $Q$ functions appearing in (34) a corresponding $Q$ function evaluated at

$$(t-2m^2-2m_\pi^2-t_c)/4pq$$

so as to introduce a cutoff on the left discontinuity of $\Gamma_0(t)$ corresponding to the one introduced on the right. The second consists in subtracting the value of the expression so obtained evaluated at $t=0$ and adding the value $-0.077$ derived from pion-nucleon forward scattering dispersion relations,[16] so that the whole amplitude $\Gamma_0(t)$ becomes correctly normalized at $t=0$. This provides the so-called "pair suppression." Once $\Gamma_0(t)$ is obtained in this way we get $g_S^2(t)$ from (30) and evaluate the integral in (29): This completes the calculation of the function $V_S^2(t)$.

Having obtained $V_S^2$, we calculate the contribution of the $I=0$, $J=0$ $N\bar{N}\to N\bar{N}$ amplitude to the $NN\to NN$ amplitude using the crossing relations:

$$\begin{aligned}
(E/m)\varphi_{1,S}^{(0)}&=(1/4m)[-m^2(1-z)V_S^2(u)+m^2(1+z)V_S^2(t)],\\
(E/m)\varphi_{2,S}^{(0)}&=(1/4m)\{[-\tfrac{1}{2}u+m^2(1+z)]V_S^2(u)+[\tfrac{1}{2}t-m^2(1-z)]V_S^2(t)\},\\
(E/m)\varphi_{3,S}^{(0)}&=[(1+z)/4m][\tfrac{1}{4}sV_S^2(u)+m^2V_S^2(t)],\\
(E/m)\varphi_{4,S}^{(0)}&=[(1-z)/4m][m^2V_S^2(u)+\tfrac{1}{4}sV_S^2(t)],\\
[1/(1-z^2)^{1/2}]\varphi_{5,S}^{(0)}&=(1/4m)[-m^2V_S^2(u)-m^2V_S^2(t)].
\end{aligned} \tag{36}$$

The $I=1$ terms are obtained by changing the sign of the $u$ contributions.

## V. PARTIAL-WAVE DISPERSION EQUATIONS

Having obtained the exchange terms we follow now the procedure outlined in Sec. II and use dispersion relations to construct scattering amplitudes which satisfy elastic unitarity conditions. These conditions can be derived from Eqs. (13) and read as follows:

$$\underset{s\geq 4m^2}{\mathrm{Im}h_J}=(mp/E)|h_J|^2, \tag{37a}$$

$$\underset{s\geq 4m^2}{\mathrm{Im}h_{JJ}}=(mp/E)|h_{JJ}|^2, \tag{37b}$$

$$\underset{s\geq 4m^2}{\mathrm{Im}h_{J-1,J}}=(mp/E)\{|h^J|^2+|h_{J-1,J}|^2\}, \tag{37c}$$

$$\underset{s\geq 4m^2}{\mathrm{Im}h_{J+1,J}}=(mp/E)\{|h^J|^2+|h_{J+1,J}|^2\}, \tag{37d}$$

$$\underset{s\geq 4m^2}{\mathrm{Im}h^J}=(mp/E)\{h_{J-1,J}{}^*h^J+h^{J*}h_{J+1,J}\}. \tag{37e}$$

If we denote by $\varphi_n{}^{(I)E}(s,t)$ the sum of the exchange terms contributing to the helicity amplitudes in the isospin ($I$) state

$$\varphi_n{}^{(I)E}(s,t)=\varphi_{n,\pi}{}^{(I)}+\varphi_{n,\eta}{}^{(I)}+\varphi_{n,\omega}{}^{(I)}+\varphi_{n,\varphi}{}^{(I)}+\varphi_{n,\rho1}{}^{(I)}+\varphi_{n,\rho2}{}^{(I)}+\varphi_{n,\rho1\rho2}{}^{(I)}+\varphi_{n,S}{}^{(I)};\quad n=1,\cdots,5, \tag{38}$$

and by $h_J{}^E(s)$, $h_{JJ}{}^E(s)$, $h_{J-1,J}{}^E(s)$, $h_{J+1,J}{}^E(s)$, $h^{JE}(s)$ the corresponding partial-wave projections, it is easily seen that these partial-wave amplitudes are analytic functions regular for $s\geq 4m^2$. To satisfy Eqs. (37), our dispersion relations will introduce discontinuities starting at the physical threshold $s=4m^2$. Let us denote by $h_J{}^R(s)$, $h_{JJ}{}^R(s)$, $h_{J-1,J}{}^R(s)$, $h_{J+1,J}{}^R(s)$, $h^{JR}(s)$ that part of the amplitudes which contains these discontinuities, we will then write for the helicity amplitudes the following expressions corresponding to Eq. (4) of Sec. II:

$$\begin{aligned}
(E/m)\varphi_1{}^{(0)}=(E/m)\varphi_1{}^{(0)E}&+[3h_1{}^{(0)R}+h_{0,1}{}^{(0)R}+2h_{2,1}{}^{(0)R}+2\sqrt{2}h^{1(0)R}]P_1\\
&+[7h_3{}^{(0)R}+3h_{2,3}{}^{(0)R}+4h_{4,3}{}^{(0)R}+(2\sqrt{12})h^{3(0)R}]P_3,\\
(E/m)\varphi_2{}^{(0)}=(E/m)\varphi_2{}^{(0)E}&+[-3h_1{}^{(0)R}+h_{0,1}{}^{(0)R}+2h_{2,1}{}^{(0)R}+2\sqrt{2}h^{1(0)R}]P_1\\
&+[-7h_3{}^{(0)R}+3h_{2,3}{}^{(0)R}+4h_{4,3}{}^{(0)R}+(2\sqrt{12})h^{3(0)R}]P_3,\\
(E/m)\varphi_3{}^{(0)}=(E/m)\varphi_3{}^{(0)E}&+[2h_{0,1}{}^{(0)R}+h_{2,1}{}^{(0)R}-2\sqrt{2}h^{1(0)R}]d_{11}{}^1\\
&+[4h_{2,3}{}^{(0)R}+3h_{4,3}{}^{(0)R}-(2\sqrt{12})h^{3(0)R}]d_{11}{}^3+5h_{22}{}^{(0)R}d_{11}{}^2+9h_{44}{}^{(0)R}d_{11}{}^4,\\
(E/m)\varphi_4{}^{(0)}=(E/m)\varphi_4{}^{(0)E}&+[2h_{0,1}{}^{(0)R}+h_{2,1}{}^{(0)R}-2\sqrt{2}h^{1(0)R}]d_{-11}{}^1\\
&+[4h_{2,3}{}^{(0)R}+3h_{4,3}{}^{(0)R}-(2\sqrt{12})h^{3(0)R}]d_{-11}{}^3-5h_{22}{}^{(0)R}d_{-11}{}^2-9h_{44}{}^{(0)R}d_{-11}{}^4,
\end{aligned}$$

[16] J. S. Ball and D. Y. Wong, Phys. Rev. Letters 6, 29 (1961).

$$
\begin{aligned}
\frac{1}{(1-z^2)^{1/2}}\varphi_5^{(0)} &= \frac{1}{(1-z^2)^{1/2}}\varphi_5^{(0)E}+\frac{m}{E}[\sqrt{2}(h_{01}^{(0)R}-h_{21}^{(0)R})+h^{1(0)R}]\frac{d_{10}^1}{(1-z^2)^{1/2}} \\
&\quad +\frac{m}{E}[(\sqrt{12})(h_{23}^{(0)R}-h_{43}^{(0)R})+h^{3(0)R}]\frac{d_{10}^3}{(1-z^2)^{1/2}}, \\
(E/m)\varphi_1^{(1)} &= (E/m)\varphi_1^{(1)E}+[h_0^{(1)R}+h_{10}^{(1)R}]P_0+[5h_2^{(1)R}+2h_{12}^{(1)R}+3h_{32}^{(1)R}+(2\sqrt{6})h^{2(1)R}]P_2 \\
&\quad +[9h_4^{(1)R}+4h_{34}^{(1)R}+5h_{54}^{(1)R}+(2\sqrt{20})h^{4(1)R}]P_4, \\
(E/m)\varphi_2^{(1)} &= (E/m)\varphi_2^{(1)}+[-h_0^{(1)R}+h_{10}^{(1)R}]P_0+[-5h_2^{(1)R}+2h_{12}^{(1)R}+3h_{32}^{(1)R}+(2\sqrt{6})h^{2(1)R}]P_2 \\
&\quad +[-9h_4^{(1)R}+4h_{34}^{(1)R}+5h_{54}^{(1)R}+(2\sqrt{20})h^{4(1)R}]P_4, \\
(E/m)\varphi_3^{(1)} &= (E/m)\varphi_3^{(1)E}+[3h_{1,2}^{(1)R}+2h_{3,2}^{(1)R}-(2\sqrt{6})h^{2(1)R}]d_{11}^2 \\
&\quad +[5h_{3,4}^{(1)R}+4h_{4,5}^{(1)R}-(2\sqrt{20})h^{4(1)R}]d_{11}^4+3h_{11}^{(1)R}d_{11}^1+7h_{33}^{(1)R}d_{11}^3, \\
(E/m)\varphi_4^{(1)} &= (E/m)\varphi_4^{(1)E}+[3h_{1,2}^{(1)R}+2h_{3,2}^{(1)R}-(2\sqrt{6})h^{2(1)R}]d_{-11}^2 \\
&\quad +[5h_{3,4}^{(1)R}+4h_{5,4}^{(1)R}-(2\sqrt{20})h^{4(1)R}]d_{-11}^4-3h_{11}^{(1)R}d_{-11}^1-7h_{33}^{(1)}d_{11}^3, \\
\frac{1}{(1-z^2)^{1/2}}\varphi_5^{(1)} &= \frac{1}{(1-z^2)^{1/2}}\varphi_5^{(1)E}+\frac{m}{E}[(\sqrt{6})(h_{12}^{(1)R}-h_{32}^{(1)R})+h^{2(1)R}]\frac{d_{10}^2}{(1-z^2)^{1/2}} \\
&\quad +\frac{m}{E}[(\sqrt{20})(h_{34}^{(1)R}-h_{54}^{(1)R})+h^{4(1)R}]\frac{d_{10}^4}{(1-z^2)^{1/2}}.
\end{aligned}
\tag{39}
$$

All the arguments in (39) have been suppressed.

Coming now to the construction of the dispersion equations we consider first the four uncoupled $S$ and $P$ waves $h_0^{(1)}$, $h_{10}^{(1)}$, $h_{11}^{(1)}$, $h_1^{(0)}$. They satisfy similar equations so we use the common symbol $h$ for any one of them and we write

$$h(s)=N(s)/D(s)\,, \tag{40}$$

where $N(s)$ contains the singularities of the exchange terms and is regular for $s>4m^2$, $D(s)$ is regular for $s<4m^2$ but has a branch point at $s=4m^2$.

From Eq. (37) it is easily seen that

$$\mathrm{Im}(1/h)=-mp/E. \tag{41}$$

Hence

$$\mathrm{Im}D(s)\underset{s\geq 4m^2}{=}-(mp/E)N(s)\,,$$

$$D(s)=1-\frac{s-4m^2}{\pi}\int_{4m^2}^{+\infty}ds'\frac{mp'}{E'}\frac{N(s')}{(s'-4m^2)(s'-s)}. \tag{42}$$

Since the analytic properties of $N$ and $D$ remain unchanged if one multiplies both by a fixed constant, we choose to normalize $D=1$ at $s=4m^2$. The dispersion relation for $N(s)$ can now be written in the form

$$N(s)=h^E(s)D(s)-h^E(4m^2)-a_1-\frac{s-4m^2}{\pi}\times\int_{4m^2}^{+\infty}\frac{h^E(s')\,\mathrm{Im}D(s')}{(s'-4m^2)(s'-s)}ds'. \tag{43}$$

The dispersion integral simply removes the right-hand cut of the product $h^E(s)D(s)$ while leaving the singularities of the exchange term $h^E(s)$ unaffected. Both $a_1$ and $h^E(4m^2)$ are zero for the $P$-wave amplitudes. For the $S$-wave amplitude $a_1$ is the proton-proton scattering length $a_1^{pp}$ (the neutron-proton scattering length $a_1^{np}$ when we calculate the $I=1$ amplitude for neutron-proton scattering). Expressing $D(s)$ in (43) by means of (42) we obtain then the integral equation for $N(s)$

$$N(s)=h^E(s)-h^E(4m^2)-a_1+\frac{s-4m^2}{\pi}\times\int_{4m^2}^{+\infty}ds'\frac{[h^E(s')-h^E(s)]}{(s'-4m^2)(s'-s)}\frac{mp'}{E'}N(s'). \tag{44}$$

Since $h^E(s)$ and its derivative are bounded in the physical region $s\geq 4m^2$ and converge sufficiently rapidly at high energy, Eq. (44) is a Fredholm integral equation with a nonsingular kernel and has a unique solution. After solving Eq. (44) for $N(s)$, one can perform the integration in (42) to obtain $D(s)$. The amplitude $h(s)$ can then be evaluated

$$h(s)=\frac{1}{D(s)}\left[h^E(s)D(s)-h^E(4m^2)-a_1-\frac{s-4m^2}{\pi}\times\int_{4m^2}^{+\infty}\frac{ds'h^E(s')\,\mathrm{Im}D(s')}{(s'-4m^2)(s'-s)}\right]. \tag{45}$$

At this point, we should point out that Eq. (42) is not the only one that satisfies the unitarity condition (41). There is the well-known *CDD* ambiguity which amounts to the addition of poles in (42) corresponding to a set of zeros in the amplitude $h(s)$. However, these poles in $D(s)$ are also associated with a set of zeros in $D(s)$ on the unphysical sheet which are usually interpreted as additional elementary particles, in this case, of nucleon number 2. We will not introduce any such poles because we believe that there are no elementary particles of baryon number 2 in this sense.[17]

Let us now consider the coupled amplitudes $h_{01}^{(0)}$, $h^{1(0)}$, $h_{21}^{(0)}$. It was first shown by Bjorken and Nauenberg[18] that the $N/D$ method can be used for the coupled amplitudes if they are considered as elements of a symmetric matrix.

Let

$$M=\begin{pmatrix} h_{01} & h^1 \\ h^1 & h_{21}\end{pmatrix}=ND^{-1}=\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22}\end{pmatrix}\begin{pmatrix} D_{22} & -D_{12} \\ -D_{21} & D_{11}\end{pmatrix}\times\frac{1}{\|D\|}, \tag{46}$$

where we have suppressed the isospin index. The imaginary part of $M^{-1}$ is simply

$$\operatorname{Im}M^{-1}=-\frac{mp}{E}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \tag{47}$$

for $s\geq 4m^2$. From this we can write a set of dispersion relations for the $D$ functions,

$$D_{ij}(s)=\delta_{ij}-\frac{(s-4m^2)}{\pi}\int_{4m^2}^{+\infty}ds'\left(\frac{mp'}{E'}\right)\frac{N_{ij}(s')}{(s'-4m^2)(s'-s)}, \tag{48}$$

and a set of somewhat more involved dispersion relations for the $N$ functions

$$N_{11}(s)=\bar{h}_{0,1}{}^E(s)D_{11}(s)+h^{1E}(s)D_{21}(s)-\frac{(s-4m^2)}{\pi}\int_{4m^2}^{+\infty}ds'\frac{\bar{h}_{0,1}{}^E(s')\operatorname{Im}D_{11}(s')+h^{1E}(s')\operatorname{Im}D_{21}(s')}{(s'-4m^2)(s'-s)}, \tag{49}$$

$$N_{12}(s)=\bar{h}_{0,1}{}^E(s)D_{12}(s)+h^{1E}(s)D_{22}(s)-\frac{(s-4m^2)}{\pi}\int_{4m^2}^{+\infty}ds'\frac{\bar{h}_{0,1}{}^E(s')\operatorname{Im}D_{12}(s')\operatorname{Im}D_{22}(s')}{(s'-4m^2)(s'-s)}, \tag{50}$$

$$N_{21}(s)=h^{1E}(s)D_{11}(s)+\bar{h}_{2,1}{}^E(s)D_{21}(s)-\frac{(s-4m^2)}{\pi}\int_{4m^2}^{+\infty}ds'\frac{h^{1E}(s')\operatorname{Im}D_{11}(s')+\bar{h}_{2,1}{}^E(s')\operatorname{Im}D_{21}(s')}{(s'-4m^2)(s'-s)}, \tag{51}$$

$$N_{22}(s)=h^{1E}(s)D_{12}(s)+\bar{h}_{2,1}{}^E(s)D_{22}(s)-\frac{(s-4m^2)}{\pi}\int_{4m^2}^{+\infty}ds'\frac{h^{1E}(s')\operatorname{Im}D_{12}(s')+\bar{h}_{2,1}{}^E(s')\operatorname{Im}D_{22}(s')}{(s'-4m^2)(s'-s)}. \tag{52}$$

We obtain $\bar{h}_{0,1}{}^E(s)$ and $\bar{h}_{2,1}{}^E(s)$ from $h_{0,1}{}^E(s)$ and $h_{2,1}{}^E(s)$ by making the following modification:

$$\bar{h}_{0,1}{}^E(s)=h_{0,1}{}^E(s)+c_0,\quad \bar{h}_{2,1}{}^E(s)=h_{2,1}{}^E+[c_2(s-4m^2)/(s-s_0)]. \tag{53}$$

The constants $c_0$ and $c_2$ are adjusted to give the correct scattering length $a_3{}^{np}$ and $D$-wave threshold behavior, respectively. The additional singularity introduced in the $D$ wave is required, as discussed in Sec. II, by the threshold condition.

We substitute now the $D$ functions given by Eq. (48) into the dispersion relations for the $N$ functions and eliminate the constants $c_0$ and $c_2$ imposing the above mentioned conditions.

The following set of integral equations for the $N$ functions is then obtained:

$$N_{11}(s)=h_{0,1}{}^E(s)-a_3{}^{np}-h_{0,1}{}^E(4m^2)+\frac{(s-4m^2)}{\pi}\int_{4m^2}^{+\infty}ds'\left(\frac{mp'}{E'}\right)\times\frac{[h_{0,1}{}^E(s')-h_{0,1}{}^E(s')]N_{11}(s')+[h^{1E}(s')-h^{1E}(s)]N_{21}(s')}{(s'-4m^2)(s'-s)}, \tag{54}$$

[17] The deuteron appears as a bound-state pole in our calculation.
[18] J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960).

$$N_{21}(s)=h^{1E}(s)+\frac{(s-4m^2)}{\pi}\int_{4m^2}^{+\infty}ds'\left(\frac{mp'}{E'}\right)$$
$$\times\frac{[h^{1E}(s')-h^{1E}(s)]N_{11}(s')+[h_{2,1}{}^{E}(s')-h_{2,1}{}^{E}(s)+(c(s'-s)(-s_0+4m^2)/(s'-s_0)(s-s_0))]N_{21}(s')}{(s'-4m^2)(s'-s)}, \quad (55)$$

$$N_{12}(s)=h^{1E}(s)+\frac{(s-4m^2)}{\pi}\int_{4m^2}^{+\infty}ds'\left(\frac{mp'}{E'}\right)\frac{[h_{0,1}{}^{E}(s')-h_{0,1}{}^{E}(s)]N_{12}(s')+[h^{1E}(s')-h^{1E}(s)]N_{22}(s')}{(s'-4m^2)(s'-s)}, \quad (56)$$

$$N_{22}(s)=h_{2,1}{}^{E}(s)+\frac{(s-4m^2)^2}{\pi(s-s_0)}\int_{4m^2}^{+\infty}ds'\left(\frac{mp'}{E'}\right)$$
$$\times\left\{\frac{\left[\left(\frac{s'-s_0}{s'-4m^2}\right)h^{1E}(s')-\left(\frac{s-s_0}{s-4m^2}\right)h^{1E}(s)\right]N_{12}(s')}{(s'-4m^2)(s'-s)}+\frac{\left[\left(\frac{s'-s_0}{s'-4m^2}\right)h_{2,1}{}^{E}(s')-\left(\frac{s-s_0}{s-4m^2}\right)h_{2,1}{}^{E}(s)\right]N_{22}(s')}{(s'-4m^2)(s'-s)}\right\}, \quad (57)$$

where

$$c=\frac{4m^2-s_0}{\pi D_{22}s_0}\int_{4m^2}^{+\infty}ds'\left(\frac{mp'}{E'}\right)\frac{1}{(s'-4m^2)^2}[h^{1E}(s')N_{12}(s')+h_{2,1}{}^{E}(s')N_{22}(s')]. \quad (58)$$

We note that Eqs. (56) and (57) can be solved independently of (54) and (55). Having obtained $N_{12}(s)$ and $N_{22}(s)$, one evaluates the integral in (58) and solves (54) and (55) to obtain $N_{11}(s)$ and $N_{21}(s)$.

The remaining $P$-wave amplitude is $h_{1,2}{}^{(1)}$. Here one may neglect the term $|h^{2(1)}|^2$ in the imaginary part given by Eq. (37c) and obtain $h_{1,2}{}^{(1)}$ by the $N/D$ method as an uncoupled amplitude because the coupling to the triplet-$F$ amplitude is very weak. The coupling amplitude $h^{2(1)}$ can then be evaluated by a dispersion integral using the unitarity condition that $h^{2(1)}$ has the same phase as $h_{1,2}{}^{(1)}$ in the physical region (neglecting $h_{32}{}^{(1)}$).

$$h^{2(1)}(s)=\frac{1}{D(s)}\left[h^{2(1)E}(s)D(s)-\frac{(s-4m^2)^2}{\pi(s-s_0)}\int_{4m^2}^{\infty}\frac{(s'-s_0)h^{2(1)E}(s')\,\mathrm{Im}D(s')}{(s'-4m^2)^2(s'-s)}ds'\right], \quad (59)$$

where $D(s)$ is the denominator functions for $h_{1,2}{}^{(1)}$. Next we consider the two uncoupled $D$-wave amplitudes $h_{22}{}^{(0)}$ and $h_2{}^{(1)}$. They can be obtained, as the uncoupled $S$ and $P$ waves, by the $N/D$ method, the only difference being the introduction of an additional singularity required by the threshold condition as we have already seen for the coupled $D$-wave $h_{2,1}{}^{(0)}$. The integral equation for the $N$ function will then read as follows for these two $D$ waves:

$$N(s)=h^E(s)+\frac{(s-4m^2)^2}{\pi(s-s_0)}\int_{4m^2}^{+\infty}\frac{\left[\left(\frac{s'-s_0}{s'-4m^2}\right)h^E(s')-\left(\frac{s-s_0}{s-4m^2}\right)h^E(s)\right]N(s)}{(s'-4m^2)(s'-s)}ds', \quad (60)$$

where the subscript of $h^E$ and the isotopic spin index have been suppressed.

Finally, we consider the intermediate partial waves, which are the remaining partial waves appearing in (39): $h_{32}{}^{(1)}(s)$, $h_{33}{}^{(1)}(s)$, $h_4{}^{(1)}(s)$, $h_{34}{}^{(1)}(s)$, $h_{54}{}^{(1)}(s)$, $h^{4(1)}(s)$, $h_3{}^{(0)}(s)$, $h_{23}{}^{(0)}(s)$, $h_{43}{}^{(0)}(s)$, $h^{3(0)}(s)$, $h_{44}{}^{(0)}(s)$. For these waves we approximate the unitarity condition by considering only the contributions of the one-pion exchange term in the evaluation of the imaginary part by Eq. (37): if we write $h_l$ for $h_{lJ}{}^{(I)}$ or $h^{J(I)}$, where $l$ is the orbital angular momentum ($l=J$ for the coupling amplitudes), Eq. (8) of Sec. II shows explicitly the form of the dispersion integrals. Thus all the partial-wave amplitudes corresponding to $J\leq 4$, $l\leq 5$ can be calculated and the right-hand part

$$h^R(s)\equiv h(s)-h^E(s) \quad (61)$$

can be substituted into Eq. (39). This completes the calculation of the helicity amplitudes $\varphi_n{}^{(I)}(s,t)$.

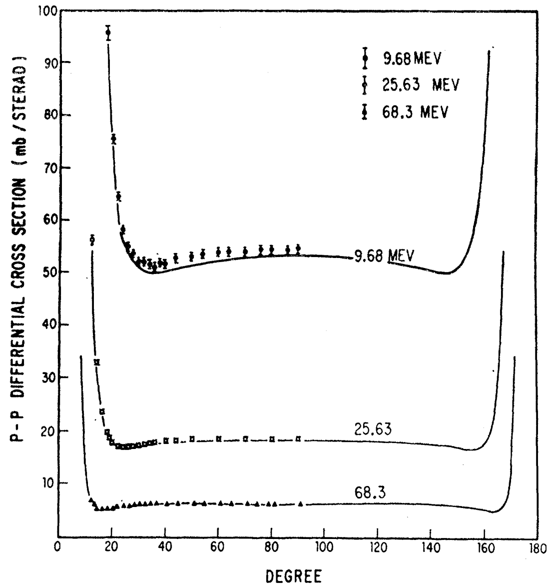


FIG. 1. *p-p* differential cross sections at 9.68, 25.63, and 68.3 MeV.

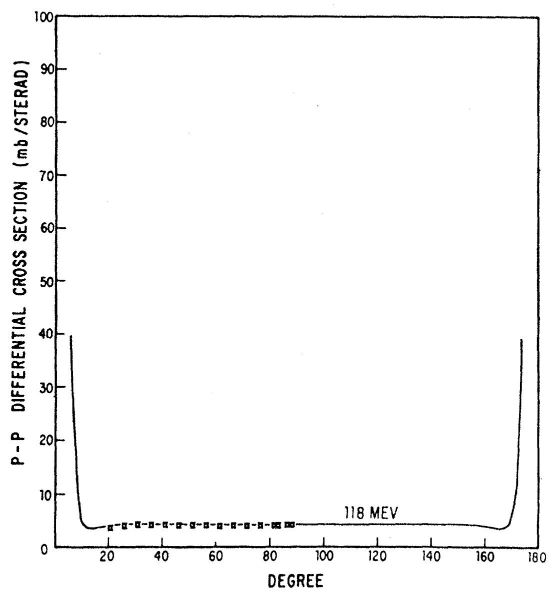


FIG. 2. *p-p* differential cross sections at 118 MeV.

## VI. RESULTS

To summarize, we have the following predetermined parameters:

$$a_{np}^{(I=0)}=5.4\text{ F}.$$
$$a_{np}^{(I=1)}=-23.74\text{ F}.$$
$$a_{pp}=-7.7\text{ F}.$$
$$g_\pi^2=14.$$
$$m_\pi=135.1\text{ MeV}.$$
$$m_\eta=548\text{ MeV}.$$
$$m_\omega=780\text{ MeV}.$$
$$m_\varphi=1020\text{ MeV}.$$

There are twelve adjustable parameters: $g_\eta$, $g_\omega$, $c_\omega$, $g_\varphi$ $c_\varphi$, $m_\rho$, $g_{\rho1}$, $g_{\rho2}$, $c_\rho$, $m_\sigma$, and $g_\sigma$ (or $a_\pi$ and $t_c$), and $s_0$.

For a given set of parameters, we calculated the *np* and *pp* partial-wave amplitudes and the direct observables $d\sigma/d\Omega$, $P$, $D$, $R$, and $A$ as described in Secs. III, IV, and V (see the Appendix for Coulomb correction). A total of 560 pieces of experimental *np* and *pp* data selected by the Livermore group[4] are used to find a set of parameters which give a best fit. In Figs. 1 to 12 we present a sample of our results. The "goodness-to-fit" parameter shows that we have an 8% fit using $m_\sigma$ and $g_\sigma$ (existence of $\sigma$) or $a_\pi$ and $t_c$ (nonexistence of $\sigma$).

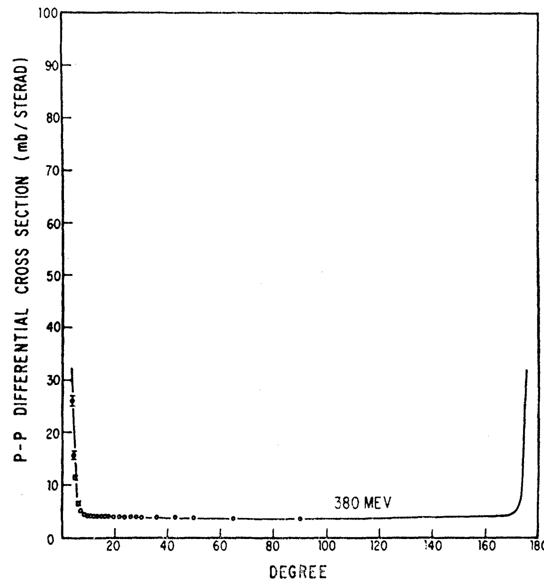


FIG. 3. *p-p* differential cross sections at 380 MeV.

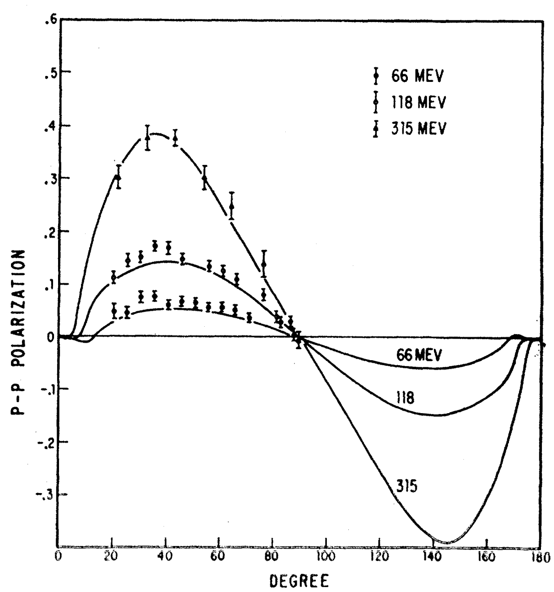


FIG. 4. *p-p* polarization at 66, 118, and 315 MeV.

Numerical values for the two cases are

(i) $m_\sigma = 437$ MeV, $g_\sigma^2 = 3.05$, $g_\eta^2 = 12.1$,
$g_\omega^2 = 2.77$, $c_\omega = 0.011\lambda_\pi^2$, $g_\varphi^2 = 2.26$,
$c_\varphi = 0.0048\lambda_\pi^2$, $m_\rho = 591$ MeV, $g_{\rho 1}^2 = 1.27$,
$g_{\rho 2}^2 = 11.4$, $c_\rho = 0.0052\lambda_\pi^2$, $s_0 = 168 m_\pi^2$;

(ii) $a_\pi = -1.0\lambda_\pi$, $t_c = 25 m_\pi^2$, $g_\eta^2 = 10.4$,
$g_\omega^2 = 2.77$, $c_\omega = 0.0028\lambda_\pi^2$, $g_\varphi^2 = 2.65$,
$c_\varphi = 0.0017\lambda_\pi^2$, $m_\rho = 591$ MeV, $g_{\rho 1}^2 = 1.27$,
$g_{\rho 2}^2 = 12.2$, $c_\rho = 0.0041\lambda_\pi^2$, $s_0 = 171 m_\pi^2$.

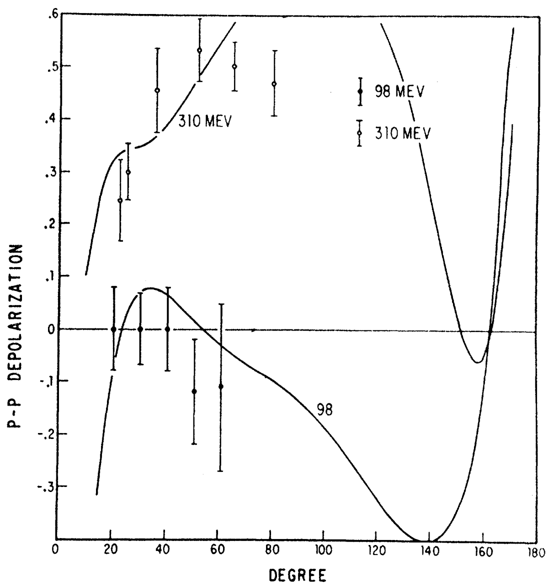


FIG. 5. $p$-$p$ depolarization at 98 and 310 MeV.

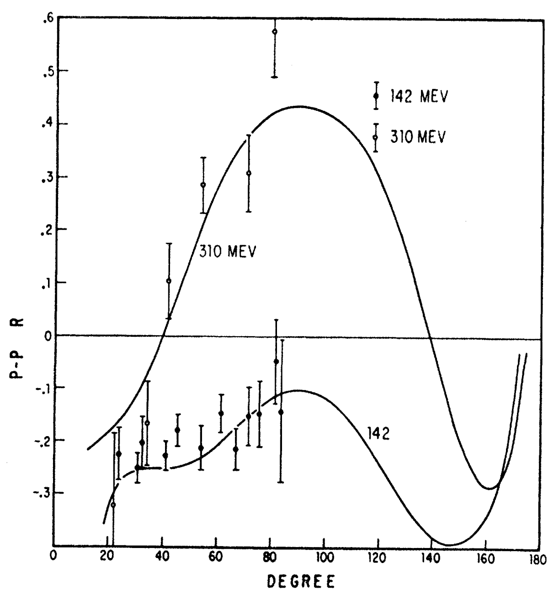


FIG. 6. $p$-$p$ $R$ parameter at 142 and 310 MeV.

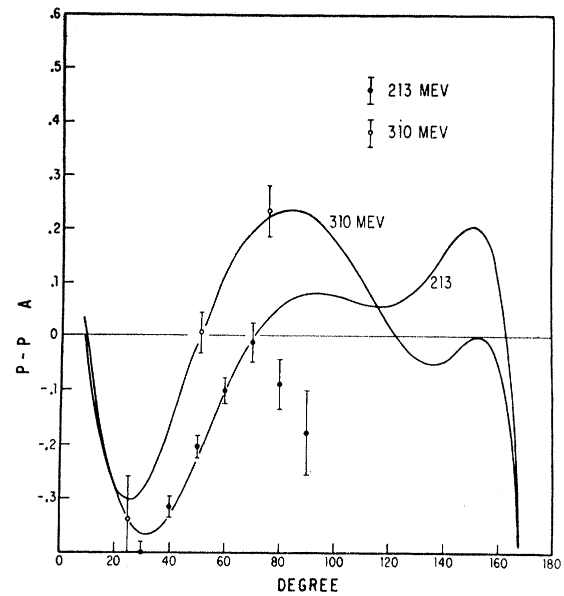


FIG. 7. $p$-$p$ $A$ parameter at 213 and 310 MeV.

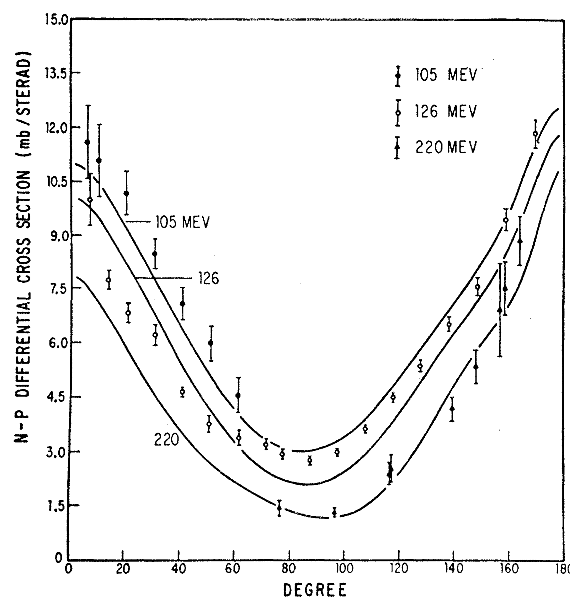


FIG. 8. $n$-$p$ differential cross sections at 105, 126, and 220 MeV.

As we mentioned before, the coupling constants within the two sets of parameters are not very different except, of course, the $S$-wave $\pi\pi$ parameters. These results give an indication that not only is the relativistic dispersion relation a practical tool for calculating scattering amplitudes, but some useful parameters such as the coupling constants can be determined this way reasonably well. Of course, one should not take the variation between sets (i) and (ii) as an absolute measure of the uncertainty of the parameters. The physical picture may be lacking, for example, in the neglect of correlated pair contribution with $\pi\pi$ relative angular

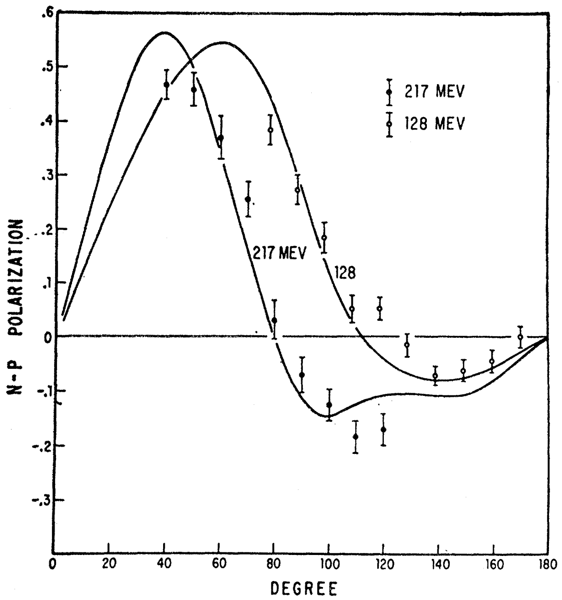


FIG. 9. $n$-$p$ polarization at 128 and 217 MeV.

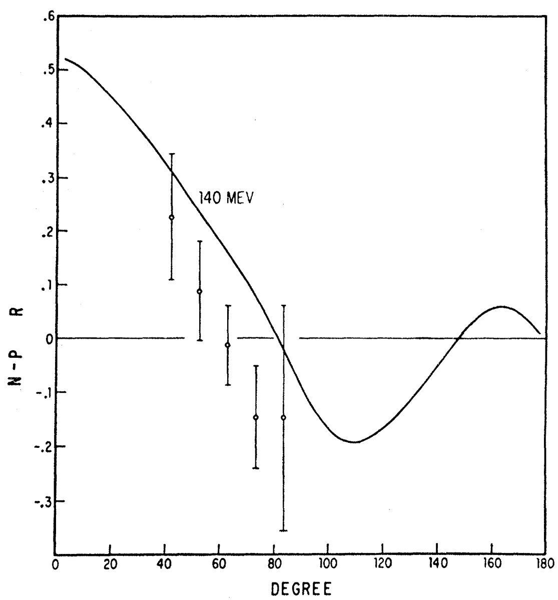


FIG. 10. $n$-$p$ depolarization at 128 MeV.

momentum $l \geq 2$ ($l \leq 1$ contributions are absorbed in the resonance parameters). Although such contributions are partially accounted for by fixing the observed scattering length and using a parameter $s_0$ for high partial waves, a certain amount of uncertainty is always present in such phenomenological fits.

Over-all speaking, our present parameterization does have considerably closer contact with elementary particle interactions than using, for instance, static potentials. On the other hand, one may raise the question of what else one can do with relativistic dispersion relations in nuclear physics. Unfortunately, the answer is that dispersion theory in its present form is not applicable to problems involving more than two particles. This severe limitation may force us to reconsider potentials with the Schrödinger equation when one is dealing, for example, with the nuclear many-body problem.

In view of the fact that the relativistic single-particle exchange term $h_l^E(s)$ plays a dominant role in the dispersion theory and that the additional requirement of unitarity condition is sufficient to yield partial-wave amplitudes that fit experimental data, it seems to us a natural way of constructing potentials is to choose the

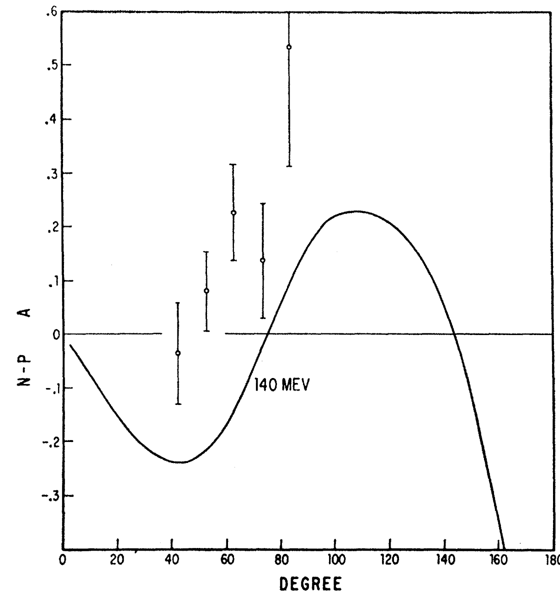


FIG. 11. $n$-$p$ $R$ parameter at 140 MeV.

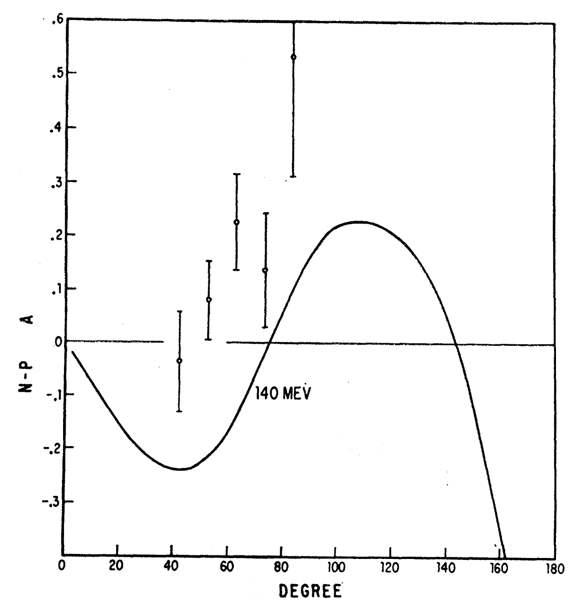


FIG. 12. $n$-$p$ $A$ parameter at 140 MeV.

one such that the Born approximation in the nonrelativistic Schrödinger theory reproduces the relativistic $h_l^E(s)$. Since solutions of the Schrödinger equation automatically satisfy the unitarity condition, the scattering amplitudes would probably be quite similar to solutions of relativistic dispersion relations, the main difference being that in the case of the Schrödinger equation, the solutions contain "genuine" multimeson branch cuts in the unphysical region. The medium-range part of these can be absorbed into the $\sigma$, $\rho$, and $\omega$ exchange terms while the short-range part is presumably shielded by the $\omega$ and $\varphi$ repulsion. One can show, however, that the potentials constructed from the relativistic Born approximation are necessarily velocity-dependent.[19] For example, the vector-meson exchange contribution to the central potential contains the term

$$U_c=g_\omega^2(1+2p^2/m^2)(e^{-m_\omega r}/r).$$

One can estimate the $(2p^2/m^2)$ term by replacing the operator $p^2$ by $k^2$. This yields a 33% variation between zero and 300 MeV.

With the full velocity dependence taken into account, one probably will find that fitting scattering data over the entire elastic scattering range ($\lesssim$400 MeV) will require a potential substantially different from existing phenomenological potentials. The application of these velocity dependent potentials to the many-body problem could give rather different results on saturation properties and binding energies.

Finally, returning to the connection between nucleon-nucleon scattering and elementary-particle interactions, it would be of considerable interest to find what values of the coupling constants are needed with the velocity-dependent potentials. The comparison between these values and our present result can give us an independent estimation of the coupling constants as deduced from nucleon-nucleon interaction.

## ACKNOWLEDGMENT

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## APPENDIX A: RELATIONS BETWEEN HELICITY AMPLITUDES AND AMPLITUDES OF STAPP, YPSILANTIS, AND METROPOLIS

The relation between the helicity amplitudes and the amplitudes using representation in the $z$ component of the initial and final spin[9] are

$$\varphi_1=\tfrac{1}{2}M_{ss}+\tfrac{1}{2}zM_{00}-(\surd\tfrac{1}{2})yM_{10}\,, \quad \text{(A1)}$$

$$\varphi_2=-\tfrac{1}{2}M_{ss}+\tfrac{1}{2}zM_{00}-(\surd\tfrac{1}{2})yM_{10}\,, \quad \text{(A2)}$$

$$\varphi_3=\tfrac{1}{2}(1+z)M_{00}+M_{1-1}+(\surd\tfrac{1}{2})(1+z)(z/y)M_{10} +(\surd\tfrac{1}{2})(1+z)/yM_{01}\,, \quad \text{(A3)}$$

$$\varphi_4=\tfrac{1}{2}(1-z)M_{00}+M_{1-1}+(\surd\tfrac{1}{2})(1-z)(z/y)M_{10} -(\surd\tfrac{1}{2})(1-z)/yM_{01}\,, \quad \text{(A4)}$$

$$\varphi_5=-\tfrac{1}{2}yM_{00}-(\surd\tfrac{1}{2})zM_{10}\,, \quad \text{(A5)}$$

where

$$z=\cos\theta\,,\quad y=\sin\theta\,.$$

The inverse of the set of equations above is

$$M_{ss}=\varphi_1-\varphi_2\,, \quad \text{(A6)}$$

$$M_{00}=z(\varphi_1+\varphi_2)-2y\varphi_5\,, \quad \text{(A7)}$$

$$M_{1-1}=\tfrac{1}{2}(\varphi_3+\varphi_4)+\tfrac{1}{2}z(\varphi_4-\varphi_3)+y\varphi_5\,, \quad \text{(A8)}$$

$$M_{10}=-(\surd\tfrac{1}{2})y(\varphi_3-\varphi_4)+\sqrt{2}z\varphi_5\,, \quad \text{(A9)}$$

$$M_{01}=(\surd\tfrac{1}{2})y(\varphi_3-\varphi_4)+\sqrt{2}z\varphi_5\,. \quad \text{(A10)}$$

Note: $M_{11}=M_{1-1}+M_{00}+\sqrt{2}(z/y)(M_{10}+M_{01})$.[9]

## APPENDIX B: COULOMB CORRECTION TO THE $N/D$ EQUATIONS

For $p$-$p$ scattering, the static Coulomb correction to the $N/D$ equations was discussed at great length by Noyes and Wong.[20] We give here a simplified version of their result. Firstly, we mention that when we evaluate the $p$-$p$ observables $\sigma$, $P$, $D$, $R$, $A$ in terms of helicity amplitudes, appropriate Coulomb functions should be added according to the formula given, for example, by Stapp, Ypsilantis, and Metropolis.[9] Secondly, our partial-wave projections of the helicity amplitudes should now be expressed in terms of nuclear-bar phase shifts with Coulomb contributions:

$$h_J=(E/2imp)[\exp(2i\bar{\delta}_J{}^N)-1]e^{2i\Phi_J}\,, \quad \text{(B1)}$$

$$h_{JJ}=(E/2imp)[\exp(2i\bar{\delta}_{JJ}{}^N)-1]e^{2i\Phi_J}\,, \quad \text{(B2)}$$

$$h_{J\pm1,J}=(E/2imp)[\cos 2\bar{\epsilon}_J{}^N\exp(2i\bar{\delta}_{J\pm1,J}{}^N)-1] \times\exp(2i\Phi_{J\pm1})\,, \quad \text{(B3)}$$

$$h^J=(E/2imp)\sin 2\bar{\epsilon}_J{}^N\exp[i(\bar{\delta}_{J+1,J}{}^N+\bar{\delta}_{J-1,J}{}^N)] \times\exp[i(\Phi_{J+1}+\Phi_{J-1})]\,, \quad \text{(B4)}$$

Thirdly, the $h$'s do not satisfy an $N/D$ equation, but we can construct some other functions of the nuclear-bar phase shifts which do satisfy an $N/D$ equation.

The work of Noyes and Wong starts with the observation that the function

$$S_J(p^2)=\prod_{\lambda=1}^{J}(1+\eta^2/\lambda^2)p^{2J}(C^2p\cot\bar{\delta}_J{}^N+Q) \quad \text{(B5)}$$

is analytic at $p^2=0$ for the singlet and the uncoupled triplet amplitudes. Here,

$$\eta=(me^2/2p)\,, \quad \text{(B6)}$$

$$C^2=2\pi\eta/[\exp(2\pi\eta)-1]\,, \quad \text{(B7)}$$

$$Q=me^2[\tfrac{1}{2}\psi(i\eta)+\tfrac{1}{2}\psi(-i\eta)-\ln\eta]\,, \quad \text{(B8)}$$

$$\psi(\pm i\eta)=\Gamma'(\pm i\eta)/\Gamma(\pm i\eta)\,. \quad \text{(B9)}$$

[19] D. Y. Wong, Nucl. Phys. **55**, 212 (1964). For a review of earlier work with potentials, see M. J. Moravcsik and H. P. Noyes, Ann. Rev. Nucl. Sci. **11**, 95 (1961), R. Cirelli and G. Stabilim, Suppl. Nuovo Cimento **20**, 157 (1961); R. Cirelli and G. M. Prosperi (to be published).

[20] D. Y. Wong and H. P. Noyes, Phys. Rev. **126**, 1866 (1962).

The basic problem is the construction of the nuclear-bar phase shift which contains a given branch cut below $p^2=-(m_\pi^2/4)$, satisfies the unitarity condition and has no other singularity except for the essential singularity at $p^2=0$ which can be removed according to (B5).

Noyes and Wong used these properties to construct an integral equation for a function $E_l(p^2)$ and obtain $S_l(p^2)$ in terms of $E_l(p^2)$. We will show, however, that the usual $N/D$ equations can be used for the function

$$\bar{h}_J=(e^{-2i\Phi_J}h_J)/\Big[C^2p^{2J}\prod_{\lambda=1}^{J}(1+\eta/\lambda^2)\Big] \qquad \text{(B10)}$$

and similarly for the uncoupled-triplet amplitude $h_{JJ}$.

Let us write

$$\bar{h}_J=\bar{N}/\bar{D}, \qquad \text{(B11)}$$

$$\bar{N}(s)=\bar{B}(s)\bar{D}(s)-\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{\bar{B}(s')\,\mathrm{Im}\bar{D}(s')}{s'-s}, \qquad \text{(B12)}$$

where $\bar{B}(s)$ is the integral over the left-hand branch cut of $\bar{h}_J$ from $p^2=-(m_\pi^2/4)$ to $-\infty$. It is clear that the Coulomb modifications causes little complication on this branch cut since $\exp(-2i\bar{\delta}_J)$, $C^2$, and $\eta^2$ are all real in this region. In fact, the one-pion exchange term for $\bar{B}(s)$ was explicitly given by Noyes and Wong. The essential question lies in the construction of $\bar{D}$.

The unitarity condition gives

$$\mathrm{Im}\bar{D}=-(2mp/E)C^2p^{2J}\Big[\prod_{\lambda=1}^{J}(1+\eta^2/\lambda^2)\Big]\bar{N};\qquad s\geq 4m^2. \qquad \text{(B13)}$$

The function $\bar{D}$ will satisfy the usual dispersion relation

$$\bar{D}(s)=1-\frac{s-4m^2}{\pi}\int_{4m^2}^{\infty}ds'\frac{\mathrm{Im}\bar{D}(s')}{(s'-4m^2)(s'-s)}, \qquad \text{(B14)}$$

if

$$S_J(p^2)\equiv(E\bar{D}/m\bar{N})+p^{2J}(iC^2p+Q)\times\Big[\prod_{\lambda=1}^{J}(1+\eta^2/\lambda^2)\Big] \qquad \text{(B15)}$$

is analytic at $p^2=0$.

From the expression (B7), one sees that $C^2$ has a pole at all integral values of $i\eta$ and hence an essential singularity in $p$. On the other hand, the function $(iC^2p+Q)$ has an accumulation of poles only on the unphysical sheet $\mathrm{Im}p\leq 0$ ($i\eta=$negative integers). It is clear that $(E\bar{D}/m\bar{N})$ also does not have an accumulation of poles on the physical sheet. The remaining question is whether $(E\bar{D}/m\bar{N})$ has an accumulation of poles on the unphysical sheet which precisely cancels the poles in the second term of (B15). This indeed is the case as can be seen by examining $\mathrm{Im}\bar{D}$ given by Eq. (B13). Each pair of poles in $C^2$ at $i\eta=\pm$integer gives rise to a single pole at $i\eta=-$integer after performing the integral (B14). The residue is just such that the cancellation takes place. Thus we have completed the Coulomb modification for the singlet and the uncoupled-triplet amplitudes. As we mentioned above, $\bar{B}(s)$ can be evaluated explicitly for the one-pion exchange term which differs from the $n$-$p$ case by approximately 3%. Corrections for the higher mass states are correspondingly smaller and will be neglected.

For the triplet-coupled amplitudes, the lowest angular momentum state which enters into $p$-$p$ scattering is $^3P_2$ and $^3F_2$ coupled by $\epsilon_2$. Since we have already made an approximation in the unitarity condition for these amplitudes, it is reasonable to treat the $^3P_2$ amplitude as uncoupled when we introduce the Coulomb correction. Coulomb corrections of the $N/D$ type can be ignored altogether for the $F$ and higher partial waves.

Finally, we remark that the method outlined above is used only for the calculation of the energy variation of the phase shifts (the scattering length being taken from experiment). The uncertainty of the Coulomb modification on the left-hand branch cut will not cause a large variation of our results since it is only a nonsingular effect of the order 3%.