# Minimum Principle for Effective Potentials*

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In the familiar Rayleigh-Ritz method, a finite-dimensional Hamiltonian matrix is constructed using a set of linearly independent trial functions. The eigenvalues of this matrix, suitably ordered, are guaranteed to lie above the corresponding true eigenvalues. An analogous situation is shown to hold in scattering theory for energies low enough so that only two-body channels are open. An effective Hamiltonian, which is a matrix in the open-channel subspace, is constructed variationally. Since the corresponding Schrödinger equation is of the two-body type, it may be solved, numerically if necessary. The eigenvalues of the reaction matrix, in a partial-wave representation, are then guaranteed to lie below the true eigenvalues. The essential point involved is that the error in the effective Hamiltonian can be shown to be a nonpositive operator, provided the trial function satisfies a certain natural constraint. Effects due to the identity of particles are easily accounted for. As an application, the three-body problem is considered in some detail, and the required constraint on the trial function is explicitly defined for this problem in terms of simple orthogonality conditions.

## 1. INTRODUCTION

In previous discussions,[1,2] confined to the three-body problem, we have shown that with the aid of certain generalized impulse approximations the elastic scattering amplitude may be determined by a two-body Lippmann-Schwinger integral equation with a relatively simple effective (or "optical") potential. A more systematic approach to this problem is taken in the present paper, where it is shown that approximations to the true effective potential can be constructed with the aid of a minimum principle of the familiar Rayleigh-Ritz type. The calculation requires evaluation of matrix elements of the Hamiltonian with trial functions which must satisfy certain subsidiary orthogonality constraints along with the appropriate boundary conditions. The trial functions may be systematically improved with the aid of the rigorous validity criterion that the true eigenvalues of the reaction matrix lie above the computed values, obtained by solving the Lippmann-Schwinger equation with the approximate effective potential. The presence of identical particles causes no difficulties. Indeed, we will see that when all three particles are identical the problem simplifies considerably.

While the above prescription can be derived directly by manipulating the three-body integral equations (see Sec. 3B for the derivation in the case of three identical particles), it is simpler to make use of the projection-operator formalism of Feshbach.[3] In fact, with the aid of this abstract technique, the minimum principle can be established for arbitrarily complicated scattering systems, provided the energy is low enough so that only two-body channels are open. It should be stated, however, that practical applications might prove to be quite difficult in general. There are, of course, the usual complexities associated with the many-body nature of the problem. In addition, the formalism requires the construction of trial functions which have no projection onto the open-channel subspace. Since the channel wave functions are not, in general, orthogonal, it is not a simple matter to construct the projection operator. In the particular case of *three*-body scattering, however, this formal obstacle may be removed. Examination of the Faddeev[4] representation of the three-body wave function immediately reveals the form of the orthogonality constraints to be placed on the trial function. Since these are quite simple, the variational construction of the effective Hamiltonian should then be no more difficult in practice than an ordinary Rayleigh-Ritz bound-state calculation.

We have emphasized the desirable practical features of the formalism, since minimum principles for multichannel scattering have already been proposed. Hahn, O'Malley, and Spruch[5] have given a generalization and improvement of an earlier version suggested by Rosenberg and Spruch.[6] Another formulation has been given more recently by Sugar and Blankenbecler.[7] In fact, a long history of the development of minimum principles in scattering theory can be traced (references can be found in the papers cited above) to the early work of Spruch and Rosenberg.[8] The challenge has always been of a twofold nature. Of course, a method which works in principle is sought. In addition, and this is crucial when the scattering systems are compound, it should be as simple as possible to use in practice. It is the purpose of this article to point out a different approach which, we hope, will add to the power and flexibility of the minimum principle as a calculational tool. For example, it is

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[1] L. Rosenberg, Phys. Rev. **131**, 874 (1963).

[2] L. Rosenberg, Phys. Rev. **135**, B715 (1964).

[3] H. Feshbach, Ann. Phys. **5**, 357 (1958); **19**, 287 (1962).

[4] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[5] Y. Hahn, T. F. O'Malley, and L. Spruch, Phys. Rev. **134**, B911 (1964).

[6] L. Rosenberg and L. Spruch, Phys. Rev. **125**, 1407 (1962).

[7] R. Sugar and R. Blankenbecler, Phys. Rev. **136**, B472 (1964). See also, M. Gaĭlitis, Zh. Eksperim. i Teor. Fiz. **47**, 160 (1964) [English transl.: Soviet Phys.—JETP **20**, 107 (1965)].

[8] L. Spruch and L. Rosenberg, Phys. Rev. **116**, 1034 (1959). Even earlier, Kato had shown how to obtain upper and lower bounds on phase shifts for scattering by static potentials [T. Kato, Progr. Theoret. Phys. (Kyoto) **6**, 394 (1951)]. This method, however, is not suitable in the more interesting many-body problems.

now quite feasible to obtain variational lower bounds on the phase shifts for neutron-deuteron scattering below the inelastic threshold, and work on this problem is in progress.

For the case of three identical particles, a second development of an effective-potential formalism is given (see Sec. 3), in which separable potentials are used rather than projection operators. This generalizes previous work[9,10] in which only the leading term of the effective potential was retained. Here again, we obtain a minimum principle for the effective potential.

## 2. FORMAL DEVELOPMENT

Feshbach[3] has shown that instead of directly solving collision problems of the type A+B → C+D, defined by the Schrödinger equation

$$(H-E)\Psi=0, \tag{2.1}$$

along with the usual boundary conditions, we may equivalently attempt to construct an effective Hamiltonian $\mathcal{H}(E)$. The scattering parameters may then be deduced from the solution of the *two*-body Schrödinger equation

$$(\mathcal{H}-E)P\Psi\equiv\{P(H-E)P+PHQG^QQHP\}P\Psi=0. \tag{2.2}$$

Here $P$ and $Q$ are projection operators satisfying $P+Q=1$ (so that $PQ=0$), where $P$ projects onto the open-channel subspace. The Green's function $G^Q$ is related to the closed-channel Hamiltonian $QHQ$ by

$$G^Q(E)=[Q(E-H)Q]^{-1}; \tag{2.3}$$

it evidently takes into account intermediate states of the scattering process in which the colliding systems are in any of those excited states which make up the closed-channel subspace. We assume that $E$ lies below the threshold $E_c$ of the continuous spectrum of $QHQ$. The usual infinitesimal imaginary addition to $E$ may therefore be omitted in Eq. (2.3). Furthermore, since the functions in $Q$ space vanish asymptotically, we may treat the Hamiltonian as a Hermitian operator in this space.

It is our object to obtain an identity of the form

$$\mathcal{H}(E)-E=\mathcal{H}_t(E)-E+\Delta[\mathcal{H}(E)-E], \tag{2.4}$$

where $\mathcal{H}_t$ is some variational estimate of the effective Hamiltonian.[11] We first note that Eq. (2.2) may be written as

$$\mathcal{H}-E=P(H-E)\omega, \tag{2.5}$$

with

$$\omega\equiv P+QG^QQHP. \tag{2.6}$$

A basic relation satisfied by $\omega$ is

$$Q(H-E)\omega\equiv QHP+Q(H-E)QG^QQHP =0, \tag{2.7}$$

which follows from Eq. (2.3) in the form $Q(H-E)QG^QQ=-Q$. Then, since $\omega^\dagger-P=PHQG^QQ$, Eq. (2.5) is equivalent to

$$\mathcal{H}-E=\omega^\dagger(H-E)\omega. \tag{2.8}$$

We now introduce

$$\omega_t=P+QG_t^QQHP, \tag{2.9}$$

where $G_t^Q$ is a trial Green's function. Accordingly, the error $\Delta\omega=\omega_t-\omega$ is in "$Q$ space", i.e., $Q\Delta\omega=\Delta\omega$. From this property, along with Eq. (2.7), we see that terms linear in $\Delta\omega$ drop out of Eq. (2.8), which then takes the desired form

$$\mathcal{H}-E=\omega_t^\dagger(H-E)\omega_t-(\Delta\omega)^\dagger(H-E)\Delta\omega. \tag{2.10}$$

The fact that the error operator

$$\Delta(\mathcal{H}-E)=-(\Delta\omega)^\dagger(H-E)\Delta\omega =-(\Delta\omega)^\dagger Q(H-E)Q\Delta\omega \tag{2.11}$$

is of second order implies that

$$\mathcal{H}_t-E=\omega_t^\dagger(H-E)\omega_t \tag{2.12}$$

is indeed a variational expression. To elevate this variational principle to the status of a minimum principle, we need only observe that the sign of diagonal matrix elements of the error operator may be determined from a knowledge of the spectrum of $QHQ$. In particular, if $QHQ$ has no discrete eigenvalues below $E_c$, then $Q(H-E)Q$ is a positive operator (recall $E<E_c$), and the error operator is negative. If discrete states do exist, they may be subtracted out, using a technique which has been described previously.[12] This technique is based on a theorem which may be stated in the context of the present problem as follows: Suppose $QHQ$ has $M$ discrete eigenvalues, and suppose further that we have been able to construct $M$ linearly independent trial eigenfunctions $\varphi_{it}$ which belong to $Q$ space and which are accurate enough so that the matrix $\mathbf{D}$ with elements

$$D_{ij}=(\varphi_{it},[H-E]\varphi_{jt}) \tag{2.13}$$

is negative. Then

$$\hat{H}-E\equiv H-E-\sum_{i,j=1}^{M}|(H-E)\varphi_{it}\rangle \times(\mathbf{D}^{-1})_{ij}\langle\varphi_{jt}(H-E)| \tag{2.14}$$

is non-negative in $Q$ space. With the aid of Eq. (2.7), we may write Eq. (2.10) as

$$\mathcal{H}(E)-E=\omega_t^\dagger(\hat{H}-E)\omega_t-(\Delta\omega)^\dagger(\hat{H}-E)\Delta\omega, \tag{2.15}$$

[9] L. Rosenberg, Phys. Rev. **134**, B937 (1964).
[10] R. D. Amado, Phys. Rev. **132**, 485 (1963).
[11] The following discussion, leading to Eq. (2.15), may be viewed as an alternative route to the minimum principle of Ref. 7. The connection is made more apparent by choosing a trial function, in the present version, in the form of a linear combination of functions with the linear parameters then evaluated variationally.
[12] L. Rosenberg, L. Spruch, and T. F. O'Malley, Phys. Rev. **118**, 184 (1960).

so that even when $QHQ$ does have discrete eigenvalues the error in the variational estimate of $\mathcal{H}-E$ may be displayed as a negative operator. The requirement of the minimum principle that we subtract out the discrete states in Eq. (2.14) has led us at the same time to explicitly introduce into the variational expression for the effective Hamiltonian those separable interactions which account for the resonances[3,13]; Eq. (2.12) is to be replaced by

$$\mathcal{H}_t-E=\omega_t^\dagger(\hat{H}-E)\omega_t. \tag{2.16}$$

The most direct way to make use of the fact that the error operator $\Delta(\mathcal{H}-E)$ is negative is to couple it with the monotonicity theorem[14] for the partial-wave projections $K_{\alpha\beta}$ of the reaction matrix elements. (For single-channel scattering, $K$ is just the tangent of the phase shift.) If we define $\mathbf{K}_t$ as the reactionmatri x deduced from Eq. (2.2) with $\mathcal{H}$ replaced by $\mathcal{H}_t$, then the theorem states that

$$\mathbf{a}^\dagger\mathbf{K}\mathbf{a}\geq\mathbf{a}^\dagger\mathbf{K}_t\mathbf{a}, \tag{2.17}$$

where $\mathbf{a}$ is an aribtrary column vector in the space of open-channel indices. (We have constructed a quadratic form for $\mathbf{K}$ in order to introduce a *diagonal* matrix element of the error operator, which is then of known sign.)

In considering variational estimates for $\omega$, the simplest first guess is $\omega_t=P$ (corresponding to $G_t^Q=0$). This choice leads to the method of resonating group structure introduced by Wheeler[15] (and often referred to subsequently as the static or close-coupling approximation) for the determination of $\mathbf{K}_t$. It was shown by Hahn *et al.*,[16] and is apparent from the present discussion as well, that, if no discrete states of $QHQ$ exist below $E$, then the inequality

$$\mathbf{a}^\dagger\mathbf{K}\mathbf{a}\geq\mathbf{a}^\dagger\mathbf{K}_S\mathbf{a} \tag{2.18}$$

holds; here $\mathbf{K}_S$ is the reaction matrix in the static approximation. Furthermore, even when such discrete states do exist, the eigenphase shifts in the static approximation provide lower bounds on the true eigenphases.[17] The utility of this latter result is somewhat diminished by the fact that when resonances are present the bound obtained will generally differ from the true value by more than $\pi$. It is clearly desirable to be able to keep track of the branch of the tangent curve on which the phase shift lies; it will, in fact, be possible provided one has some additional information. Specifically, we state, as an extension of the above theorem, that the inequality of Eq. (2.18) will be valid even when resonances are present provided that all the resonances are accounted for in the static approximation.[18] To see this, consider a sequence of projection operators $\{P^{(m)}\}$, and the related sequence $\{Q^{(m)}\}$, defined such that the set of open channels associated with $P^{(m)}$ is contained in the set associated with $P^{(m+1)}$. If in the present formulation of the minimum principle we choose $\omega_t=P^{(m)}$, then the modified static problem, defined by

$$P^{(m)}(\hat{H}-E)P^{(m)}\hat{\Psi}_{S_m}=0 \tag{2.19}$$

(this differs from the usual form by the replacement $H\rightarrow\hat{H}$), leads to a reaction matrix $\hat{\mathbf{K}}_{S_m}$ which satisfies

$$\mathbf{a}^\dagger\mathbf{K}\mathbf{a}\geq\mathbf{a}^\dagger\hat{\mathbf{K}}_{S_m}\mathbf{a}. \tag{2.20}$$

This is subject to the usual conditions on the trial bound-state functions $\varphi_{it}$ which appear in $\hat{H}$ [see Eq. (2.14)]; we require, in particular, that $P^{(m)}\varphi_{it}=0$. The inequality will remain valid if we repeat the calculation with the same set of functions $\varphi_{it}$ but with $\omega_t$ chosen as $P^{(n)}$, $n>m$. Now suppose that[19] $Q^{(n)}\varphi_{it}=0$. Then, since the $\varphi_{it}$ are in $P^{(n)}$ space,

$$\langle\varphi_{it}(H-E)P^{(n)}|\Psi_{S_n}=0, \tag{2.21}$$

so that the distinction between $H$ and $\hat{H}$ may be ignored in the solution of the modified static problem. This immediately gives the desired inequality

$$\mathbf{a}^\dagger\mathbf{K}\mathbf{a}\geq\mathbf{a}^\dagger\mathbf{K}_{S_n}\mathbf{a}. \tag{2.22}$$

It is possible, and often convenient, to cast Eq. (2.2) into the form of a set of coupled Lippmann-Schwinger integral equations for the $T$-matrix or $K$-matrix elements. For example, the $T$ matrix (in the center-of-mass system) can be found, once the effective potential is known, by solving the set of equations

$$T_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)=V_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)+\sum_\gamma{}^{(P)}\int d\mathbf{k}_\gamma V_{\alpha\gamma}(\mathbf{k}_\alpha,\mathbf{k}_\gamma;E)\times\frac{1}{E+i\eta-(\hbar^2k_\gamma^2/2\mu_\gamma)+\epsilon_\gamma}T_{\gamma\beta}(\mathbf{k}_\gamma,\mathbf{k}_\beta;E). \tag{2.23}$$

Here all channels $\alpha$, $\beta$, and $\gamma$ which appear explicitly are of the two-body type associated with $P$ space; the many-body intermediate states are accounted for in the structure of the effective potential matrix $\mathbf{V}$. The sum of the binding energies of the two composite systems

[13] S. Weinberg, Phys. Rev. **130**, 776 (1963); **131**, 440 (1963); **133**, B232 (1964).

[14] The theorem follows directly from the $K$-matrix version of Eq. (2.35) below; one applies the comparison theorem, Eq. (2.34), with $G_A=G_B$ and $V_A-V_B$ an infinitesimal matrix of definite sign. Actually, the bound obtained should be stated in terms of the eigenphases rather than the reaction matrix since the latter changes discontinuously as a new resonance is introduced.

[15] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937).

[16] Y. Hahn, T. F. O'Malley, and L. Spruch, Phys. Rev. **128**, 932 (1962).

[17] Y. Hahn, T. F. O'Malley, and L. Spruch, Phys. Rev. **134**, B397 (1964).

[18] This generalizes a theorem obtained previously for zero-energy scattering; it states that the static approximation provides an upper bound on the scattering length, provided the Hamiltonian in the static approximation has the same number of negative-energy bound states as the true Hamiltonian. See L. Spruch and L. Rosenberg, Nucl. Phys. **17**, 30 (1960).

[19] Of course, with $P$ space limited to two-body channels, it may be impossible in principle to account for all the resonances in the static approximation merely by enlarging $P$ space sufficiently. In such cases, the theorem will not apply.

which scatter in channel $\gamma$ is denoted by $\epsilon_\gamma$ ($\epsilon_\gamma>0$), and $\mu_\gamma$ is the reduced mass of the two systems. The $K$-matrix elements are determined by equations of the same form, with the principal value prescription used to avoid the singularities in the integrands. The matrix $\mathbf{V}$ is defined by

$$V_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)=(\Phi_\alpha,[\mathfrak{IC}(E)-E]\Phi_\beta), \quad (2.24)$$

where $\Phi_\alpha$ is the unperturbed wave function in channel $\alpha$, i.e.,

$$\Phi_\alpha=(2\pi)^{-3/2}X_\alpha\exp(i\mathbf{k}_\alpha\cdot\mathbf{r}_\alpha); \quad (2.25)$$

$X_\alpha$ is the product of the bound-state wave functions of the two composite systems. The wave equation for $\Phi_\alpha$ is

$$(H-E)\Phi_\alpha=V_\alpha\Phi_\alpha, \quad (2.26)$$

where $V_\alpha$ is the interaction potential which acts between the two composite systems and which vanishes for $r_\alpha\to\infty$. According to Eqs. (2.6) and (2.8), we also have

$$\begin{aligned}V_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)&=(\Psi_\alpha{}^Q,[H-E]\Psi_\beta{}^Q)\\&=(\Psi_\alpha{}^Q,[\hat{H}-E]\Psi_\beta{}^Q),\end{aligned} \quad (2.27)$$

with

$$\Psi_\alpha{}^Q=\Phi_\alpha+QG^QQHP\Phi_\alpha. \quad (2.28)$$

The replacement of $H$ with $\hat{H}$ in Eq. (2.27) is justified by Eq. (2.7). It is easy to show that Eq. (2.24) is equivalent to the definition

$$V_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)=(\Phi_\alpha,V_{\alpha\beta}(E)\Phi_\beta), \quad (2.29)$$

where the effective potential operator $V_{\alpha\beta}(E)$ is given by

$$V_{\alpha\beta}(E)=V_\alpha+V_\alpha G^Q(E)V_\beta. \quad (2.30)$$

For comparison, we recall that

$$T_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)=(\Phi_\alpha,T_{\alpha\beta}(E)\Phi_\beta), \quad (2.31)$$

with

$$T_{\alpha\beta}(E)=V_\alpha+V_\alpha G(E)V_\beta. \quad (2.32)$$

It is worthwhile to remark that while we have assumed that $E<E_c$, so that the channels in $Q$ space are closed, Eq. (2.23) is still valid at higher energies. We need only continue Eq. (2.30) in energy by adding to $E$ a small positive imaginary part which eventually tends to zero. In this somewhat broader context, independent of the minimum principle, we wish to emphasize the utility of Eq. (2.23) as the basis for approximation procedures. We first show that these coupled integral equations give rise to unitary scattering amplitudes. This result follows directly from a very useful comparison identity which was derived previously[2] and will be used several times in the present paper. Suppose $T_A$ and $T_B$ satisfy

$$\begin{aligned}T_A&=V_A+V_AG_AT_A\equiv V_A\Omega_A,\\T_B&=V_B+V_BG_BT_B\equiv V_B\Omega_B.\end{aligned} \quad (2.33)$$

The identity

$$\begin{aligned}T_A{}^{(+)}=T_B{}^{\dagger(-)}&+\Omega_B{}^{\dagger(-)}(V_A{}^{(+)}-V_B{}^{\dagger(-)})\Omega_A{}^{(+)}\\&+T_B{}^{\dagger(-)}(G_A{}^{(+)}-G_B{}^{\dagger(-)})T_A{}^{(+)}\end{aligned} \quad (2.34)$$

then holds. To make use of this relation, we write Eq. (2.23) in the matrix-operator form

$$\mathbf{T}(E)=\mathbf{V}(E)+\mathbf{V}(E)\mathbf{g}^{(P)}(E)\mathbf{T}(E), \quad (2.35)$$

where $(\mathbf{V})_{\alpha\beta}$ and $(\mathbf{T})_{\alpha\beta}$ are given by Eqs. (2.30) and (2.32), respectively, and $\mathbf{g}^{(P)}$ is a diagonal matrix describing propagation in intermediate states in $P$ space. If we denote the analogous $Q$-space propagator by $\mathbf{g}^{(Q)}$, then Eq. (2.30) may be shown to imply the "unitarity" relation

$$\mathbf{V}-\mathbf{V}^\dagger=\mathbf{V}^\dagger(\mathbf{g}^{(Q)}-\mathbf{g}^{(Q)\dagger})\mathbf{V}, \quad (2.36)$$

with $(\mathbf{V}^\dagger)_{\alpha\beta}\equiv(\mathbf{V})_{\beta\alpha}{}^\dagger$. If, in Eq. (2.34), we set

$$\begin{aligned}&V_A{}^{(+)}=V_B{}^{(-)}=\mathbf{V}(E);\ G_A{}^{(+)}=G_B{}^{(-)}=\mathbf{g}^{(P)}(E);\\&T_A{}^{(+)}=T_B{}^{(-)}=\mathbf{T}(E),\end{aligned} \quad (2.37)$$

we obtain

$$\mathbf{T}-\mathbf{T}^\dagger=\mathbf{\Omega}^\dagger\mathbf{V}^\dagger(\mathbf{g}^{(Q)}-\mathbf{g}^{(Q)\dagger})\mathbf{V}\mathbf{\Omega}+\mathbf{T}^\dagger(\mathbf{g}^{(P)}-\mathbf{g}^{(P)\dagger})\mathbf{T}. \quad (2.38)$$

With $\mathbf{T}=\mathbf{V}\mathbf{\Omega}$ and $\mathbf{g}=\mathbf{g}^{(P)}+\mathbf{g}^{(Q)}$, we arrive at the correct form of the unitarity relation, namely,

$$\mathbf{T}(E)-\mathbf{T}^\dagger(E)=\mathbf{T}^\dagger(E)[\mathbf{g}(E)-\mathbf{g}^\dagger(E)]\mathbf{T}(E). \quad (2.39)$$

We note that if the same analysis is applied to the original form of the Lippmann-Schwinger equation,[20] one finds that unitarity is not satisfied; in that case, not all channels are represented in the sum over intermediate states. Some substitute, such as Eq. (2.23), is therefore a *necessity*. The derivation of Eq. (2.39) has an immediate practical application. Thus, let $\mathbf{V}_t(E)$ be some approximation to the true effective potential which satisfies the correct "unitarity" relation, Eq. (2.36). If we then use $\mathbf{V}_t$ in Eq. (2.35) to determine an approximate $T$-matrix $\mathbf{T}_t$, it follows that $\mathbf{T}_t$ is unitary. The utility of this result derives, of course, from the fact that the truncated unitarity relation given by Eq. (2.36) will be easier to satisfy in practice, as opposed to attempting to construct a unitary $T$ matrix directly. This discussion may be viewed as a simple variation of the usual $K$-matrix theory which leads to the Heitler equation.[21] We note, finally, that Eq. (2.35) will be valid even when $P$ space contains many-body channels. Of course, in that case, the solution of the integral equation will be far from trivial due to the necessity of summing over many-body intermediate states.

Returning to our formulation of the minimum principle, we define a trial effective potential according to

$$V_{\alpha\beta t}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)=(\Psi_{\alpha t}{}^Q,[\hat{H}-E]\Psi_{\beta t}{}^Q), \quad (2.40)$$

where

$$\Psi_{\alpha t}{}^Q=\Phi_\alpha+QG_t{}^QQHP\Phi_\alpha. \quad (2.41)$$

[20] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[21] See, e.g., M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), Sec. 5.6.

Then the error is given by

$$\Delta V_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta; E) = -(\Delta\Psi_\alpha{}^Q, [\hat{H}-E]\Delta\Psi_\beta{}^Q), \quad (2.42)$$

with $\Delta\Psi_\alpha{}^Q=\Psi_{\alpha t}{}^Q-\Psi_\alpha{}^Q$. Since $\Delta\Psi_\alpha{}^Q$ is in $Q$ space, $\hat{H}-E$ may be regarded as a non-negative operator in Eq. (2.42). Once the trial effective potential is constructed, the reaction matrix $\mathbf{K}_t$ may be determined by a routine numerical solution of the coupled two-body Lippmann-Schwinger equations. The variational bound, Eq. (2.17), is then guaranteed. It is fortunately not necessary to actually construct the trial Green's function. We need only choose $\Psi_{\alpha t}{}^Q=\Phi_\alpha+\tilde{\Psi}_{\alpha t}$ with $\tilde{\Psi}_{\alpha t}$ in $Q$ space. There will then exist some operator $G_t{}^Q$ such that $\tilde{\Psi}_{\alpha t}=QG_t{}^QQHP\Phi_\alpha$; $G_t{}^Q$ itself need not be explicitly exhibited. The statement that the trial functions $\tilde{\Psi}_{\alpha t}$ and $\varphi_{it}$ are in $Q$ space means that they vanish for large interparticle separations and satisfy

$$\begin{aligned} P\tilde{\Psi}_{\alpha t}&=0, \\ P\varphi_{it}&=0, \end{aligned} \quad (2.43)$$

but are otherwise arbitrary. We note that in the present version of the minimum principle the projection operator appears in the constraint conditions, Eqs. (2.43), but nowhere else.

To illustrate the treatment of identical-particle effects, suppose we are interested in a problem such as the scattering of electrons by hydrogen atoms at energies below the first excitation threshold. To take into account both direct and exchange effects, we consider the $2\times2$ $K$ matrix

$$K=\begin{pmatrix} K_D & K_E \\ K_E & K_D \end{pmatrix}. \quad (2.44)$$

If we choose

$$\mathbf{a}_\pm=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm1 \end{pmatrix} \quad (2.45)$$

for scattering in the triplet $(-)$ or singlet $(+)$ spin states, we get a *single*-channel Lippmann-Schwinger integral equation for the properly symmetrized $K$-matrix element of interest, namely,

$$\begin{aligned} K_{t\pm}&=\mathbf{a}_\pm{}^\dagger\mathbf{K}_t\mathbf{a}_\pm \\ &=K_{Dt}\pm K_{Et}, \end{aligned} \quad (2.46)$$

which represents a lower bound on the true value. The effective potential in the integral equation is a sum of direct and exchange potentials, i.e.,

$$V_t=V_{Dt}\pm V_{Et}=V_{\alpha\alpha t}\pm V_{\alpha\beta t}. \quad (2.47)$$

The simplicity of this result stems from the fact that, since $\mathbf{a}_\pm$ is independent of the matrix elements, it is an eigenvector of both the $K$ matrix *and* the effective potential matrix, on and off the energy shell. Coupled equations will, of course, be unavoidable when the particles are distinguishable (in which case $K_{11}\neq K_{22}$, and the above simplification does not obtain). More generally, in the elastic scattering of a particle by a system containing $N$ identical bound particles, the $(N+1)\times(N+1)$ $K$ matrix (before any spin decomposition is introduced) will satisfy

$$\begin{aligned} K_{ii}&=K_D, \\ K_{ij}&=K_{ji}, \\ K_{ij}&=\pm K_{i,j+1} \quad \text{for} \quad i\neq j, \quad i\neq j+1, \end{aligned} \quad (2.48)$$

where we have taken into account the Bose-Einstein $(+)$ or Fermi-Dirac $(-)$ symmetry of the target wave function. If we define $K_{12}=K_E$, it is easy to see that the eigenvector $\mathbf{a}_\pm$ which has the physically relevant eigenvalue $K=K_D\pm NK_E$ has the components

$$a_{i\pm}=(N+1)^{-1/2}(\pm1)^{i+1}, \quad i=1, 2, \cdots, N+1,$$

independent of $K_D$ and $K_E$. The integral equations therefore decouple, and the effects due to exchange of identical particles are accounted for in a simple way.

## 3. THREE-BODY PROBLEM

### A. Distinguishable Particles

In any application of the formalism described above, there still remains the task of giving an explicit realization of the projection operators $P$ and $Q$. This is possible in principle[3] but generally will be quite difficult in practice. Of course, if the Wigner-Eisenbud picture is applicable,[22] i.e., if configuration space can be separated into "inside" and "outside" regions, such that the Schrödinger equation is soluble in the outside region, one may set $P=1$ in the outside region and $P=0$ inside. Bounds on the eigenvalues of the $K$ matrix may be obtained by performing an ordinary variational calculation with a trial function $\Psi_t$ which satisfies the subsidiary constraint

$$P(H-E)\Psi_t=0. \quad (3.1)$$

This is the basis of the minimum principle described previously.[6] The results of Ref. 5 show (in effect) that the constraint Eq. (3.1) leads to a variational bound for the entire class of projection operators defined by Feshbach.[3] This is a major improvement, but one still must construct the operator $P$.

The above remarks are meant to motivate the following analysis of the three-body problem. As we have seen, the projection operator $P$ enters our formulation of the minimum principle only in the constraint conditions Eqs. (2.43). We now show that the Faddeev[4] integral equations for the three-body wave function enable us to give explicit meaning to Eqs. (2.43) in terms of simple orthogonality conditions.

According to Faddeev, the wave function may be written as $\Psi=\Psi^{(1)}+\Psi^{(2)}+\Psi^{(3)}$, with the $\Psi^{(i)}$ deter-

[22] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).

mined by the set of coupled integral equations

$$\begin{pmatrix}\Psi^{(1)}\\ \Psi^{(2)}\\ \Psi^{(3)}\end{pmatrix}=\begin{pmatrix}a_1\Phi_{23}\\ a_2\Phi_{13}\\ a_3\Phi_{12}\end{pmatrix}+\begin{pmatrix}0 & G_{23}V_{23} & G_{23}V_{23}\\ G_{13}V_{13} & 0 & G_{13}V_{13}\\ G_{12}V_{12} & G_{12}V_{12} & 0\end{pmatrix}\begin{pmatrix}\Psi^{(1)}\\ \Psi^{(2)}\\ \Psi^{(3)}\end{pmatrix}, \tag{3.2}$$

where for each pair $(i,j)$,

$$G_{ij}(E)=[E+i\eta-K-V_{ij}]^{-1} \tag{3.3}$$

($K$ is the total-kinetic-energy operator) and, e.g., $\Phi_{23}$ is an unperturbed wave corresponding to particle 1 incident on a bound (2,3) pair. Since we are ultimately interested in the *eigenvalues* of the reaction matrix [see Eq. (2.17)], we have allowed for waves in the other two entrance channels as well. The significant feature of the Faddeev equation is that the continuous spectrum can be obtained by inspection of the kernel. [We have indicated in Sec. 2 that the ordinary Lippmann-Schwinger equation[20] does not share this feature and is therefore inadequate.] In fact, it is the set of branch cuts in the Green's functions $G_{ij}(E)$ which determines the continuous spectrum. As usual, these branch cuts run along the real energy axis from the physical threshold positions to infinity and may be exhibited in principle by introducing the eigenfunction expansions of the $G_{ij}$. We now wish to write down the linear integral equation which determines the continuum function $\Psi_\alpha{}^Q$ defined in Eq. (2.28). Solutions to the homogeneous version of this integral equation will give rise to the eigenfunctions $\varphi_i$ of the discrete spectrum. The basic requirement is that the portion of the continuous spectrum of $H$ associated with the open channels should be missing from the spectrum of $QHQ$. According to the above remarks, we need only introduce the decomposition

$$G_{ij}=G_{ij}{}^{(P)}+G_{ij}{}^{(Q)}, \tag{3.4}$$

where $G_{ij}{}^{(P)}$ consists of those contributions to the eigenfunction expansion which involve the bound states of the $(ij)$ pair associated with the open channels. Then the functions $\Psi_\alpha{}^Q$ are obtained by solving Eq. (3.2) with the $G_{ij}$ replaced by the $G_{ij}{}^{(Q)}$. The discrete states are obtained from the solution of the homogeneous integral equation

$$\begin{pmatrix}\varphi_i{}^{(1)}\\ \varphi_i{}^{(2)}\\ \varphi_i{}^{(3)}\end{pmatrix}=\begin{pmatrix}0 & G_{23}{}^{(Q)}V_{23} & G_{23}{}^{(Q)}V_{23}\\ G_{13}{}^{(Q)}V_{13} & 0 & G_{13}{}^{(Q)}V_{13}\\ G_{12}{}^{(Q)}V_{12} & G_{12}{}^{(Q)}V_{12} & 0\end{pmatrix}\begin{pmatrix}\varphi_i{}^{(1)}\\ \varphi_i{}^{(2)}\\ \varphi_i{}^{(3)}\end{pmatrix}, \tag{3.5}$$

with $\varphi_i=\varphi_i{}^{(1)}+\varphi_i{}^{(2)}+\varphi_i{}^{(3)}$. Alternatively, the discrete eigenvalues may be characterized by the maximum-minimum property,[23] allowing for a Rayleigh-Ritz variational construction. The trial functions $\varphi_{it}=\varphi_{it}{}^{(1)}+\varphi_{it}{}^{(2)}+\varphi_{it}{}^{(3)}$ would then have to satisfy certain orthogonality conditions in order that they lie in the same space (i.e., $Q$ space) as the true functions $\varphi_i$. It is the purpose of this discussion to arrive at the explicit form of these conditions.

We denote the bound-state wave functions of the (2,3) pair, corresponding to states in $P$ space, as $\chi_{23,u}$, with $u=1, 2, \cdots, N_{23}(P)$. Then, according to Eq. (3.5),

$$\varphi^{(1)}=\sum_{u>N_{23}(P)}\chi_{23,u}(\boldsymbol{\varrho}_{23})f_{23,u}(\mathbf{q}_1) \tag{3.6}$$

(integration over continuum states is implied), where, in terms of the position vectors $\mathbf{r}_i$ and masses $m_i$, we have

$$\begin{aligned}\boldsymbol{\varrho}_{23}&=\mathbf{r}_2-\mathbf{r}_3,\\ \mathbf{q}_1&=\mathbf{r}_1-[(m_2\mathbf{r}_2+m_3\mathbf{r}_3)/(m_2+m_3)].\end{aligned} \tag{3.7}$$

Similar expansions may be written down for $\varphi^{(2)}$ and $\varphi^{(3)}$. In deducing the orthogonality conditions, there is a certain degenerate case which should be discussed first. Thus, suppose $m_3=\infty$, with $m_1$ and $m_2$ finite. Then $\mathbf{q}_1\rightarrow\boldsymbol{\varrho}_{13}$ and the functions $\varphi^{(1)}$ and $\varphi^{(2)}$ should be taken together, i.e.,

$$\varphi^{(1)}+\varphi^{(2)}=\sum_{u>N_{23}(P)}\sum_{v>N_{13}(P)}b_{uv}\chi_{23,u}(\boldsymbol{\varrho}_{23})\chi_{13,v}(\boldsymbol{\varrho}_{13}). \tag{3.8}$$

Evidently, the orthogonality conditions are

$$(\chi_{23,u},[\varphi^{(1)}+\varphi^{(2)}])=0,\quad u=1, 2, \cdots, N_{23}(P), \tag{3.9a}$$

$$(\chi_{13,u},[\varphi^{(1)}+\varphi^{(2)}])=0,\quad u=1, 2, \cdots, N_{13}(P), \tag{3.9b}$$

along with

$$(\chi_{12,u},\varphi^{(3)})=0,\quad u=1, 2, \cdots, N_{12}(P). \tag{3.9c}$$

Note that for the electron-hydrogen problem we may represent $\varphi$ in the form given in Eq. (3.8), i.e., with $\varphi^{(3)}=0$. Then Eqs. (3.9) reproduce the prescription given previously[16,24] for this problem. In general, when all three masses are finite, the orthogonality conditions become

$$(\chi_{ij,u},\varphi^{(k)})=0,\quad u=1, 2, \cdots, N_{ij}(P), \tag{3.10}$$

for $k=1$, 2, and 3, with $i\neq j\neq k$. It is now clear how to interpret the constraint conditions Eqs. (2.43). We choose a trial function

$$\Psi_t{}^Q=\sum_{i=1}^{3}\Psi_t{}^{Q(i)}, \tag{3.11}$$

with

$$\Psi_t{}^{Q(i)}=a_i\Phi_{jk}+\tilde{\Psi}_t{}^{(i)}. \tag{3.12}$$

The functions $\tilde{\Psi}_t{}^{(i)}$ must vanish asymptotically and must satisfy

$$(\chi_{ij,u},\tilde{\Psi}_t{}^{(k)})=0,\quad u=1, 2, \cdots, N_{ij}(P), \tag{3.13}$$

in the finite mass case, or else conditions such as shown in Eqs. (3.9) when one mass is infinite. Since the true function $\Psi^Q$ is of the same form, the error $\Delta\Psi^Q=\Psi_t{}^Q-\Psi^Q$

[23] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. 1, Chap. 1. This property constitutes the mathematical basis of our minimum principle.

[24] We emphasize that even with the same form for the projection operators the present approach is quite distinct from the method of Hahn *et al.*

will belong to $Q$ space and hence will be an allowable trial function in the $Q$-space bound-state problem. The error in the variational estimate of the effective potential is of the form [see Eq. (2.42)]

$$-(\Delta\Psi^Q, [\hat{H}-E]\Delta\Psi^Q).$$

The preceding discussion establishes that $\hat{H}-E$ is positive in the space of functions of the type $\Delta\Psi^Q$, so that the formal developments of Sec. 2 apply. This establishes the minimum principle for the general three-body problem.

### B. Indistinguishable Particles

The simplifications which arise when the three particles are identical are significant enough to merit separate discussion. Faddeev[4] has shown that when the three particles are spinless bosons Eqs. (3.2) may be written as a single, uncoupled, integral equation. It is more convenient for our purpose to study the corresponding integral equation for the elastic scattering amplitude $T=T_D+2T_E$. The transformation has been carried out previously[2] for the general three-body problem and for all channels. When applied to the special case at hand, we found[9] that the amplitude could be written as $T=2\mathcal{T}$, with

$$\mathcal{T}(\mathbf{k}_f,\mathbf{k}_i;E)=B(\mathbf{k}_f,\mathbf{k}_i;E)+\tau(\mathbf{k}_f,\mathbf{k}_i;E). \quad (3.14)$$

Here $B(\mathbf{k}_f,\mathbf{k}_i;E)$ is the Born approximation to the exchange amplitude [see Eqs. (3.33) and (3.38) below], and

$$\tau(\mathbf{k}_f,\mathbf{k}_i;E)=(\Phi_{jk},\tau^{(k)}\Phi_{jk}), \quad (3.15)$$

with

$$\tau^{(k)}=2t_{ij}+2t_{ij}G_0I_{ik}\tau^{(k)}. \quad (3.16)$$

The particle indices $(i,j,k)$ can be any one of the permutations of (1,2,3). In Eq. (3.16), $t_{ij}$ is the two-body scattering operator which satisfies

$$t_{ij}=V_{ij}+V_{ij}G_{ij}V_{ij}, \quad (3.17)$$

$G_0$ is the "free" Green's function, and $I_{ik}$ is an interchange operator which guarantees that an interaction between a pair of particles, described by the operator $t$, is always followed by an interaction between a *different* pair. [We have used the language of multiple scattering expansions, but the meaning of Eq. (3.16) itself should be clear.] We now show how an effective potential may be directly exhibited, without recourse to the projection operator formalism of Sec. 2.

The two-body scattering operator $t$ (we omit all particle indices now for ease of writing) may be represented as

$$t=t^{(P)}+t^{(Q)}, \quad (3.18)$$

according to Eqs. (3.4) and (3.17). If, for simplicity, we assume there is only one two-body bound state, then $t^{(P)}$ is of the separable form

$$t^{(P)}(E)=|V\chi\rangle[1/(E+\epsilon)]\langle\chi V|. \quad (3.19)$$

It is seen from the comparison identity Eq. (2.34) that the solutions to the two equations

$$\tau=2t+2tG_0I\tau \quad (3.16')$$

and

$$\tau^{(Q)}=2t^{(Q)}+2t^{(Q)}G_0I\tau^{(Q)} \quad (3.20)$$

are related by

$$\tau=\tau^{(Q)}+(1+\tau^{(Q)}IG_0)2t^{(P)}(1+G_0I\tau). \quad (3.21)$$

From Eqs. (3.14) and (3.21), we obtain an integral equation for $\mathcal{T}(\mathbf{k}_f,\mathbf{k}_i;E)$ of the desired form, namely, (with $\hbar^2=2m=1$)

$$\mathcal{T}(\mathbf{k}_f,\mathbf{k}_i;E)=\mathcal{U}(\mathbf{k}_f,\mathbf{k}_i;E)+2\int d\mathbf{k}\,\mathcal{U}(\mathbf{k}_f,\mathbf{k};E)\times\frac{1}{E-\frac{3}{2}k^2+\epsilon}\mathcal{T}(\mathbf{k},\mathbf{k}_i;E). \quad (3.22)$$

Here the effective potential is given by

$$\mathcal{U}(\mathbf{k}_f,\mathbf{k}_i;E)=B(\mathbf{k}_f,\mathbf{k}_i;E)+\tau^Q(\mathbf{k}_f,\mathbf{k}_i;E), \quad (3.23)$$

as might have been anticipated from our general considerations in Sec. 2. In arriving at Eq. (3.22) we have adopted the prescription[9] that in continuing an amplitude off the energy shell we make the replacement $\Phi\rightarrow G_0V\Phi$. These two forms are equal on the energy shell, i.e., for $E=\frac{3}{2}k^2-\epsilon$, by virtue of the eigenvalue equation satisfied by the two-body bound-state wave function. Since the bound-state singularities have been removed in the construction of the effective potential, the formalism of Sec. 2, which leads to a minimum principle for the effective potential and a variational lower bound on the phase shift, applies directly here.

There has been considerable interest lately in obtaining approximations to three-body scattering amplitudes by replacing the two-body interactions by separable potentials.[9,10,25] The resulting equations (we continue to consider the system of three spinless bosons) are of the two-body type, with a slightly modified propagator and an effective potential which is just the Born term $B(\mathbf{k}_f,\mathbf{k}_i;E)$. To assess the accuracy of the approximation, and improve it if possible, it is desirable to obtain an expression for the exact effective potential. This will now be done, with a slight modification of the procedure leading to Eq. (3.22).

We write the two-body potential as $V=V_1+V_s$, with

$$V_s=V|\chi\rangle\langle\chi|V/\langle\chi|V|\chi\rangle; \quad (3.24)$$

we again assume that only one two-body bound state exists. Introducing the operator $t_1$ as

$$t_1=V_1+V_1G_0t_1\equiv V_1\Omega_1, \quad (3.25)$$

[25] A. N. Mitra, Nucl. Phys. **32**, 529 (1962); C. Lovelace, Phys. Rev. **135**, B1225 (1964).

we may replace Eq. (3.18) by the decomposition

$$t=t_1+\Omega_1^{\dagger(-)}V_s\Omega, \tag{3.26}$$

where we have used Eq. (2.34). The wave operators $\Omega$ and $\Omega_1$, corresponding to the potentials $V$ and $V_1$, respectively, are related, according to the Lippmann-Schwinger equation, by

$$\Omega=\Omega_1+\Omega_1G_0V_s\Omega. \tag{3.27}$$

If we define $t_s$ by

$$t_s=V_s+V_s\Omega_1G_0t_s, \tag{3.28}$$

Eq. (3.27) implies that $t_s\Omega_1=V_s\Omega$, so that Eq. (3.26) may be written as

$$t=t_1+\Omega_1^{\dagger(-)}t_s\Omega_1^{(+)}. \tag{3.29}$$

Since $V_s$ is separable, Eq. (3.28) may be solved explicitly; we obtain

$$\langle\mathbf{k}'|\Omega_1^{\dagger(-)}(E)t_s(E)\Omega_1^{(+)}(E)|\mathbf{k}\rangle =\tfrac{1}{2}[g^{(+)}(\mathbf{k}',E)g^{(-)*}(\mathbf{k},E)/(E+\epsilon)]S_1^{(+)}(E). \tag{3.30}$$

Here we have defined

$$g^{(\pm)}(\mathbf{k},E)=\langle\mathbf{k}|\Omega_1^{\dagger(\mp)}(E)V|\chi\rangle \tag{3.31}$$

and

$$[S_1^{(+)}(E)]^{-1}=\frac{1}{2}\int d\mathbf{k}\frac{|g^{(+)}(\mathbf{k},2k^2)|^2}{(2k^2-E-i\eta)(2k^2+\epsilon)}. \tag{3.32}$$

We note the on-shell relation

$$g(\mathbf{k},-\epsilon)=g(\mathbf{k})=\langle\mathbf{k}|V\chi\rangle, \tag{3.33}$$

which may be derived using the eigenvalue equation satisfied by $\chi$ along with the property $V_1\chi=0$. In place of Eq. (3.21), we now have [once again we use the comparison identity Eq. (2.34)]

$$\tau=\tau_1+(1+\tau_1IG_0)[\Omega_1^{\dagger(-)}2t_s\Omega_1](1+G_0I\tau), \tag{3.34}$$

where $\tau_1$ is obtained from Eq. (3.16′) with the replacement of $t$ by $t_1$. The exact integral equation for the scattering amplitude then becomes

$$\mathcal{T}(\mathbf{k}_f,\mathbf{k}_i;E)=\mathcal{U}_1(\mathbf{k}_f,\mathbf{k}_i;E) +\int d\mathbf{k}\mathcal{U}_1(\mathbf{k}_f,\mathbf{k};E)\frac{S_1^{(+)}(E-\frac{3}{2}k^2)}{E-\frac{3}{2}k^2+\epsilon}\mathcal{T}(\mathbf{k},\mathbf{k}_i;E), \tag{3.35}$$

with

$$\mathcal{U}_1(\mathbf{k}_f,\mathbf{k}_i;E)=B(\mathbf{k}_f,\mathbf{k}_i;E)+\tau_1(\mathbf{k}_f,\mathbf{k}_i;E). \tag{3.36}$$

The prescription for continuing these amplitudes off the energy shell is now

$$\begin{aligned}|\Phi_i(E)\rangle&\rightarrow|G_0(E)\Omega_1^{\dagger(-)}(E)V\Phi_i(E)\rangle,\\ \langle\Phi_f(E)|&\rightarrow\langle\Phi_f(E)V\Omega_1^{(+)}(E)G_0(E)|.\end{aligned} \tag{3.37}$$

For example, the Born term is

$$B(\mathbf{k}_f,\mathbf{k}_i;E)=\frac{g(\mathbf{k}_i+\frac{1}{2}\mathbf{k}_f)g(\mathbf{k}_f+\frac{1}{2}\mathbf{k}_i)}{E-(\mathbf{k}_i+\mathbf{k}_f)^2-k_i^2-k_f^2}, \tag{3.38}$$

for $\frac{3}{2}k_i^2=\frac{3}{2}k_f^2=E+\epsilon$. When $k_f^2$ is varied with $E$ and $k_i^2$ fixed, we must make the replacement

$$g(\mathbf{k}_i+\tfrac{1}{2}\mathbf{k}_f)\rightarrow g^{(-)*}(\mathbf{k}_i+\tfrac{1}{2}\mathbf{k}_f,E-\tfrac{3}{2}k_f^2)$$

in Eq. (3.38). Similarly, when $k_i^2$ is varied, with $E$ and $k_f^2$ fixed, the replacement

$$g(\mathbf{k}_f+\tfrac{1}{2}\mathbf{k}_i)\rightarrow g^{(+)}(\mathbf{k}_f+\tfrac{1}{2}\mathbf{k}_i;E-\tfrac{3}{2}k_i^2)$$

is required.

The above approach will be a reasonable one when the potential $V_1$ is sufficiently weak so that it cannot form a two-body bound state. If this is the case, the minimum principle for the effective potential $\mathcal{U}_1$ can be formulated in analogy with the procedure described in Sec. 2. In fact, the situation is somewhat simpler here, since no orthogonality constraints need be placed on the trial functions; the continuous spectrum of the Hamiltonian obtained by the replacement of $V$ with $V_1$ will begin at the inelastic threshold $E=0$. As observed above we may translate the minimum principle for the effective potential into a minimum principle for the phase shift using the monotonicity theorem[14]; the presence of the modified propagator in Eq. (3.35) does not alter the argument. We note, finally, that, in addition to variational estimates of the effective potential, a Born expansion should be useful, provided the strength of the potential $V_1$ has been sufficiently weakened by the removal of the separable part $V_s$. This is the basic idea of the quasiparticle technique originally introduced by Weinberg[13] for two-body scattering and here applied in the context of a three-body problem.

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