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Eigenvalues of Fermion Density Matrices*

FUKASHI SASAKI†
Quantum Chemistry Group, Uppsala University, Uppsala, Sweden
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The least upper bound of the eigenvalues of second-order reduced density matrices for a system of fermions is proved to be n for a system of $2n$ or $2n+1$ identical fermions. It is also shown that this limiting state may be interpreted as a system of identical pairs behaving as quasibosons.

1. INTRODUCTION

It is known that certain features of a system are illuminated by the spectrum of its first-order density matrix. For example, an eigenvalue of this matrix may be interpreted as the occupation number of the corresponding spin orbital, and if all the eigenvalues are equal to 1, the state can be described by a single Slater determinant.[1] We might expect that the spectrum of a higher order reduced density matrix would also characterize the structure of the system. However, it seems that little has been done along this line.[2] In this paper, we discuss the range of the eigenvalues of a many-particle density matrix in order to approach this problem.

For this purpose, it is convenient to use a wave function expanded in terms of the eigenfunctions of density matrices.[3] The expansion is obtained by the use of the following theorems.[3–5]

Theorem 1. If A is a linear and completely continuous transformation[4] of one Hilbert space into another, and f is an element of the first Hilbert space, Af can be written in the form

$$Af=\sum_i \mu_i g_i(f_i,f)\,.$$

Here $\{f_i\}$ and $\{g_i\}$ are orthonormal sets in the two Hilbert spaces involved, and $\{\mu_i\}$ is a nonincreasing sequence of positive numbers. The sequence can be finite or infinite, and in the latter case it tends to zero.

Corollary 1:

$$\sup|(Af,g)|/[(f,f)(g,g)]^{1/2}=\mu_1\,.$$

Theorem 2. If there exists a normal operator S such that $AS=A$, every f_i is an eigen-element of S, i.e., $Sf_i=f_i$.

A normalized wave function $\Psi(x_1,x_2,\cdots x_N)$ of N fermions may be regarded as a kernel of the operator A, which transforms absolute-square-integrable functions of M fermions into functions of $N-M$ fermions:

$$g(x_1,\cdots,x_{N-M}) =\int\cdots\int dx_1'\cdots dx_M' \times\Psi(x_1,\cdots,x_{N-M},x_1',\cdots,x_M')f(x_1',\cdots,x_M')\,,$$

or, in a brief form

$$g(x)=\int\Psi(x,y)f(y)dy\,,$$

where x and y denote $(x_1\cdots x_{N-M})$ and $(x_1'\cdots x_M')$, respectively. Since the wave function $\Psi(x,y)$ is normalized:

$$\int\int|\Psi(x,y)|^2dxdy=1\,,$$

it corresponds necessarily to a completely continuous transformation. By the use of theorem 1, we obtain the following expansion of the wave function Ψ:

$$\Psi(x,y)=\sum_i \mu_i g_i(x) f_i(y)\,, \tag{1.1}$$

where

$$\int g_i^*(x)g_j(x)dx=\delta_{ij}\,,$$

$$\int f_i^*(y)f_j(y)dy=\delta_{ij}\,,$$

and

$$\mu_i\geq\mu_j>0 \quad \text{for} \quad i<j\,.$$

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1 Per-Olov Löwdin, Phys. Rev. **97**, 1474 (1955).

2 See, however, the preliminary report of A. J. Coleman, Can. Math. Bull. **4**, No. 2 (1961). Further progress has been reported by him at Sanibel Island, Winter Institute in Quantum Chemistry and Solid State Physics, January 1962 (unpublished).

3 J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Julius Springer, Verlag, Berlin, 1932) (English transl.: Princeton University Press, Princeton, New Jersey, 1955, Chap. VI); B. C. Carlson, and J. M. Keller, Phys. Rev. **121**, 659 (1961).

4 See, e.g., F. Riesz and B. Sz-Nagy, *Functional Analysis* (Frederick Ungar Publishing Company, New York, 1955), p. 206.

5 A. T. Amos and G. G. Hall, Proc. Roy. Soc. (London) **A263**, 483 (1961). Detailed proofs of these theorems have been given in F. Sasaki, Technical Report 77, Uppsala Quantum Chemistry Group, 1962 (unpublished).

Since the density matrix of order M of this pure state is defined by

$$\Gamma_M(y,y')=\binom{N}{M}\int \Psi(x,y)\Psi^*(x,y')dx,$$

we obtain immediately the diagonal expansion of the density matrix from (1.1) in the form

$$\Gamma_M(y,y')=\binom{N}{M}\sum_i \mu_i^2 f_i(y)f_i^*(y').$$

Similarly, the density matrix of order $N-M$ is found to be

$$\Gamma_{N-M}(x,x')=\binom{N}{M}\sum_i \mu_i^2 g_i(x)g_i^*(x').$$

In order to evaluate the symmetry property of f_i, it is convenient to introduce the antisymmetry projection operator defined with respect to the coordinates $y=(x_1'\cdots x_M')$:

$$O_{\mathrm{AS},y}=\frac{1}{M!}\sum_P \epsilon_P P.$$

Here P is a permutation operator which permutes only the coordinates y and ϵ_P is its parity. It is easy to see that $O_{\mathrm{AS},y}$ is self-adjoint and that

$$\int \Psi(x,y)O_{\mathrm{AS},y}f(y)dy = \int O_{\mathrm{AS},y}\Psi(x,y)f(y)dy=\int \Psi(x,y)f(y)dy.$$

Thus by using theorem 2, it follows that $O_{\mathrm{AS},y}f=f$, i.e., that if the function Ψ is antisymmetric, f_i and g_i in the expansion (1.1) should also be antisymmetric.

2. THE LEAST UPPER BOUND

The largest eigenvalue of a density matrix of order M may be regarded as a functional of Ψ:

$$\lambda_{M,N}(\Psi)=\binom{N}{M}\mu_1^2.$$

Introducing a projection operator O_Ψ which projects out the state Ψ:

$$O_\Psi=\Psi)(\Psi,$$

we obtain the following equality from (1.1);

$$\lambda_{M,N}(\Psi)=\binom{N}{M}(g_1f_1O_\Psi g_1f_1).$$

Introducing the total antisymmetry projection operator

$$O_{\mathrm{AS}}=\frac{1}{N!}\sum_P \epsilon_P P,$$

it is found for any function $\omega=\omega(x,y)$ that

$$(\omega(O_{\mathrm{AS}}-O_\Psi)\omega)=(\omega(1-O_\Psi)O_{\mathrm{AS}}(1-O_\Psi)\omega) = ((1-O_\Psi)\omega O_{\mathrm{AS}}(1-O_\Psi)\omega)\geq 0.$$

Thus we obtain the following inequality

$$\lambda_{M,N}(\Psi)\leq\binom{N}{M}(g_1f_1O_{\mathrm{AS}}g_1f_1) \leq\binom{N}{M}\sup_{f,g}(gfO_{\mathrm{AS}}gf), \tag{2.1}$$

where f and g are normalized functions of M and $N-M$ particles, respectively. Since the last term of (2.1) does not depend on Ψ, it follows that

$$\lambda_{M,N}\equiv\sup_\Psi\lambda_{M,N}(\Psi)\leq\binom{N}{M}\sup_{f,g}(gfO_{\mathrm{AS}}gf). \tag{2.2}$$

We shall now prove that the last term of (2.2) is equal to $\lambda_{M,N}$. Let $\{f^{(k)}\}$ and $\{g^{(k)}\}$ be the sets of normalized functions which give a solution of the above extremum problem:

$$0<\lambda^{(k)}\equiv\binom{N}{M}(g^{(k)}f^{(k)}O_{\mathrm{AS}}g^{(k)}f^{(k)})\rightarrow\binom{N}{M}\sup(gfO_{\mathrm{AS}}gf)$$

as $k\rightarrow\infty$.

Since a set of functions $\{\Psi^{(k)}\}$ defined by the equation

$$\Psi^{(k)}=\left(\binom{N}{M}\Big/\lambda^{(k)}\right)^{1/2}O_{\mathrm{AS}}g^{(k)}f^{(k)}$$

consists of normalized antisymmetric functions, it follows from corollary 1 that

$$\lambda_{M,N}(\Psi^{(k)})\geq\binom{N}{M}|(g^{(k)}f^{(k)},\Psi^{(k)})|^2=\lambda^{(k)},$$

i.e., that

$$\lim_{k\rightarrow\infty}\lambda_{M,N}(\Psi^{(k)})\geq\lim_{k\rightarrow\infty}\lambda^{(k)}. \tag{2.3}$$

By comparing (2.3) with (2.2), it is found that

$$\lambda_{M,N}=\binom{N}{M}\sup_{f,g}(gfO_{\mathrm{AS}}gf).\text{[6]} \tag{2.4}$$

[6] We note that for a system of identical bosons, the whole argument is valid after replacing the antisymmetry projection operators by the symmetry projection operators. Thus the least upper bound of the eigenvalues for bosons is given by the equation

$$\lambda_{M,N}=\binom{N}{M}\sup_{f,g}(gfO_{\mathrm{S}}gf) \tag{2.5}$$

under the condition $(f,f)=(g,g)=1$. Here the total symmetry projection operator O_{S} is given by the equation

$$O_{\mathrm{S}}=(1/N!)\Sigma_P P.$$

It is readily seen from (2.5) that

$$\lambda_{M,N}=\binom{N}{M}\ \text{(for a system of bosons)}.$$

3. UPPER BOUNDS

It is convenient to write the antisymmetry projection operator in the form

$$\binom{N}{M} O_{AS}(1,\cdots,N) = O_{AS}(1,\cdots,M) O_{AS}(M+1,\cdots,N) \times \sum_{i=0}^{\min\{N-M,M\}} (-)^i \binom{N-M}{i}\binom{M}{i} \times P\{(1,M+1)(2,M+2)\cdots(i,M+i)\} \times O_{AS}(1,\cdots,M) O_{AS}(M+1,\cdots,N), \quad (3.1)$$

where $O_{AS}(\cdots)$ denotes the antisymmetry projection operator defined with respect to the coordinates in the parenthesis, and $P\{(1,M+1)(2,M+2)\cdots(i,M+i)\}$ denotes the operation of replacing the coordinate 1 by $M+1$, $M+1$ by 1, $\cdots$, i by $M+i$ and $M+i$ by i. This shows that the f and g that give the extremum in Eq. (2.4) should be antisymmetric. Therefore we may introduce the density matrices of the ith order $\Gamma_{i,f}$ and $\Gamma_{i,g}$ reduced from f and g. By the use of these density matrices, we obtain

$$\lambda_{M,N} = 1 + \sup \sum_{i=1}^{\min(M,N-M)} (-)^i \operatorname{tr}\Gamma_{i,f}\Gamma_{i,g}. \quad (3.2)$$

Since density matrices are positive definite, it is easy to see that

$$0 \le \operatorname{tr}\Gamma_{i,f}\Gamma_{i,g} \le \min(\operatorname{tr}\Gamma_{i,f}\lambda_{i,N-M};\ \operatorname{tr}\Gamma_{i,g}\lambda_{i,M}) = \min\left[\binom{M}{i}\lambda_{i,N-M};\ \binom{N-M}{i}\lambda_{i,M}\right]. \quad (3.3)$$

From (3.2) and (3.3) we obtain an upper bound of the eigenvalues by the recurrence equation[7]

$$\lambda_{M,N} \le \Lambda_{M,N} = 1 + \sum_{i=1}^{\min\{[\frac{1}{2}M],[\frac{1}{2}(N-M)]\}} \min\left[\binom{M}{2i}\Lambda_{2i,N-M};\ \binom{N-M}{2i}\Lambda_{2i,M}\right]. \quad (3.4)$$

The solutions of (3.4) are

$$\Lambda_{0,N} = 1,$$
$$\Lambda_{1,N} = 1,$$
$$\Lambda_{2,N} = [\tfrac{1}{2}N], \quad N \ge 4$$
$$\Lambda_{3,N} = 1 + 3[\tfrac{1}{2}(N-3)], \quad N \ge 6$$
$$\Lambda_{4,N} = 1 + ([\tfrac{1}{4}N]-1)(1+6[\tfrac{1}{2}N]-6[\tfrac{1}{4}N]) + (N-4[\tfrac{1}{4}N])(N-4[\tfrac{1}{4}N]-1), \quad N \ge 8$$
$$\Lambda_{5,N} = 1 + 10[\tfrac{1}{2}(N-5)] + 5\Lambda_{4,N-5}, \quad N \ge 10$$
$$\vdots$$

[7] $[x]$ stands for the integral part of x.

It should be noticed that

$$\Lambda_{M,N} = O(N^{[M/2]}).$$

This is the same order of magnitude as the largest eigenvalue of $\Gamma_{[M/2]}$ for a system of $[N/2]$ bosons.[8]

Since the eigenvalues of the first-order density matrix of a single-determinant wave function are 1, it follows that $\Lambda_{1,N}$ is equal to $\lambda_{1,N}$. It is shown in Sec. 5 that $\Lambda_{2,N}$ is also equal to $\lambda_{2,N}$.

4. EXTREME PROPERTIES OF WAVE FUNCTIONS

In this section we study the case where the largest eigenvalue of the Mth-order density matrix is almost equal to the least upper bound $\lambda_{M,N}$. Suppose we have a wave function Ψ such that

$$(f\Gamma_{M,\Psi}f) = \lambda_{M,N} - \epsilon,$$

where ϵ is a small non-negative number and f is a normalized function of M particles. It should be noted that f may or may not be an eigenfunction of $\Gamma_{M,\Psi}$. Define a function Φ by the equation

$$\Phi(1,\cdots,N) = \left[\binom{N}{M}\Big/\lambda_{M,N}\right]^{1/2} \times O_{AS} f(1,\cdots,M) g(M+1,\cdots,N),$$

where

$$g(M+1,\cdots,N) = \left[\binom{N}{M}\Big/(\lambda_{M,N}-\epsilon)\right]^{1/2} \times \int\cdots\int \Psi(1,\cdots,N) f^*(1,\cdots,M) dx_1\cdots dx_M.$$

It is easy to see that

$$(g,g) = (\lambda_{M,N}-\epsilon)^{-1}(f\Gamma_{M,\Psi}f) = 1,$$

$$(\Psi,\Phi) = \left[\binom{N}{M}\Big/\lambda_{M,N}\right]^{1/2} (\Psi O_{AS} fg) = \left[\binom{N}{M}\Big/\lambda_{M,N}\right]^{1/2} (\Psi, fg) = (1-\epsilon/\lambda_{M,N})^{1/2},$$

[8] This was conjectured by C. N. Yang, Rev. Mod. Phys. 34, 694 (1962). (See also footnote 9.)

and

$$(\Phi,\Phi)=\binom{N}{M}(fgO_{\mathrm{AS}}fg)/\lambda_{M,N}\leq 1.$$

Then it follows that

$$\begin{aligned}0\leq&(\Psi-\Phi,\Psi-\Phi)\\ &=(\Psi,\Psi)+(\Phi,\Phi)-(\Psi,\Phi)-(\Phi,\Psi)\\ &\leq 2-2(1-\epsilon/\lambda_{M,N})^{1/2}\leq 2\epsilon/\lambda_{M,N}. \quad (4.1)\end{aligned}$$

From the first three terms of (4.1), we obtain

$$(\Phi,\Phi)\geq 2(1-\epsilon/\lambda_{M,N})^{1/2}-1\geq 1-2\epsilon/\lambda_{M,N}.$$

Summarizing the results obtained above, we have a theorem.

Theorem 3. If a normalized M-particle function f satisfies the following equation

$$(f\Gamma_{M,\Psi}f)=\lambda_{M,N}-\epsilon,$$

the wave function Ψ can be expressed as

$$\Psi=\left[\binom{N}{M}\Big/\lambda_{M,N}\right]^{1/2}\times O_{\mathrm{AS}}f(1,\cdots,M)g(M+1,\cdots N)+h(1,\cdots,N),$$

where $(g,g)=1$, $(h,h)\leq 2\epsilon/\lambda_{M,N}$, and $1\geq(\Psi-h,\Psi-h)\geq 1-2\epsilon/\lambda_{M,N}$.

We apply the above theorem to the first-order density matrix. We know that some of the eigenvalues of the first-order density matrix can be $\lambda_{1,N}(=1)$,

$$\Gamma_{1,\Psi}(1,1')=\sum_{i=1}^{p}f_i(1)f_i^*(1')+\sum_{i=m+1}\lambda_i f_i(1)f_i^*(1'). \quad (4.2)$$

In such a case, it follows from theorem 3 that the wave function Ψ can be expressed as

$$\Psi=N^{1/2}O_{\mathrm{AS}}f_1(1)g_1(2,\cdots,N). \quad (4.3)$$

Using (3.1), we obtain

$$1=(\Psi,\Psi)=N(f_1g_1O_{\mathrm{AS}}f_1g_1)=1-\int\cdots\int dx_2\cdots dx_{N-1}\times\left|\int f_1^*(1)g_1(1,2,\cdots,N-1)dx_1\right|^2,$$

i.e.,

$$\int dx_1 f_1^*(1)g_1(1,2,\cdots,N-1)\equiv 0. \quad (4.4)$$

The first-order density matrix of g is found from (4.3) and (4.4) to be

$$\Gamma_{1,g_1}=\Gamma_{1,\Psi}-\Gamma_{1,f_1}. \quad (4.5)$$

Comparing (4.2) with (4.5), we see that the largest eigenvalue of Γ_{1,g_1} is also 1 if $p>1$. Thus, by repeated application of the previous discussion, it is found that

$$\Psi=\left[\frac{N!}{(N-p)!}\right]^{1/2}\times O_{\mathrm{AS}}f_1(1)\cdots f_p(p)g(p+1,\cdots,N), \quad (4.6)$$

$$\Gamma_{1,\Psi}=\sum_{i=1}^{p}f_i)(f_i+\Gamma_{1,g} \quad (4.7)$$

and

$$\int dx_1 f_i^*(1)g(1,2,\cdots,N-p)\equiv 0 \quad (i=1,\cdots,p). \quad (4.8)$$

5. THE LEAST UPPER BOUND OF THE EIGENVALUES OF THE SECOND-ORDER DENSITY MATRICES

In this section, we prove that the upper bound $\Lambda_{2,N}$ derived in Sec. 3 is actually the smallest.

Define functions $F_{2n}(1\cdots 2n)$ and $F_{2n+1}(1\cdots 2n+1)$ by the equations

$$\begin{aligned}F_{2n}(1,\cdots,2n)&=O_{\mathrm{AS}}f(1,2)f(3,4)\cdots f(2n-1,2n),\\ F_{2n+1}(1,\cdots,2n+1)&=O_{\mathrm{AS}}f(1,2)f(3,4)\cdots\times f(2n-1,2n)g(2n+1),\end{aligned} \quad (5.1)$$

where $f(1,2)$ is a normalized antisymmetric function of two particles and $g(1)$ is an arbitrary normalized function of a particle.

Then it is found that

$$(F_{2n},F_{2n})=2^n n!/(2n)!+O(\epsilon^2), \quad (5.2)$$

$$(F_{2n+1},F_{2n+1})=2^n n!/(2n+1)!+O(\epsilon), \quad (5.2')$$

where

$$\epsilon^2=\mathrm{tr}(\Gamma_{1,f})^2=\int\int\int\int f(1,2)f(3,4)f^*(1,3)\times f^*(2.4)dx_1dx_2dx_3dx_4. \quad (5.3)$$

Proof:

$$\begin{aligned}(F_{2n},F_{2n})&=(O_{\mathrm{AS}}f\cdots f,O_{\mathrm{AS}}f\cdots f)\\ &=(f\cdots f,O_{\mathrm{AS}}f\cdots f)\\ &=(1/(2n)!)\textstyle\sum_P \epsilon_P(f\cdots f,Pf\cdots f)\\ &=(1/(2n)!)\textstyle\sum_P a_P,\end{aligned}$$

where $a_P=\epsilon_P(f\cdots fPf\cdots f)$. In order to evaluate the above sum, we consider a subgroup H of the symmetric group S_{2n}. The subgroup H is generated from n transpositions, (12), (34), $\cdots$, $(2n-1,2n)$, and two other permutations, (13)(24) and $(135\cdots 2n-1)(246\cdots 2n)$. There are $2^n n!$ permutations in H. It is easy to see that $a_P=1$ for such a permutation since $\epsilon_P Pf\cdots f=f\cdots f$, but otherwise a_P is of the order of $\mathrm{tr}[(\Gamma_{1,f})^2]$.

For odd N, (5.2′) can be similarly proven. In this case some of the permutations will give integrals of the order $\mathrm{tr}(\Gamma_{1,f}\Gamma_{1,g})$, but

$$0\leq\mathrm{tr}\Gamma_{1,f}\Gamma_{1,g}\leq[\mathrm{tr}(\Gamma_{1,f})^2\,\mathrm{tr}(\Gamma_{1,g})^2]^{1/2}=[\mathrm{tr}(\Gamma_{1,f})^2]^{1/2}=\epsilon. \quad \text{Q.E.D.}$$

Let

$$\Psi^N(1,\cdots,N)=F_N(1,\cdots,N)/[(F_N,F_N)]^{1/2}. \quad (5.4)$$

Using (3.1), (5.2), and (5.2′) we obtain

$$(\Psi^N,\Psi^N)=1$$
$$=\frac{(F_{N-2},F_{N-2})}{(F_N,F_N)}(f\Psi^{N-2},O_{\mathrm{AS}}f\Psi^{N-2})$$
$$=\frac{1-\mathrm{tr}\Gamma_{1,f}\Gamma_{1,\Psi^{N-2}}+(f\Gamma_{2,\Psi^{N-2}}f)}{[N/2]+O(\epsilon)}.$$

Here we have used a trivial equality

$$O_{\mathrm{AS}}(1,\cdots,N)=O_{\mathrm{AS}}(3,\cdots,N)O_{\mathrm{AS}}(1,\cdots,N)O_{\mathrm{AS}}(3,\cdots,N).$$

Since $\mathrm{tr}\Gamma_{1,f}\Gamma_{1,\Psi^{N-2}}=O(\epsilon)$, we finally obtain

$$(f\Gamma_{2,\Psi^N}f)=[\tfrac{1}{2}N]+O(\epsilon). \quad (5.5)$$

It is possible to make $\mathrm{tr}(\Gamma_{1,f})^2$ as small as we wish, and therefore the largest eigenvalue of Γ_{2,Ψ^N} can be arbitrarily close to $\Lambda_{2,N}$.

It is found further that a wave function Ψ can be approximated by the form (5.4), if the largest eigenvalue is close to $\Lambda_{2,N}$. To prove this, suppose we have an N-particle wave function Ψ and a 2-particle function f such that

$$(f\Gamma_{2,\Psi}f)=[N/2]-\epsilon. \quad (5.6)$$

Then using theorem 3 and (3.1), we obtain

$$\Psi=\left[\binom{N}{2}\bigg/\Lambda_{2,N}\right]^{1/2}O_{\mathrm{AS}}fg_1+h_1,$$

where $(h_1|h_1)<2\epsilon/\Lambda_{2,N}$, and

$$1-\frac{2\epsilon}{\Lambda_{2,N}}\leq\binom{N}{2}(fg_1O_{\mathrm{AS}}fg_1)/\Lambda_{2,N}$$
$$=\{1-\mathrm{tr}\Gamma_{1,f}\Gamma_{1,g_1}+(f\Gamma_{2,g_1}f)\}/\Lambda_{2,N}.$$

Since $\mathrm{tr}\Gamma_{1,f}\Gamma_{1,g}>0$, we see that

$$(f\Gamma_{2,g_1}f)\geq\Lambda_{2,N-2}-2\epsilon, \quad (5.7)$$

showing that the function g_1 can be again expressed as

$$g_1=\left[\binom{N-2}{2}\bigg/\Lambda_{2,N-2}\right]^{1/2}O_{\mathrm{AS}}fg_2+h_2,$$

where $(h_2|h_2)<4\epsilon/\Lambda_{2,N-2}$. Repeating the procedure, we obtain a decomposition of the total wave function Ψ:

$$\Psi=\left[\binom{N}{2}\bigg/\Lambda_{2,N}\right]^{1/2}O_{\mathrm{AS}}f\left[\left(\binom{N-2}{2}\bigg/\Lambda_{2,N-2}\right)^{1/2}\times O_{\mathrm{AS}}f[\cdot[\cdots]\cdot\cdot]+h_2\right]+h_1$$
$$=\frac{N!}{2^{N/2}(N/2)!}O_{\mathrm{AS}}f(1,2)f(3,4)\cdots\times f(N-1,N)+h(1,\cdots,N),\quad (N\text{ even})$$
$$=\frac{N!}{2^{(N-1)/2}[(N-1)/2]!}O_{\mathrm{AS}}f(1,2)f(3,4)\cdots\times f(N-2,N-1)g(N)+h(1,\cdots,N),\quad (N\text{ odd}), \quad (5.8)$$

where

$$(h|h)=O(\epsilon).\quad \text{Q.E.D.}$$

By using (5.3), the first-order density matrix can be written in the form

$$\Gamma_{1,\Psi}\simeq\tfrac{1}{2}N\Gamma_{1,f},\quad (N\text{ even})$$
$$\Gamma_{1,\Psi}\simeq\tfrac{1}{2}(N-1)\Gamma_{1,f}+g)(g.\quad (N\text{ odd}) \quad (5.9)$$

The expressions (5.8) and (5.9) suggest that such a state may be interpreted as a system of fermion pairs which occupy the same state. These electron pairs behave like quasibosons and, since they are all in the same state, the limiting wave function corresponds to a situation with complete Bose-Einstein condensation.[9]

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[9] This paper was originally submitted for publication in the proceedings of the Hylleraas symposium together with the papers by A. J. Coleman, Rev. Mod. Phys. **35**, 668 (1963), and by T. Ando, *ibid.* **35**, 690 (1963), containing similar results. However, since the paper was accidentally omitted in the printed volume, it was later resubmitted to the Physical Review. During the interim, there has been an intense discussion of the interpretation of the results as to Bose-Einstein condensation; see particularly M. Girardeau, J. Math. Phys. **4**, 1096 (1963); J. M. Blatt, Math. Rev. **28**, 178 (1964); A. J. Coleman, Phys. Rev. Letters **13**, 406 (1964); F. Bloch, Phys. Rev. **137**, A787 (1965); A. J. Coleman, J. Math. Phys. (to be published).