term of the same structure,

$$(I/\mathscr{I})^2 \sum_i \mathcal{E}_i(c_{ii}^{(2)*}+c_{ii}^{(2)}) = -\sum_{a,i} |c_{ai}^{(1)}|^2 \mathcal{E}_i. \quad \text{(A21)}$$

The term $\rho_1^{(2)}$ contributes

$$(I/\mathscr{I})^2 \sum_{ai} \mathcal{E}_a |c_{ai}|^2. \quad \text{(A22)}$$

In deriving (A20)–(A22) essential use has been made of the Hartree-Fock equations. There is finally a second-order contribution arising from the interaction term in the energy depending on $(\rho^{(1)})^2$:

$$\tfrac{1}{2}\sum_{ai}\sum_{a'i'} [c_{ai}^{(1)*}c_{a'i'}^{(1)*}(aa'|V|ii') + c_{ai}^{(1)}c_{a'i'}^{(1)}(ii'|V|aa') + c_{ai}^{(1)*}c_{a'i'}^{(1)}(ai'|V|ia') + c_{ai}^{(1)}c_{a'i'}^{(1)*}(a'i|V|i'a)]. \quad \text{(A23)}$$

If we add (A19)–(A23) we obtain Eq. (98) and sequel, the formulas given in the text for the moment of inertia.

We add a final remark about the solution of the equation

$$\sum_{k'i} \mathbf{M}_{ki;k'i'}\mathbf{C}_{k'i'} = \mathbf{j}_{ki}. \quad \text{(A24)}$$

One must in fact question the existence of any solution to this equation, since it is well known (cf. Appendix C of I) that the corresponding homogeneous equation

$$\sum_{k'i} \mathbf{M}_{ki;k'i'}\xi_{k'i'} = 0, \quad \text{(A25)}$$

possesses, for a rotationally invariant Hamiltonian, a solution

$$\xi_{ki} = i\begin{pmatrix} j_{ki} \\ -j_{ki}^* \end{pmatrix}. \quad \text{(A26)}$$

Since

$$\sum_{ki} \xi_{ki}^\dagger \mathbf{j}_{ki} = 0, \quad \text{(A27)}$$

however, we may anticipate the existence of a solution of (A24).

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Brownian-Motion Model of Nonrelativistic Quantum Mechanics

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In Feynman's space-time approach to nonrelativistic quantum mechanics, the wave function may be considered to be the sum of path integrals over Brownian-motion trajectories. This formalism is entirely equivalent to the Schrödinger-equation treatment with its intrinsic indeterminacy. Pursuing the analogy with Brownian motion further, one may introduce a hidden variable corresponding to particle velocity in order to represent the wave function as a sum of path integrals taken over phase-space trajectories. By means of a hidden parameter corresponding to the time scale of fictitious interactions of the particle with the vacuum, one can define generalized propagators which over very short times produce wave functions which are localizable in phase space and hence are deterministic. For long times the conventional Feynman formalism is recovered. This short-time causality does not violate the von Neumann injunction against hidden variables since the new velocity variable turns out to be non-Hermitian over the long time scales normally considered. In contrast to the models proposed by Bohm and others, the present Brownian model is linear and preserves the usual statistical interpretation of the wave function for sufficiently long times.

I. INTRODUCTION

To avoid the mathematical and epistemological difficulties of high-energy quantum field theory it has often been suggested[1–3] that quantum theory be reformulated in terms of a fundamental cutoff length. Arguments have been advanced by Landau and others for introducing a natural cutoff equal to the short-range distance over which the electromagnetic and gravitational interactions become of comparable magnitude. This would serve to eliminate the divergences occurring in quantum electrodynamical and meson processes involving the exchange of virtual quanta. However, in the words of DeWitt,[4] "There exists as yet absolutely no concrete mathematical evidence either to support or to deny these speculations. A long program of formalism building and calculations is an unavoidable prerequisite."

In a recent book,[5] Bohm has also argued eloquently

[1] L. D. Landau, L. Rosenfeld, and O. Klein, in *Niels Bohr and the Development of Physics*, edited by W. Pauli, L. Rosenfeld, and V. Weisskopf (Pergamon Press Ltd., London, 1955).
[2] S. Deser, Rev. Mod. Phys. **21**, 417 (1957).
[3] B. S. DeWitt, Phys. Rev. Letters **13**, 114 (1964).
[4] B. S. DeWitt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 8.
[5] D. Bohm, *Causality and Chance in Modern Physics* (Harper & Brothers, New York, 1957).

for the development of new models for quantum theory, especially along deterministic lines, since these seem to contain the most directly available physical insight. Accordingly, Bohm has reintroduced into quantum mechanics the concept of continuous particle trajectories defined as orthogonal to the constant phase surfaces of the Schrödinger waves.[6–8] These Schrödinger waves are regarded as real force fields, rather than merely abstract probability amplitudes, which are capable of influencing the particle trajectories by means of new quantum potentials inserted into the classical equations. A short-range modification in this picture of field theory would involve the introduction into Schrödinger's equation of additional terms making it possible for the guided particles to react back, in a nonlinear way, upon the guiding force field. Bohm's point of view has been criticized on several grounds. First, as Heisenberg[9] has emphasized, the current form of the theory is experimentally indistinguishable from conventional quantum mechanics and merely amounts to an alternative language. Second, the nonlinear character of the suggested short-range modifications has been questioned by Feynman,[10] who points out that the superposition principle appears to be useful even for high-energy events involving neutral kaons. Third, the particle trajectories resulting from Bohm's theory are continuous but rapidly fluctuating in time and space, somewhat like Brownian motion trajectories, thereby vitiating their usefulness as aids to the intuition. More recent attempts by Wesley[11] to construct a deterministic model for quantum mechanics in which the notions of particle trajectories and of probability differ substantially from the previous work are also not free from mathematical and conceptual difficulties.

The situation can be brought into sharper focus by means of Feynman's space-time (propagator) approach to nonrelativistic quantum mechanics[12]; according to that approach, the probability amplitude for a particle reaching the point $(\mathbf{x},t)$ is the sum of path integrals taken over all possible trajectories leading to this space-time point. In the correspondence limit $\hbar=0$, the dominant trajectories are indeed just those of classical Hamilton-Jacobi theory; however, for finite values of $\hbar$ they come to resemble the continuous but nondifferentiable trajectories of Brownian motion theory. Hence the reintroduction of determinism at this level of description seems as unprofitable as the reintroduction of ray concepts into optical diffraction theory. On the other hand, such arguments cannot exclude the possibility of determinism existing at some underlying subquantum-mechanical level of description.[5]

The similarities between quantum mechanics and Brownian motion are not surprising and have been noted by Fürth[13] and others.[12,14–17] Now, in Brownian motion the uncertainty principle governing fluctuations in position and velocity is such that these fluctuations may be reduced indefinitely by lowering the ambient temperature. In addition, if one goes to times comparable to the collision period of the ambient medium, the trajectories become relatively smooth. Over very short times the motion is entirely rectilinear (in the absence of external forces) and deterministic.

It is natural to assume that the Brownian motion analogy is complete, and that a kinetic model can be used to introduce a fundamental time scale into quantum mechanics. This time scale, corresponding to the collision period of fictitious interactions between the particle and the vacuum, is chosen so that for sufficiently long times conventional quantum mechanics is recovered. This idea is present, in embryonic form, in the work of Fenyes,[14] Weizel,[15] Guth,[18,19] and Bohm.[5,7]

In the present paper, a hidden parameter β, corresponding to the above-mentioned time-scale parameter, is introduced into the nonrelativistic propagator formalism, in conjunction with a hidden variable $\boldsymbol{\xi}$, which plays the role of particle velocity. The space-time approach in Sec. II is then generalized so that the probability amplitude becomes the sum of path integrals taken over all possible trajectories in the phase space. For short times the dominant trajectories indeed tend to be deterministic and causal in violation of the uncertainty principle, and for long times the trajectories once again take on the Brownian motion appearance of the Feynman formalism. The crucial point in the discussion is whether or not in nature there actually exists a subquantum-mechanical time scale, characterized by finite values of the hidden parameter β. If so, there should appear, for energies of order $\hbar\beta$, shifts in the effective mass of the electron proportional to these energy values.[20] The current theory does not violate von Neumann's famous theorem[21,22] against hidden variable formulations of quantum theory, since the hidden variable in this case turns out to be non-

[6] D. Bohm, Phys. Rev. **85**, 166, 180 (1952).
[7] D. Bohm and J. P. Vigier, Phys. Rev. **96**, 208 (1954).
[8] For a survey of the deterministic theories by Bohm and de Broglie, see H. Freistadt, Nuovo Cimento Suppl. **5**, 1 (1957).
[9] W. Heisenberg, essay in book cited in Ref. 1.
[10] R. P. Feynman, *Theory of Fundamental Processes* (W. A. Benjamin, Inc., New York, 1961), p. 50.
[11] J. P. Wesley, Phys. Rev. **122**, 1932 (1961).
[12] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).
[13] R. Fürth, Z. Physik **81**, 143 (1933).
[14] I. Fenyes, Z. Physik **132**, 81 (1952).
[15] W. Weizel, Z. Physik **134**, 264 (1953); **135**, 270 (1953).
[16] N. Wiener, *Nonlinear Problems in Random Theory* (MIT Press, Cambridge, Massachusetts, 1958), Chap. 9; N. Wiener and A. Siegel, Phys. Rev. **91**, 1551 (1953).
[17] E. Nelson, J. Math. Phys. **5**, 332 (1964).
[18] E. Guth, Phys. Rev. **126**, 1213 (1962).
[19] E. Guth, in Proceedings of the Midwestern Conference on Theoretical Physics, Argonne National Laboratory, 1962 (unpublished).
[20] In this connection see H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1953), p. 314.
[21] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), p. 323 ff.
[22] J. M. Jauch and C. Piron, Helv. Phys. Acta **36**, 827 (1963).

Hermitian for large values of βt, as is discussed in detail in Sec. III. It should be emphasized that the present Brownian model differs from its predecessors in that it is completely linear, and that it preserves the statistical interpretation of the Schrödinger waves for long times. The determinism only appears in the subquantum level of description for times too short for the randomizing effects of the particle-vacuum interaction to operate. As a consequence, certain unphysical results contained in Weizel's work[15] do not arise here, as discussed in Sec. IV.

II. GENERALIZATION OF NONRELATIVISTIC PROPAGATOR THEORY

In the space-time approach of Feynman[12] the wave function $\psi(\mathbf{x},t)$ for a single nonrelativistic particle of mass m in a potential field $V(\mathbf{x})$ is given by the integral relation

$$\psi(\mathbf{x},\,t+\Delta t)=\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{3/2}\times\int\exp\left[\frac{im(\Delta\mathbf{x})^2}{2\hbar\Delta t}-i\frac{V\Delta t}{\hbar}\right]\psi(\mathbf{x}-\Delta\mathbf{x},\,t)d\Delta\mathbf{x}. \quad (1)$$

This expression is equivalent to the usual Schrödinger equation

$$i\hbar(\partial\psi/\partial t)=(-\hbar^2/2m)\nabla^2\psi+V\psi\,, \quad (2)$$

as may be seen by expanding all functions in Taylor series about the point $(\mathbf{x},t)$, performing the integration over $\Delta\mathbf{x}$, and passing to the limit $\Delta t\to 0$. Equation (1) may also be extended to the many-particle problem by introducing additional position coordinates. The exponential factor is just the Green's function or propagator which takes the particle from $(\mathbf{x}-\Delta\mathbf{x},\,t)$ to $(\mathbf{x},\,t+\Delta t)$. The free-particle propagator $(m/2\pi i\hbar\Delta t)^{3/2}\times\exp[im(\Delta\mathbf{x})^2/2\hbar\Delta t]$ resembles the Brownian-motion[23] Green's function $(4\pi D\Delta t)^{-3/2}\exp[-(\Delta\mathbf{x})^2/4D\Delta t]$, where the diffusion constant D depends on the state of the fluid in which the Brownian particle is immersed. In fact, we have the formal equivalence

$$D=i\hbar/2m\,, \quad (3)$$

first noted by Fürth.[13]

Using the mathematical apparatus of Brownian motion as a guide, we may introduce as a hidden variable the particle velocity $\boldsymbol{\xi}$, and a distribution function for the probability amplitude, namely Φ, defined such that

$$\psi(\mathbf{x},t)=\int\Phi(\mathbf{x},\boldsymbol{\xi},t)d\boldsymbol{\xi}. \quad (4)$$

That $\boldsymbol{\xi}$ may be consistently thought of as classical particle velocity, at least under limited circumstances, is explored more thoroughly in the next section of this paper. We further assume that the distribution of probability Φ is given by the Green's function expression

$$\Phi(\mathbf{x},\,\boldsymbol{\xi},\,t+\Delta t)=\int\int K(\Delta\mathbf{x},\boldsymbol{\xi},\boldsymbol{\xi}',\Delta t)e^{-iV(\mathbf{x})\Delta t/\hbar}\times\Phi(\mathbf{x}-\Delta\mathbf{x},\,\boldsymbol{\xi}',\,t)d\Delta\mathbf{x}d\boldsymbol{\xi}', \quad (5)$$

where the generalized free-particle propagator $K(\mathbf{x},\boldsymbol{\xi},t)$ is governed by the Fokker-Planck equation[23,24]

$$\partial K/\partial t+\boldsymbol{\xi}\cdot\nabla K=\beta\nabla_{\xi}\cdot(\boldsymbol{\xi}K+v_0^2\nabla_{\xi}K)\,. \quad (6)$$

Here the hidden parameter β plays the role of a fictitious collision frequency, as is discussed in Sec. III, and v_0^2 plays the role of a fictitious thermal speed in analogy with Brownian motion; the latter quantity may be eliminated by means of the Einstein relation

$$v_0^2=\beta D\,, \quad (7)$$

where the diffusion constant D is given by the Fürth relation (3). The equivalent differential equation is

$$\frac{\partial\Phi}{\partial t}+\frac{iV}{\hbar}\Phi+\boldsymbol{\xi}\cdot\nabla\Phi=\beta\nabla_{\xi}\cdot(\boldsymbol{\xi}\Phi+v_0^2\nabla_{\xi}\Phi)\,, \quad (8)$$

which follows from Eq. (5) upon expanding both sides about $(\mathbf{x},\boldsymbol{\xi},t)$ and utilizing Eq. (6) in passing to the limit $\Delta t\to 0$.

It has been shown by Smoluchowski[25] that for sufficiently large values of βt, the distribution attains the asymptotic form

$$\Phi(\mathbf{x},\boldsymbol{\xi},t)=(2\pi v_0^2)^{-3/2}\exp(-\xi^2/2v_0^2)\psi(\mathbf{x},t)\,, \quad (9)$$

where ψ satisfies

$$\partial\psi/\partial t=\nabla\cdot(D\nabla\psi)-(i/\hbar)V\psi. \quad (10)$$

But this is just the Schrödinger equation (2) in slightly disguised form, hence we have arrived at a correspondence principle between the generalized theory and ordinary quantum theory for sufficiently long times. Now, Smoluchowski also showed that for a Brownian particle in a slightly inhomogeneous potential field U, one obtains the asymptotic equation

$$\frac{\partial\psi}{\partial t}=\nabla\cdot\left(D\nabla\psi+\frac{1}{\beta}\psi\nabla U\right)-\frac{i}{\hbar}V\psi. \quad (11)$$

Even for large values of β, however, this last expression does not seem to be useful, since the additional gradient term in Schrödinger's equation would constitute a dissipative mechanism and thereby render quantum mechanics non-Hamiltonian. Consequently, the only sensible choice is to set $U=0$.

[23] For a comprehensive review of Brownian motion, see S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943), especially Chap. II.

[24] Of course, a consistency requirement for (5) is that at time $\Delta t=0$, $K=\delta(\Delta x)\delta(\xi-\xi')$, where δ is the three-dimensional Dirac delta function.

[25] Reference 20, p. 41.

Another consequence of the asymptotic solution (9) is that the generalized propagator K, defined by Eq. (6), eventually becomes ($\beta t \gg 1$)

$$K=\frac{1}{(4\pi D\Delta t)^{3/2}}\exp\left(-\frac{(\Delta\mathbf{x})^2}{4D\Delta t}\right)\frac{1}{(2\pi v_0^2)^{3/2}}\times\exp\left(-\frac{(\boldsymbol{\xi})^2}{2v_0^2}\right), \quad (12)$$

and so from Eqs. (4) and (5) we arrive at the Feynman formalism

$$\psi(\mathbf{x},t+\Delta t)=\int\bar{K}(\Delta\mathbf{x},\Delta t)\psi(\mathbf{x}-\Delta\mathbf{x},t)d\Delta\mathbf{x}, \quad (13)$$

where

$$\bar{K}=\int K(\Delta\mathbf{x},\boldsymbol{\xi},\Delta t)d\boldsymbol{\xi}=(4\pi D\Delta t)^{-3/2}\times\exp[-(\Delta\mathbf{x})^2/4D\Delta t]\exp[-iV(\mathbf{x})\Delta t/\hbar]. \quad (14)$$

For finite values of βt one may perform numerical calculations of the generalized wave functions by means of Eqs. (4) and (5), since the solutions of the free-particle Green's function K can be written in closed form.[26] In certain simple cases the differential equation (8) itself may be solved by separation of variables and transform techniques, as discussed in the next section. It must be stressed that in all such calculations the hidden variable $\boldsymbol{\xi}$ plays an essential part.

Before concluding the present section, we observe that from Brownian motion theory[27] we may write an effective diffusion constant

$$D_*=\frac{i\hbar}{2m_*}=\frac{i\hbar}{2m}\left(1-\frac{1-e^{-\beta\Delta t}}{\beta\Delta t}\right). \quad (15)$$

For long times $\beta\Delta t\gg 1$ this expression gives a small change in the effective mass of a particle amounting to

$$(m_*-m)/m=\Delta m/m=1/\beta\Delta t. \quad (16)$$

Utilizing the energy-time uncertainty relation, we obtain, for energies $E=(\hbar/\Delta t)=\hbar\omega$, the approximate expression

$$\Delta m/m\sim E/\hbar\beta. \quad (17)$$

The observation of such a mass shift would constitute a determination of the hitherto unspecified hidden parameter β, in the same sense that Brownian motion measurements lead to a determination of the Boltzmann constant of statistical mechanics. It should be noted that in a proper relativistic treatment additional mass shifts $\Delta m/m\sim E/mc^2$ would probably arise. Clearly, the subquantum-mechanical-energy scale $\hbar\beta$ must greatly exceed the particle rest energy mc^2; consequently, these relativistic effects would tend to mask the shift given by Eq. (17) in an actual high-energy experiment.

[26] Reference 20, Eq. (286).
[27] Reference 20, Eq. (175′).

III. OCCURRENCE OF CAUSALITY ON THE SUBQUANTUM-MECHANICAL LEVEL OF DESCRIPTION

For times short compared to β^{-1}, causality or determinism enters the picture. To see how this follows from the generalized Schrödinger equation (8), we observe that for small values of βt the right-hand side may be neglected in comparison with the left. This is just the collisionless approximation so often used in rarefied-gas dynamics. In the case of a free particle (i.e., $V=0$), we have

$$\partial\Phi/\partial t+\boldsymbol{\xi}\cdot\nabla\Phi=0, \quad (18)$$

which has the solution $\Phi(\mathbf{x},\boldsymbol{\xi},t)=f(\mathbf{x}-\boldsymbol{\xi}t,\boldsymbol{\xi})$, where the function f is determined by the initial distribution $\Phi(\mathbf{x},\boldsymbol{\xi},0)=f(\mathbf{x},\boldsymbol{\xi})$. Clearly, special solutions may easily be written down which violate the uncertainty principle; for example,

$$\Phi(\mathbf{x},\boldsymbol{\xi},t)=\delta(\mathbf{x}-\boldsymbol{\xi}t)\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0). \quad (19)$$

Substitution of (19) into (4) yields the traveling wave function

$$\psi(\mathbf{x},t)=\delta(\mathbf{x}-\boldsymbol{\xi}_0 t), \quad (20)$$

which should be contrasted with the Schrödinger solution for the initial condition $\psi(\mathbf{x},0)=\delta(\mathbf{x})$, namely: $\psi=(m/2\pi i\hbar\Delta t)^{3/2}\exp[im\mathbf{x}^2/(2\hbar\Delta t)]$. The Schrödinger solution responds to the perfect initial localization by spreading infinitely fast; that is to say, $|\psi|^2=$const for nonzero times. On the other hand, once solution (20) is localized in position and velocity it remains so. Furthermore, the Ehrenfest theorem corresponding to (20) is just $d\langle\mathbf{x}\rangle/dt=\boldsymbol{\xi}_0$, rather than the form dictated by uncertainty, namely: $d\langle\mathbf{x}\rangle/dt=\langle -i\hbar\partial/\partial\mathbf{x}\rangle$.

To gain deeper insight into the physical and mathematical properties of the Brownian-motion model let us consider the stationary solutions corresponding to a free particle in a perfectly reflecting box of length L. Fourier-analyzing away the spatial dependence by setting

$$\Phi=\exp(i\mathbf{k}_p\cdot\mathbf{x}-i\omega t)F(\boldsymbol{\xi}), \quad (21)$$

where $\mathbf{k}_p=\mathbf{p}\pi/L$, $|\mathbf{p}|=0,\pm1,\pm2,\cdots$, we get for (8)

$$(-i\omega+i\mathbf{k}_p\cdot\boldsymbol{\xi})F(\boldsymbol{\xi})=\beta\nabla_\xi\cdot(\boldsymbol{\xi}F+v_0^2\nabla_\xi F). \quad (22)$$

By means of the transformations $\boldsymbol{\xi}=V_0^2\boldsymbol{\zeta}$, $V_0^2=2v_0^2$, $F=\exp(-\frac{1}{2}\zeta^2)G$, this expression becomes

$$\nabla_\zeta^2 G+[\gamma-(\boldsymbol{\zeta}+\mathbf{a})^2]G=0, \quad (23)$$

where $\gamma=3+2i\omega/\beta-k_p^2V_0^2/\beta^2$, $\mathbf{a}=i\mathbf{k}_pV_0/\beta$. But (23) is merely the Schrödinger equation (in velocity space) for a harmonic oscillator. Hence the eigenvalues are $\gamma=2l+2m+2n+3$ ($l,m,n=0,1,2,\cdots$) and the corresponding eigenfunctions are $G_{lmn}(\boldsymbol{\zeta})=G_{1l}(\zeta_x)G_{2m}(\zeta_y)$

$\times G_{3n}(\zeta_z)$, where, for k chosen along the z axis, we have

$$G_{1l}(\zeta_x)=N_l\exp(-\tfrac{1}{2}\zeta_x^2)H_l(\zeta_x)\,,$$
$$G_{2m}(\zeta_y)=N_m\exp(-\tfrac{1}{2}\zeta_y^2)H_m(\zeta_y)\,,\qquad(24)$$
$$G_{3n}(\zeta_z)=N_n\exp[-\tfrac{1}{2}(\zeta_z+a)^2]H_n(\zeta_z+a)\,,$$

in which $a=ik_pV_0/\beta$, H_l is the Hermite polynomial satisfying the equation $H''-2\zeta H'+2lH=0$, the normalization constant is $N_l=(l!2^l\pi^{1/2})^{-1/2}$, and so forth. The dispersion relation connecting ω and k_p must therefore be of the form $\omega=-i\beta(k_p^2V_0^2/2\beta^2+l+m+n)$. Stationary solutions correspond to $l=m=n=0$, so that

$$\omega=-ik_p^2V_0^2/2\beta=\hbar k_p^2/2m\,.\qquad(25)$$

Notice that (25) is just the eigenvalue predicted by the Schrödinger equation for an electron in a box. Of course, one would not expect a stationary solution to exhibit any time scale.

However, the situation is not entirely trivial as may be seen by combining (21), (22), and (24) for the case $l=m=n=0$:

$$\Phi(\mathbf{x},\boldsymbol{\xi},t)=\left(\frac{m}{2i\pi\beta\hbar}\right)^{3/2}\exp\left[-\frac{m}{2i\hbar\beta}\left(\boldsymbol{\xi}-\frac{\hbar\mathbf{k}_p}{m}\right)^2\right]\times\exp(i\mathbf{k}_p\cdot\mathbf{x}-i\omega_pt)\,.\qquad(26)$$

The above relation is completely equivalent to the usual wave function, and the hidden variable $\boldsymbol{\xi}$ is seen to be merely a dummy variable which is eliminated upon substitution of (26) into (4). Yet it is interesting to note that $m\langle\boldsymbol{\xi}\rangle=\hbar\mathbf{k}_p$ in accordance with the de Broglie relation. Moreover, the mean-square fluctuation $\langle(\Delta\boldsymbol{\xi})^2\rangle$ is proportional to β, indicating that the hidden variable $\boldsymbol{\xi}$ *strictly* represents particle velocity only for small values of β and is spread out greatly for large values of β. This behavior is in harmony with the role assigned to $\boldsymbol{\xi}$ in the preceding discussion. For the general nonstationary case, initial value solutions must be built up by means of linear superpositions of the above Hermite velocity-eigenfunctions.[28] From the unnormalized version of (24) it is clear that the spread in $\boldsymbol{\xi}$ is always governed by the parameter β.

The Fokker-Planck formalism presented in Sec. II is entirely equivalent to the complex Langevin equation[23]

$$d\boldsymbol{\xi}/dt=-\beta\boldsymbol{\xi}+i^{1/2}\mathbf{A}(t)\,,\qquad(27)$$

in which the quantity $\mathbf{A}$ is a real stochastic Gaussian process representing the effects of collisions on a Brownian particle. Due to the presence of the complex coefficient $i^{1/2}$, we must conclude that the randomizing process in quantum mechanics involves fictitious collisions described by trajectories in a phase space with complex values of velocity $\boldsymbol{\xi}$. This indicates that the statistical interpretation of quantum theory is real, not apparent, and that the deterministic structure of the subquantum-mechanical level of description can never reveal itself by measurements taken over long times. It is only over time scales so short that the randomizing mechanism has not sufficient time to operate that one can detect the deterministic microstructure underlying quantum mechanics. In other words, initially, wave functions may be localized without spreading and feature real velocities satisfying the relation $d\langle\mathbf{x}\rangle/dt=\boldsymbol{\xi}_0$. Eventually the randomization mechanism, here visualized in terms of fictitious collisions of the particles with the vacuum, begins to operate forcing the hidden variable $\boldsymbol{\xi}$ to take on complex values and to develop a spread about its initial value. This is accompanied by a corresponding spread in the wave function according to the usual uncertainty relation $d\langle\mathbf{x}\rangle/dt=\langle-i\hbar\nabla\rangle$. As $\boldsymbol{\xi}$ ceases to be strictly real, it ceases to be a dynamical observable in the quantum-mechanical sense, since, although it continues to commute with the position $\mathbf{x}$, it no longer is Hermitian. For this reason the present hidden variable model lies outside the conditions of the von Neumann theorem,[21,22] which states that quantum mechanics is logically complete since no further specification of Hermitian hidden variables can cause it to become deterministic.

One may consider the present model a mathematical bridge leading from an intrinsically statistical quantum-mechanical world of low-energy processes to an intrinsically deterministic subquantum-mechanical world of very high-energy processes. The curious feature is that imaginary diffusion and complex collisions must be introduced into the discussion. This is analogous to the phase-space, probability distribution functions used by Wigner[29] to provide a bridge between quantum statistical mechanics and classical statistical mechanics. The difficulty in the case of Wigner's distribution functions is that they do not necessarily represent strictly positive-definite probabilities for finite values of $\hbar$, as is well known.

IV. COMPARISON WITH THE HIDDEN VARIABLE MODELS OF WEIZEL AND OF WIENER AND DELLA RICCIA

The present model may be considered an improvement on the Brownian-motion model of Weizel,[15] which in turn is based on the deterministic models of Bohm[6,7] and Fenyes.[14] As was discussed previously, the present model is not built up completely from deterministic notions, but only displays causality on the subquantum-mechanical level. On the other hand, in the work of Weizel one assumes that for Schrödinger waves of the form $\psi=R\exp(iS/\hbar)$, where R and S are real functions of space and time, there exists a meaningful and actual particle momentum given by $\mathbf{p}=\nabla S$. The Schrödinger equation (2) can then be used to derive an equation of

[28] The necessary mathematical techniques were developed in a different connection by E. C. Taylor and G. G. Comisar, Phys. Rev. 132, 2379 (1963).

[29] E. P. Wigner, Phys. Rev. 40, 749 (1932).

continuity

$$\partial\rho/\partial t+\nabla\cdot(\rho\mathbf{U})=0\,, \tag{28}$$

where an effective fluid density is given by $\rho=R^2$, and an effective flow velocity by $\mathbf{U}=m^{-1}\nabla S$. In addition, one gets a momentum conservation equation

$$m\frac{d\mathbf{U}}{dt}=-\nabla\left\{V-\frac{\hbar^2}{8m}\left[\frac{\nabla^2\rho}{\rho}-\left(\frac{\nabla\rho}{\rho}\right)^2\right]\right\}\,, \tag{29}$$

where the extra potential terms are intrinsically quantum mechanical in nature.

To shed light on the deterministic structure of his model, Weizel then shows that if one begins with (28) and the relation

$$\mathbf{U}_{\mathrm{av}}=\mathbf{v}_0-\hbar/2m\nabla\ln\rho\,, \tag{30}$$

which he derives from a kinetic argument in which $\mathbf{U}_{\mathrm{av}}$ and $\mathbf{v}_0$ stand, respectively, for the average particle velocity and the velocity of that fraction of particles actually suffering collisions per unit time, then with the help of these two relations one can deduce the momentum equation (29). This would seem to constitute an actual deterministic proof of the Schrödinger equation. However, the physical consistency of expression (30) seems doubtful.

In particular, let us substitute $\mathbf{U}_{\mathrm{av}}=\mathbf{U}=m^{-1}\nabla S$ into Eq. (30). This reduces to

$$\mathbf{v}_0=(\hbar/2m)\nabla\ln(R^2\exp 2S/\hbar)\,, \tag{31}$$

which, for the case of a particle between two perfectly reflecting walls, turns out to be equal in magnitude to

$$v_0=\frac{\hbar k_n}{m}\cot k_n x\,, \tag{32}$$

where, as before, $k_n=\pi n/L$, $n=0,\ \pm1,\ \pm2,\cdots$. But this expression has poles at all the nodal planes, which seems unreasonable owing to the fact that v_0 is a quantity that has been averaged over a supposedly well-behaved ensemble of particles. The present treatment avoids such difficulties by investigating ψ rather[30] than $|\psi|^2$.

There exists another interesting approach to the hidden variable problem due to Wiener and Della Riccia.[31] Starting with the many-particle Liouville equation of classical statistical mechanics

$$\frac{\partial F_N}{\partial t}+\sum_{i=1}^{N}\frac{\mathbf{p}_i}{m}\cdot\frac{\partial F_N}{\partial\mathbf{q}_i}-\sum_{i=1}^{N}\frac{\partial V}{\partial\mathbf{q}_i}\cdot\frac{\partial F_N}{\partial\mathbf{p}_i}=0\,, \tag{33}$$

where $F_N(\mathbf{q},\mathbf{p},t)$ is the N-body distribution function in the $6N$-dimensional phase space $(\mathbf{q},\mathbf{p})$ and V is the potential energy for the system, one may project out the momentum variables $\mathbf{p}_i$ by means of the expression

$$\psi(\mathbf{q},t)=\int F_{\mathrm{N}}(\mathbf{q},\mathbf{p},t)\exp\left(-\lambda\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m}\right)d\mathbf{p}\,, \tag{34}$$

where λ is an initially unspecified parameter. After a great deal of straightforward algebra, one arrives at

$$\lambda\frac{\partial^2\psi}{\partial t^2}=\frac{1}{2m}\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial\mathbf{q}_i^2}+P(\mathbf{q})\psi\,, \tag{35}$$

where

$$P(\mathbf{q})=\frac{\lambda}{2m}\left[\sum_{i=1}^{N}\frac{\partial^2 V}{\partial\mathbf{q}_i^2}-\lambda\left(\frac{\partial V}{\partial\mathbf{q}_i}\right)^2\right]. \tag{36}$$

Except for the potential term, this is just the Klein-Gordon equation of relativistic quantum theory. By the usual trick of subtracting off the rest energy, one obtains the N-body Schrödinger equation

$$2\lambda\, mc^2\frac{\partial\psi}{\partial t}=\frac{1}{2m}\sum_{i=1}^{N}\frac{\partial^2\psi}{\partial\mathbf{q}_i^2}+P(\mathbf{q})\psi\,. \tag{37}$$

We can find several points of contact between this approach and that presented in Sec. II. First, the hitherto unspecified parameter λ turns out to be imaginary and, in fact, corresponds to an imaginary temperature of the sort implied by Eq. (7). This rules out a direct causal passage from classical statistical mechanics to quantum mechanics. Second, the initial classical distribution corresponds to ψ rather than $|\psi|^2$. Third, there is lacking the kind of correspondence principle that would always leave the potential energy invariant in passing from the subquantum to the quantum domain.[32]

The difference between the Wiener model and the present one lies in the fact that the former starts from a reversible point of view, while the latter starts from an irreversible kinetic formalism. This is the reason that the Wiener model leads to a differential equation that is hyperbolic, while the present leads to a parabolic (diffusion) equation. Since irreversibility has been introduced into the Brownian motion formalism from the outset, it is not surprising that this particular system tends to become Maxwellian with the passage of time. On the other hand, in the Liouville approach, into which no irreversibility has yet been inserted either explicitly or implicitly, it is difficult to see the justification for using Maxwellian-type solutions as was done in Eq. (34).

[30] These remarks also apply to the recent work of D. Kershaw [Phys. Rev. **136**, B 1850 (1964)]; for example, in his text compare the equation following Eq. (15) with our Eq. (30).

[31] N. Wiener and G. Della Riccia (private communication).

[32] Compare Eqs. (11) and (36).