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Regge Trajectories versus Vanishing Renormalization Constants as Dynamical Criteria*

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The connection between continuous Regge trajectories and the vanishing of renormalization constants is explored. It is found that if, in a field theory $Z_1 \to 0$ and $Z_3 \to 0$ in such a way that $Z_1/Z_3 \to 0$, then a Regge trajectory moves smoothly under an elementary particle pole so that the particle becomes dynamical in the Regge sense. Thus a bootstrapped world may perhaps equally well be defined by its satisfying a field theory with all renormalization constants set equal to zero, as by saying that all particles lie on Regge trajectories.

I. INTRODUCTION

WHEN Chew and Frautschi first suggested the bootstrap hypothesis,[1] they formulated their idea in terms of Regge poles. They postulated that each strongly interacting particle should appear on a Regge trajectory, and there should be no fixed poles in l, i.e., no Kronecker delta in the S matrix. Such a requirement was presumed to guarantee the dynamical nature of each particle.

Within the framework of conventional Lagrangian field theories, it has been suggested by several people[2,3] that an elementary particle may be made to look dynamical by imposing the requirement that its wave function renormalization constant vanish. Thus a normal type of field theory, with this added condition on all wave function renormalization constants, might also define a bootstrapped world.

We should like to explore the connection between these two languages. In particular, we should like to follow in some detail the effects, in a Lagrangian field theory, of turning off the renormalization constants of a given elementary particle. We shall find that as $Z_1/Z_3 \to 0$, where Z_1 is the vertex renormalization associated with the coupling of the elementary particle and Z_3 is its wave function renormalization, a Regge trajectory moves smoothly under the elementary particle pole, and passes through it, thus making the particle dynamical. If at the same time $Z_3 \to 0$, then the residue associated with the trajectory smoothly approaches the same residue as the pole had when it was elementary. The combined limit thus replaces the elementary particle by a superficially indistinguishable dynamical one. Our analysis will be restricted by the assumption of elastic unitarity, but is otherwise general. To be specific, we shall discuss a theory of, say, spinless "pions" and spinless "nucleons" and shall focus our attention on the "pion" pole in "nucleon-antinucleon" scattering.

Our conclusion is thus that both definitions of "dynamical" agree, so that the bootstrap hypothesis can indeed be stated equally well in the context of (an unfortunately not yet completely well-defined) "S-matrix theory," with Regge boundary conditions, or in the context of Lagrangian field theory, with *all* renormalization constants set equal to zero.

There is one clarification to be made with respect to this conclusion. For the case of certain problems with spin, the condition that a *given* particle lie on a Regge trajectory is *not* by itself sufficient to guarantee that the particle is dynamical.[4] However, it is conjectured that in spite of this, if *all* particles in the theory have to lie on trajectories, then the theory is still a bootstrapped one.[5] Possibly analogous statements apply to the $Z=0$ conditions.

Finally, it may be worth remarking on the fact that neither of these ways of defining a bootstrap theory is as yet entirely satisfactory. Intuitively, a bootstrap theory should be something like the following: One should start with some set of information sufficient to specify a theory, e.g., the number of particles, their quantum numbers, masses, couplings, etc. The theory is then defined in the sense that the given input is sufficient, when implemented by some method of calculation, to predict all observable quantities. These quantities, which constitute the output, will include the number and type of all particles, stable and unstable, and their masses and couplings, as well as all scattering amplitudes, production amplitudes, etc. The bootstrap requirement is the requirement that the input and output be the same. Now it is obvious that one needs a precise mathematical condition which can replace this somewhat nebulous intuitive idea, but which will ensure the principle expressed by it. It is this which either the

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1 G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 7, 394 (1962).

2 A. Salam, Nuovo Cimento 25, 224 (1962).

3 B. Lee, K. Mahanthappa, I. Gerstein, and M. Whippman, Ann. Phys. (N. Y.) 28, 466 (1964). Other references are given in this paper.

4 M. Gell-Mann, M. Goldberger, F. Low, V. Singh, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

5 S. Mandelstam, Phys. Rev. 137, B949 (1965).

condition of Regge trajectories or the conditions $Z=0$ purport to do. However, the condition of Regge behavior, which is a condition on the analytic properties of S-matrix elements, presupposes that one knows how to write down *all* these matrix elements sufficiently explicitly to permit the analytic properties to be exhibited, and this cannot yet be done. On the other hand, the condition $Z=0$ probably presupposes that a Lagrangian field theory with well-defined Z's can be constructed which includes high spins, and this we do not know how to do either.

II. DERIVATION OF THE DYNAMICAL CONDITIONS

Let us suppose we are given some input for an N over D calculation of nucleon-antinucleon scattering in a field theory (we assume spinless nucleons) which includes an elementary particle pole representing a spinless pion of mass m coupled to the nucleons with a renormalized coupling constant g. We may write the input as

$$\bar{B}_l(s)=B_l(s)-[g^2/(s-m^2)]\delta_{l0}, \qquad (2.1)$$

where $B_l(s)$ is presumed to be an analytic function of l except for certain well-defined singularities. If, for example, we use as an illustration the standard type of bootstrap calculation, the two terms in (2.1) will represent the diagrams of Fig. 1, and $B_l(s)$ will be proportional to a Q_l function with poles at all negative integer values of l.

Our input will generate an N function of the form

$$\bar{N}_l(s)=N_l(s)+\left(-\frac{g^2\bar{D}_0(m^2)}{s-m^2}+F(s)\right)\delta_{l0}, \qquad (2.2)$$

where

$$F(s)=\frac{1}{\pi}\int\frac{\rho(s')ds'}{s'-s}(B_0(s')-B_0(s))\left(-\frac{g^2\bar{D}_0(m^2)}{s'-m^2}+F(s')\right)$$

and

$$N_l(s)=B_l(s)+\frac{1}{\pi}\int\frac{\rho(s')ds'}{s'-s}(B_l(s')-B_l(s))N_l(s'), \qquad (2.3)$$

so that $F(s)$ is proportional to $\bar{D}_0(m^2)$. Finally, from Eq. (2.2) we can construct the D function

$$\bar{D}_l(s)=D_l(s)+G(s)\delta_{l0}, \qquad (2.4)$$

where

$$D_l(s)=1-\frac{1}{\pi}\int\frac{\rho(s')ds'}{s'-s}N_l(s'), \qquad (2.5)$$

and

$$G(s)=\frac{g^2\bar{D}_0(m^2)}{\pi}\int\frac{\rho(s')ds'}{(s'-m^2)(s'-s)}-\frac{1}{\pi}\int\frac{\rho(s')ds'}{s'-s}F(s'). \qquad (2.6)$$

In these equations, $\rho(s)$ is a phase-space factor, resulting from the normalization we use for the partial-wave scattering amplitude,

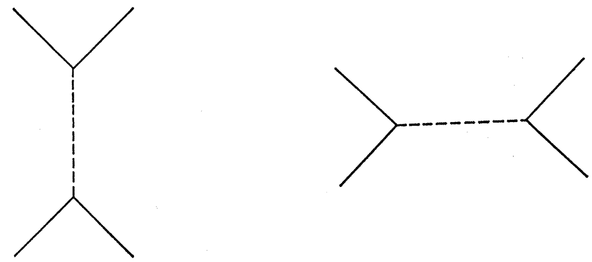

FIG. 1. Diagrams representing the input $\bar{B}_l(s)$ in the usual type of model.

$$t_l(s)=\frac{\bar{N}_l(s)}{\bar{D}_l(s)}=\frac{1}{\rho(s)}\sin\delta_l(s)e^{i\delta_l(s)}. \qquad (2.7)$$

No subtraction is explicitly indicated in the D functions (i.e., we have subtracted at infinity) but we could have made one at any other point at no cost, since the results of the N over D method are well known to be independent of a subtraction in D.

If we had been doing the calculation of nucleon-antinucleon scattering without the presence of the elementary pion, our resulting scattering amplitude would have been simply

$$t_l(s)=N_l(s)/D_l(s). \qquad (2.8)$$

As is easily seen from the model indicated by Fig. 1, this $t_l(s)$ may be expected to be a smooth function of l, containing only Regge poles. Thus, we may write

$$N_l(s)/D_l(s)=\textstyle\sum_n\beta_n(s)/(l-\alpha_n(s)) \qquad (2.9)$$

plus other terms containing cuts or what have you in l, but at least containing no Kronecker deltas.

Next we may separate out the δ_{l0} terms in Eq. (2.7) explicitly. A minimal amount of algebra yields

$$t_l(s)=\sum_n\frac{\beta_n(s)}{l-\alpha_n(s)}+\delta_{l0}\left[-\frac{g^2\bar{D}_0(m^2)}{s-m^2}+F(s)-\frac{N_0(s)G(s)}{D_0(s)}\right]\frac{1}{\bar{D}_0(s)} \qquad (2.10)$$

plus whatever other terms occurred in Eq. (2.9).

Dynamical bound states of $t_0(s)$ occur when $\bar{D}_0(s)=0$. Hence, when $\bar{D}_0(m^2)$ is small, we can expect there to be a bound state $s=\bar{s}_0$ near m^2. Furthermore, since $\bar{D}_0(s)=D_0(s)+G(s)$, and since $G(s)\to 0$ as $\bar{D}_0(m^2)\to 0$, associated with the point $\bar{s}_0$ there will be a zero of $D_0(s)$; call it s_0. This point must correspond to some trajectory, call it $\alpha(s)$, passing through $l=0$ at $s=s_0$, and s_0 will also approach m^2 as $\bar{D}_0(m^2)\to 0$. Finally, because of the existence of the pole in $\bar{N}_0(s)$, there will be a zero of $\bar{N}_0$ at a point $s=m_0^2$ which gets near to m^2 as the residue of the pole weakens.

Let us summarize the geometry:

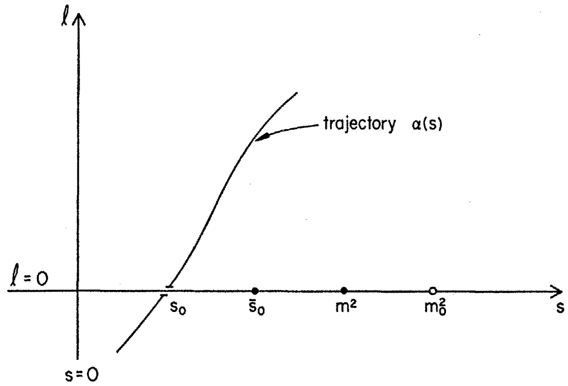

FIG. 2. Poles and zeros of the partial wave amplitude $t_l(s)$. There are two cancelling poles on s_0, poles on $\bar{s}_0$ and m^2, and a zero on m_0^2.

(i) s_0 is the point at which the trajectory $\alpha(s)$ passes through $l=0$: $\alpha(s_0)=0$. The residue here is $R\equiv-\beta(s_0)/\alpha'(s_0)$.

(ii) $\bar{s}_0$ is a zero of $\bar{D}_0(s):\bar{D}_0(\bar{s}_0)=0$. The residue of this pole in $t_0(s)$ is $\bar{R}=\bar{N}(\bar{s}_0)/\bar{D}'(\bar{s}_0)$.

(iii) m_0^2 is a zero of $\bar{N}(s):\bar{N}(m_0^2)=0$.

Obviously, we do not insist that $\bar{s}_0$ is the *only* zero of $\bar{D}_0(s)$; it is the zero which approaches m^2 in the limit $\bar{D}_0(m^2)\rightarrow 0$.

Note that, although $\alpha(s_0)=0$, the amplitude does *not* in fact have a pole at s_0, since the δ_{l0} term in (2.8) also contains a pole at s_0 exactly cancelling the one on the Regge trajectory. Altogether, then, we have the structure indicated in Fig. 2. $t_0(s)$, as expressed by (2.10), has a dynamical bound state at $s=\bar{s}_0$, another pole at $s=m^2$, and two *cancelling* poles at s_0, and it has a zero at m_0^2. This zero at $s=m_0^2$ is the position of the Castillejo-Dalitz-Dyson (CDD) pole expressing the existence of the elementary particle at $s=m^2$. In spite of the notation, we do not mean to imply that m_0^2 has anything to do with the bare mass of the elementary particle.

As we have already said, in the limit $\bar{D}_0(m^2)\rightarrow 0$, the entire effect of the elementary particle goes away. This not very subtle remark should be no surprise to anyone remembering Eq. (2.2). Nevertheless, it is of some interest to watch the transition from the elementary case to $\bar{D}_0(m^2)=0$. For small $\bar{D}_0(m^2)$, we may write

$$\bar{s}_0=m^2+\bar{\epsilon}, \tag{2.11}$$

with

$$\bar{\epsilon}=-\bar{D}_0(m^2)/\bar{D}_0'(m^2); \tag{2.12}$$

$$s_0=m^2+\epsilon,$$

with

$$\epsilon=-\frac{\bar{D}_0(m^2)-G(m^2)}{\bar{D}_0'(m^2)-G'(m^2)}; \tag{2.13}$$

and finally

$$m_0^2=m^2+\delta, \tag{2.14}$$

with

$$\delta=g^2\bar{D}_0(m^2)/N_0(m^2). \tag{2.15}$$

When we remember that $G(m^2)\rightarrow 0$ as $\bar{D}_0(m^2)\rightarrow 0$, it is clear that s_0, $\bar{s}_0$, and m_0^2 *all* approach m^2 in the limit. Using, for example, (2.11) and (2.12), it is easy to show that as $\bar{D}_0(m^2)\rightarrow 0$,

$$R\rightarrow\bar{R}-g^2. \tag{2.16}$$

What happens as the elementary particle is removed is then clear. As $\bar{D}_0(m^2)\rightarrow 0$, the dynamical bound state at $s=\bar{s}_0$ moves over to $s=m^2$, and so does the Regge pole at $s=s_0$ and its cancelling δ_{l0}-like companion. The dynamical pole from $s=\bar{s}_0$, the elementary pole, and the companion pole all eat each other, leaving no δ_{l0} term. The theory is now purely Regge-like, with a dynamical particle of mass m^2 on the trajectory $\alpha(s)$. The residue of this pole is

$$R=\bar{R}-g^2.$$

In order for the *same residue* to appear now as in the elementary case, we must in addition require $R\rightarrow -g^2$ or $\bar{R}\rightarrow 0$ in the dynamical limit. This final condition is

$$-\beta(m^2)/\alpha'(m^2)=N_0(m^2)/D_0'(m^2)\rightarrow -g^2. \tag{2.17}$$

The *two* conditions $\bar{D}_0(m^2)\rightarrow 0$ *and* $\bar{R}\rightarrow 0$ thus define a dynamical limit in which the elementary particle has disappeared but a Regge pole with the *same* parameters has taken its place. The transition to the limit is entirely smooth, and no discontinuous behavior is to be expected for any of the relevant functions or numbers.

III. INTERPRETATION OF THE DYNAMICAL CONDITIONS

Let us next investigate the meaning of the dynamical conditions in terms of normal field-theoretic quantities. First, the condition $\bar{D}_0(m^2)=0$.

We know that the form factor $F(s)$ for the pion, in the two-particle unitarity approximation, can be written

$$F(s)=g(\bar{D}_0(m^2)/\bar{D}_0(s)). \tag{3.1}$$

We also know that

$$F(s)=g\Gamma_1(s)\Delta_{F1}(s)/\Delta_F(s), \tag{3.2}$$

where $\Gamma_1(s)$ is the renormalized proper vertex function, $\Delta_{F1}(s)$ is the renormalized pion propagator, and $\Delta_F(s)$ is the free pion propagator. Finally, we know that[6]

$$\Delta_{F1}(s)/\Delta_F(s)\rightarrow 1/Z_3 \quad \text{as} \quad s\rightarrow\infty \tag{3.3}$$

and[7]

$$\Gamma_1(s)\rightarrow Z_1 \quad \text{as} \quad s\rightarrow\infty. \tag{3.4}$$

[6] H. Lehmann, Nuovo Cimento **11**, 342 (1954). See also Ref. 7.

[7] G. Källen, Helv. Phys. Acta **25**, 417 (1952), and *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. V, p. 358. Källen's proof is actually given for quantum electrodynamics. The validity of his proof has been challenged by S. G. Gasiorowicz, D. R. Yennie, and H. Suura, Phys. Rev. Letters **2**, 513 (1959). Nevertheless, the result is valid at least in the first few orders of perturbation theory, and sloppy pseudoproofs of it may easily be given. [See, for example, M. Gell-Mann and F. Zachariasen, Phys. Rev. **123**, 1065 (1961), Appendix A.] In any case we may use Eq. (3.4) as a *definition* of a quantity Z_1, which has field-theoretic significance, and which may or may not be identical with Z_1 as usually defined.

Thus we have, from (3.2),

$$F(s) \to gZ_1/Z_3 \quad \text{as} \quad s \to \infty. \tag{3.5}$$

On the other hand, for our (subtracted at infinity) $\bar{D}_0$, we know $\bar{D}_0(s) \to 1$ and $s \to \infty$. Hence, it is also the case that

$$F(s) \to g\bar{D}_0(m^2), \tag{3.6}$$

so we get

$$\bar{D}_0(m^2) = Z_1/Z_3 \tag{3.7}$$

and the dynamical limit is[8]

$$Z_1/Z_3 \to 0. \tag{3.8}$$

The other condition, $\bar{R} \to 0$, is equivalent to $Z_3 \to 0$. To see this, we note that in the field-theory case, with two-particle unitarity,

$$\Delta_{F1}(s) = \frac{1}{s-m^2} + \frac{\lambda^2}{(\bar{s}_0 - m^2)^2}\frac{1}{s-\bar{s}_0} - \frac{1}{\pi}\int \frac{\rho ds'}{s'-s}\frac{|F(s')|^2}{(s'-m^2)^2}. \tag{3.9}$$

The pole at $s = \bar{s}_0$ occurs because of the existence of the dynamical bound state. The constant λ is the effective "coupling constant" connecting the elementary pion directly to the bound state.

From (3.9) and using (3.3), we can find Z_3:

$$Z_3^{-1} = 1 + \frac{\lambda^2}{(\bar{s}_0 - m^2)^2} + \frac{1}{\pi}\int \rho ds' \frac{|F(s')|^2}{(s'-m^2)^2}. \tag{3.10}$$

We can relate λ to $\bar{R}$ through the dispersion relation for the form factor $F(s)$, again assuming two-particle unitarity. We have

$$F(s) = gZ_1/Z_3 + \frac{\lambda(-\bar{R})^{1/2}}{s-\bar{s}_0} + \frac{1}{\pi}\int \frac{\rho(s')ds'}{s'-s} F^*(s')t_0(s'), \tag{3.11}$$

and therefore

$$g = gZ_1/Z_3 + \frac{\lambda(-\bar{R})^{1/2}}{m^2 - \bar{s}_0} + \frac{1}{\pi}\int \frac{\rho ds'}{s'-m^2} F^*(s')t_0(s'). \tag{3.12}$$

Now let us go to the dynamical limit. As $\bar{D}_0(m^2) \to 0$, we see that $F(s) \to 0$ from Eq. (3.1). Then (3.12) tells us that

$$\lambda/(m^2 - \bar{s}_0) \to g/(-\bar{R})^{1/2}, \tag{3.13}$$

and hence

$$Z_3^{-1} \to 1 - g^2/\bar{R}. \tag{3.14}$$

Finally, if we in addition demand that $\bar{R} \to 0$, we get

$$Z_3 \to 0. \tag{3.15}$$

In conventional field theoretic language, then, the limit described at the end of Sec. II is the limit $Z_1 \to 0$, $Z_3 \to 0$, such that $Z_1/Z_3 \to 0$.

Finally, it may be worth remarking that the passage to the dynamical limit in some toy models does *not* exhibit the behavior described here, even though these models have often been used to guess at what the dynamical conditions should be. In this connection, it is helpful to look at Levinson's theorem.[9] Define the function $\tilde{D}_0(s)$ by

$$\tilde{D}_0(s) = \frac{1}{(s-s_0)\cdots(s-s_n)}\bar{D}_0(s), \tag{3.16}$$

where the s_n are all the zeros of $\bar{D}_0(s)$ in the physical plane (it has no poles). $\bar{N}_0(s)$ is divided by the same polynomial to form $\tilde{N}_0(s)$. The function $\tilde{D}_0(s)$ now goes like s^{-n} as $s \to \infty$. On the other hand,

$$\tilde{D}_0(s) = \tilde{D}_0(\bar{s}) \exp -\frac{s-\bar{s}}{\pi}\int_{s_t}^{\infty} \frac{\delta_0(s')}{(s'-s)(s'-\bar{s})}ds', \tag{3.17}$$

from which we can conclude that

$$\delta_0(s_t) - \delta_0(\infty) = n\pi. \tag{3.18}$$

However, $\tilde{N}_0(s)$ now has $n+1$ poles, the n poles we have introduced plus the pole at $s = m^2$. Levinson's theorem now tells us that one pole is elementary. If $\bar{D}_0(m^2) = Z_1/Z_3 \to 0$, $\tilde{N}_0$ has only n poles and the theory is dynamical. The phase shift still changes by $n\pi$ as before. The limit $Z_3 \to 0$ is not necessary to make the amplitude dynamical, but is only required to ensure that the dynamical pole has the same residue as the elementary one had.

On the other hand, in a toy model,[10] the dynamic limit is not reached until $Z_3 \to 0$, ($Z_1 = 0$ in any case). This is so because $\tilde{N}_0(s)$ in that model has one pole at $s = m^2$ and $\bar{D}_0(s) \xrightarrow[s\to\infty]{} -(\lambda/16\pi^2)\ln|s| + Z_3$. For a dynamical amplitude, $\bar{D}_0(s)$ must go as s^{r-1}, where $r < 1$. This limit is reached when $\lambda \to 0$ and $Z_3 \to 0$. Also, the CDD zero in this model is given by $m_0^2 = m^2 - g^2/\lambda$. Therefore, as the dynamic limit is approached, $m_0^2 \to \infty$ rather than to the pole, leaving the pole nakedly behind at m^2, looking dynamical.

[8] That $Z_1 = 0$ and $Z_3 = 0$ are equivalent to bootstrap conditions on a particle has been shown in an approximation to lowest order perturbation theory by R. M. Rockmore, Phys. Rev. **132**, 878 (1963). The $Z_3 = 0$ condition is discussed more generally in Ref. 3. It should, however, be noted that the model and conditions we are employing do *not* coincide with those of Ref. 3. Lee *et al.* explicitly assume that there are no dynamical bound states in the field-theory case, and they therefore do not obtain a dynamical theory until $Z_3 = 0$. For us, on the other hand, a dynamical situation occurs as soon as $Z_1/Z_3 = 0$.

[9] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1949).

[10] F. Zachariasen, Phys. Rev. **121**, 1851 (1961).