Phase Representation and High-Energy Behavior of the Forward Scattering Amplitude*

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By making use of the phase representation, the relation between the high-energy behavior of the symmetric forward scattering amplitude $F(x)$ and the asymptotic properties of the ratio $\cot\delta = \mathrm{Re}F/\mathrm{Im}\,F$ is discussed. Starting assumptions are dispersion relations and the Greenberg-Low bound. Lower bounds as well as upper bounds are derived. Under the assumption of the Froissart bound, it is shown that $\cot\delta$ cannot stay indefinitely greater than an arbitrarily small positive number. Also if the total cross section decreases steadily to a finite limit, but slower than const/E, the real part must tend to $-\infty$. The results are discussed in connection with those of Khuri and Kinoshita. The unsymmetric case is also treated by the same methods.

1. INTRODUCTION

It is well known that a certain analyticity of the scattering amplitude in the momentum transfer t plane, unitarity, and polynomial boundedness in the energy variable s imply the existence of an upper bound on the scattering amplitude at high energies.[1,2] In these derivations, however, no use of the dispersion relation in s for fixed t has been made. Recently Khuri and Kinoshita[3] have taken into account analyticity in the s plane to improve these bounds by making use of a theorem on harmonic measure. In general, it is obvious that those bounds in Refs. 1 and 2 cannot be improved by means of dispersion relation in s alone, without further assumption, since the physical amplitude is the boundary value of an analytic function, while in the t plane the physical point is always in the interior of the analyticity domain. In fact the improvement in Ref. 3 is based upon an assumption on the ratio $\mathrm{Re}F(s)/\mathrm{Im}F(s) = \cot\delta(s)$ where $F(s)$ is the forward scattering amplitude. In general one cannot exclude the cases in which those bounds are saturated.

If one likes to relate the high-energy behavior with the asymptotic ratio $\mathrm{Re}F/\mathrm{Im}F$, the phase representation [which is essentially a dispersion relation for $\ln F(s)$] is most useful, for $\delta(s)$ is nothing but the imaginary part of $\ln F(s)$. In this paper, we shall investigate the high-energy behavior of the forward scattering amplitude in terms of the phase representation. Throughout the paper, unless otherwise stated, we shall assume only the consequences of axiomatic field theory, namely, dispersion relation for forward-scattering amplitude and the Greenberg-Low bound. Apart from its simplicity the use of the phase representation has the advantages of giving (i) lower as well as upper bound on $F(s)$, (ii) bounds which depend on the sign of $\mathrm{Re}F$, and (iii) a way of treating the nonsymmetric case as well.

In Sec. 2 we give a proof of the phase representation under very general conditions with a restriction much weaker than that of Sugawara and Tubis.[4] In Sec. 3, high-energy upper and lower bounds are derived for the symmetric case in which one of the particles is self-conjugate. The best lower and upper bounds in any complex direction are obtained in terms of lower and upper bounds on the phase. On the real axis we have a result which is sufficient to ensure that one can write a dispersion relation for $F(x)$ [in the variable $x = (s-m^2-\mu^2)^2/4m^2\mu^2$] with only one subtraction, provided $\tan\delta$ never vanishes as $x \to \infty$. We also give lower and upper bounds which, however, may not be the best possible under the assumptions made. It follows from our analysis that $\cot\delta(x)$ cannot be persistently larger than an arbitrarily small positive number without the Froissart bound being violated. In Sec. 4 the absorptive high-energy scattering, in which the phase tends to $\pi/2$ is discussed in detail and the methods of Sec. 3 are used to provide restrictions on the power of the factor $\ln s$. It is also shown that $\mathrm{Re}F(s) \to -\infty$, if the total cross section decreases steadily to a limit $\sigma(\infty)$ but not faster than const/s. Finally, in Sec. 5 the unsymmetric case is briefly discussed, and in Sec. 6 we make some comments on the results obtained.

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[1] M. Froissart, Phys. Rev. **123**, 1053 (1961); A. Martin, Phys. Rev. Letters **9**, 410 (1962).

[2] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).

[3] N. N. Khuri and T. Kinoshita, Phys. Rev. **137**, B720 (1965). In this paper the theorem on harmonic measure as applied to the function $g(E)$ only implies that there exists some neighborhood of an infinite sequence of points $\{E_n\}$, $(E_n \to \infty)$, on which $|g(E)| < C(E/E_0)^{-\alpha/2}$, but it does not follow that the bound is valid everywhere; there may in fact be infinite sequences of points or even of finite nonoverlapping intervals where the bound is not valid. It should also be remarked that although by construction $\mathrm{Im}g(E) > 0$ for large E, so that the curves Γ_1 and Γ_2 do not intercept, it is not so for $g(E)^{1/2\alpha}$ if $\alpha \leq \frac{1}{3}$ (or more generally $\delta_2 \geq 2\delta_1$) because then the phase could become larger than π. Thus the conditions of the theorem would not be fulfilled. In order to overcome this difficulty one should consider instead the function $\{(1-E^2)^{-(\delta_2-2\delta_1+\eta)/2\pi} g(E)\}^{\pi/2(\delta_2-\delta_1+\eta)}$. One then obtains the bound $|g(E)| < C(E/E_0)^{-(7\delta_1-3\delta_2)/2\pi}$ [or $C(E/E_0)^{(3/2)-5\alpha}$] in the neighborhood of the infinite sequence $\{E_n\}$. These points have also been overlooked by Meiman {N. N. Meiman, Zh. Eksperim. i Teor. Fiz. **43**, 2277 (1962) [English transl.: Soviet Phys.—JETP **16**, 1609 (1963)]}. It is apparent that some consequences drawn by the authors do not follow from the theorem.

[4] M. Sugawara and A. Tubis, Phys. Rev. **130**, 2127 (1963).

2. THE PHASE REPRESENTATION

Let us consider a function $F(z)$ with the following properties:

(i) Analytic in the z plane cut along the real interval $(1,\infty)$.
(ii) Bounded by a power of $|z|$ as $z\to\infty$ in any direction.
(iii) Real in the sense that $F(z^*)=F(z)^*$.
(iv) The discontinuity across the cut is continuous and has not more than a finite number of zeros.

It has been shown that such a function has only a finite number of zeros,[5] say $z_1\cdots z_n$. Then the function $\ln G(z)$, where

$$G(z)=F(z)/\prod_{i=1}^{n}(z-z_i)\,,$$

is clearly an analytic function of z in the cut plane and the discontinuity across the cut is given by

$$\delta(x)=\frac{1}{2i}[\ln G(x+i\epsilon)-\ln G(x-i\epsilon)]=\frac{1}{2i}\ln\frac{F(x+i\epsilon)}{F(x-i\epsilon)}=\tan^{-1}\frac{\operatorname{Im}F(x)}{\operatorname{Re}F(x)}\,. \quad (1)$$

Since the phase of $F(x+i\epsilon)$ is not uniquely defined we shall, following Sugawara and Tubis,[4] adopt the convention that $\delta(1)=0$. It follows from (iv) that $\delta(x)$ is continuous and bounded above and below:

$$\delta_2\geq\delta(x)\geq\delta_1\,. \quad (2)$$

Let us consider the function $H(z)\equiv G(z)\exp[-\varphi(z)]$, where

$$\varphi(z)=\frac{z}{\pi}\int_1^\infty\frac{\delta(x)}{x-z}\frac{dx}{x}\,, \quad (3)$$

the integral being convergent on account of (2). $H(z)$ is an entire function and has no zeros (except possibly at infinity). Now writing $z=re^{i\theta}$ we have (for $\theta\neq0$)

$$\operatorname{Re}\varphi(z)=\frac{r}{\pi}\int_1^\infty\frac{(x\cos\theta-r)\delta(x)}{x^2+r^2-2xr\cos\theta}\frac{dx}{x}\,. \quad (4)$$

If $\cos\theta\leq0$, then $(x\cos\theta-r)$ is always negative, and applying the mean value theorem we have

$$\operatorname{Re}\varphi(z)=-\frac{\delta(\xi)}{2\pi}\ln(1+r^2-2r\cos\theta)\,. \quad (5)$$

On the other hand, if $\cos\theta>0$, the integrand changes sign at $x=r/\cos\theta$ and we shall divide up the interval of integration so as to write

$$\operatorname{Re}\varphi(z)=\frac{r}{\pi}\left[\int_1^{r/\cos\theta}+\int_{r/\cos\theta}^\infty\right]\frac{(x\cos\theta-r)\delta(x)}{x^2+r^2-2xr\cos\theta}\frac{dx}{x}$$
$$=-\frac{\delta(\xi_1)}{2\pi}\ln(1+r^2-2r\cos\theta)+\frac{1}{\pi}[\delta(\xi_1)-\delta(\xi_2)]\ln|\sin\theta|\,. \quad (6)$$

Taking into account (2), it follows from (5) and (6) that

$$|\exp[-\varphi(z)]|<(r+1)^{\delta_2/\pi}(\sin\tfrac{1}{2}\theta)^{(\delta_1-\delta_2)/\pi}\,. \quad (7)$$

Also for large $|z|$, $|G(z)|$ is bounded by a power of $|z|$ so that one can find a positive number N such that $|G(z)|<C(r+1)^N$. In order to investigate the asymptotic behavior of $H(z)$ near the positive real axis, let us take a circle with center at the origin and radius $R=r+1$. Then applying Cauchy's theorem to the function $H(z)^\nu$ where $0<\nu<[(\delta_2-\delta_1)/\pi]^{-1}$, we have

$$H(z)^\nu=\frac{1}{2\pi i}\int\frac{H(z')^\nu}{z'-z}dz'=\frac{R}{2\pi}\int_0^{2\pi}\frac{H(Re^{i\pi})^\nu}{Re^{i\omega}-z}d\omega$$

and

$$|H(z)|^\nu<\frac{R}{2\pi}\int_0^{2\pi}|H(Re^{i\omega})|^\nu d\omega<\frac{C^\nu}{2\pi}(R+1)^{1+\nu(N+\delta_2/\pi)}\times\int_0^{2\pi}(\sin\tfrac{1}{2}\omega)^{\nu(\delta_1-\delta_2)/\pi}d\omega<C'^\nu(r+2)^{1+\nu(N+\delta_2/\pi)}\,,$$

or

$$|H(z)|<C'(r+2)^{1/\nu+N+\delta_2/\pi}\,.$$

Therefore, for large $|z|$, $H(z)$ is bounded by a power of $|z|$ in all directions. But $H(z)$ is an entire function and has no zeros; therefore, $H(z)$ is a constant. Hence one can write

$$F(z)=A\prod_{i=1}^{n}(z-z_i)\exp\left(\frac{z}{\pi}\int_1^\infty\frac{\delta(x)}{x-z}\frac{dx}{x}\right)\,, \quad (8)$$

which is the phase representation for $F(z)$.

This derivation can be trivially extended to the case in which $F(z)$ has a finite number of poles. An essential point in the above derivation is condition (iv), that the discontinuity across the cut does not go through zero an infinite number of times. Actually the representation (8) holds under more general conditions. One could, for nstance, replace (i) and (iv) by:

(ii-a) For large $|z|$, both $F(z)$ and $F(z)^{-1}$ are $O(\exp|z|^{1/(1+\beta)})$, where β is an arbitrarily small positive number.
(iv-a) $F(z)$ has not more than a finite number of zeros.

The proof of representation (8) under these assumptions is given in the Appendix.

[5] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

3. THE FORWARD SCATTERING AMPLITUDE

Let us consider the symmetric combination of amplitudes for forward elastic scattering of two particles of mass m and μ, respectively, say a nucleon and a pion, as a function of the variable $z=(s-m^2-\mu^2)^2/4m^2\mu^2$ where s is the square of the energy in the center-of-mass system of the particles. From axiomatic field theory one can deduce (under certain restrictions on the masses) that the forward scattering amplitude $F(z)$ satisfies conditions (i)–(iii) of the preceding section except for a pole on the real axis at $z_0=\mu^2/4m^2$. From analyticity in the Lehmann ellipse and unitarity the following bound obtains for large $|z|$:[2]

$$|F(z)|<C|z|(\ln|z|)^2. \tag{9}$$

Also unitarity ensures that $\mathrm{Im}F(x)\geq 0$, but we shall further make the plausible assumption that $\mathrm{Im}F(x)$ does not vanish infinitely many times. Then $F(z)$ will admit a representation (8) but for a factor $1/(z-z_0)$. So we shall write

$$F(z)=\frac{A}{z-z_0}\prod_{i=1}^{n}(z-z_i)\exp\left(\frac{z}{\pi}\int_1^\infty\frac{\delta(x)}{x-z}\frac{dx}{x}\right). \tag{10}$$

Let us first suppose that $F(x)$ has no zeros in the physical region $x>1$. Then according to the convention $\delta(1)=0$ and the condition $\mathrm{Im}F(x)\geq 0$ we have

$$(1)\quad F(1)>0,\quad 0\leq\delta(x)\leq\pi,$$
$$(2)\quad F(1)<0,\quad 0\geq\delta(x)\geq-\pi.$$

It then follows from (5), (9), and (10) that $n\leq 3$ or $n\leq 2$ for the first and second case, respectively. If $F(x)$ had p zeros in the physical region, the phase for x above the last zero would be shifted by $(p-2q)\pi$ where q is an integer in the interval $0\leq q\leq p$. Likewise we would then have the restriction $n-p+2q\leq 3$ or 2, for $F(1)>0$ or $F(1)<0$, respectively, the equality holding *only* if $\mathrm{Re}F(x)/\mathrm{Im}F(x)\to-\infty$ when $x\to\infty$. We remark that, since the residue at the pole is negative, $F(x)$ has a zero in the interval $(x_0,1)$ when $F(1)>0$. Then, unless $\mathrm{Re}F(x)/\mathrm{Im}F(x)\to-\infty$, we have $q=0$ and $n-p\leq 2$ or 1.

We shall from now on suppose, unless otherwise stated, that $F(x)$ *has no zeros in the physical region and* $F(1)$ *is positive*. Our results, however, can easily be seen to be *independent* of such restrictions. We shall also make the following relevant physical assumption:

$$\text{For } x>\bar{x},\ \left|\frac{\mathrm{Re}F(x)}{\mathrm{Im}F(x)}\right| \text{ is bounded by a constant.}$$

Then the number of zeros of $F(x)$ is either *one or two for case* (1) [*zero or one for case* (2)]. Let δ_1 and δ_2 be the greatest lower bound (g.l.b.) and the least upper bound (l.u.b.) of $\delta(x)$ for $x>\bar{x}$, so that by the above assumption

$$2\eta<\delta_1\leq\delta(x)\leq\delta_2<\pi-2\eta, \tag{11}$$

where η is an arbitrarily small positive number.[6] One can obtain an upper and lower bound for $|F(z)|$ as $z\to\infty$ in any complex direction ($\theta\neq 0$), directly from (5) and (6). We have

$$C_2|z|^{\nu-\delta_2/\pi}(\sin\tfrac{1}{2}\theta)^{(\delta_2-\delta_1)/\pi}<|F(z)|<C_1|z|^{\nu-\delta_1/\pi}(\sin\tfrac{1}{2}\theta)^{(\delta_1-\delta_2)/\pi}, \tag{12}$$

where $\nu=0,1$ [or $\nu=-1,0$, in case (2)] is the difference between the number of zeros and poles.

In order to study the asymptotic behavior along the positive real axis let us introduce $\Delta_1(x)=\delta(x)-\delta_1+\eta$, so that

$$\eta<\Delta_1(x)<\pi-\eta. \tag{13}$$

Then one can write

$$\Psi(z)=\exp\varphi(z)=(1-z)^{-(\delta_1-\eta)/\pi}\times\phi_1(z)\exp\left(\frac{z}{\pi}\int_1^{\bar{x}}\frac{\Delta_1(x)}{x-z}\frac{dx}{x}\right), \tag{14}$$

where

$$\phi_1(z)=\exp\left(\frac{z}{\pi}\int_{\bar{x}}^\infty\frac{\Delta_1(x)}{x-z}\frac{dx}{x}\right) \tag{15}$$

is an analytic function of z with a cut along $(\bar{x},\infty)$ and has the following properties:

(a) $\phi_1(z^*)=\phi_1(z)^*$.

(b) the imaginary part of $\phi_1(z)$ is $\mathrm{Im}\phi_1(z)=|\phi_1(z)|\times\sin\Lambda_1$, with

$$\Lambda_1=\frac{1}{\pi}\int_{\bar{x}}^\infty\frac{\Delta_1(x)r\sin\theta}{x^2+r^2-2xr\cos\theta}dx=\pm\frac{\Delta_1(\xi)}{\pi}\left[\frac{\pi}{2}-\tan^{-1}\frac{x-r\cos\theta}{r\sin\theta}\right], \tag{16}$$

where the sign is that of $\sin\theta$. Therefore $0<|\Lambda_1|<\pi-\eta$ and $\sin\Lambda_1$ has the sign of $\sin\theta$. Hence $\phi_1(z)$ is a Herglotz function.

(c) For real $x<\bar{x}$, $\phi_1(x)$ is real and positive.

Since $\phi_1(z)$ is a Herglotz function it admits the representation

$$\phi_1(z)=1+Bz+\frac{z}{\pi}\int_{\bar{x}}^\infty\frac{\mathrm{Im}\phi_1(x)}{x-z}\frac{dx}{x},$$

where $B\geq 0$. For real negative $z=-x<-\bar{x}$, we have

$$0<\phi_1(z)<1-Bx-\frac{x}{\pi}\int_{\bar{x}}^x\frac{\mathrm{Im}\phi_1(x')}{x'+x}\frac{dx'}{x'}<1-Bx-\frac{1}{2\pi}\int_{\bar{x}}^x\mathrm{Im}\phi_1(x')\frac{dx'}{x'}.$$

[6] We recall that δ_1 and δ_2 are, respectively, monotonic increasing and decreasing functions of $\bar{x}$ and they have a limit. When $\delta(x)$ has a limit δ as $x\to\infty$, the quantities δ_1 and δ_2 have the same limit δ and we shall write $\delta_1\to\delta_2\to\delta$.

For sufficiently large x the inequality will be violated unless $B=0$ and

$$\frac{1}{2\pi}\int_{\bar{x}}^{x}|\phi_1(x')|\sin\Delta_1(x')\frac{dx'}{x'}<1; \tag{17}$$

but by (13) $\sin\Delta_1(x)>\sin\eta$, then

$$[\phi_1(x)/x]\in L(1,\infty). \tag{18}$$

Hence from (10), (14), and (18) one concludes that

$$[F(x)/x^p]\in L(1,\infty);\; p>1+\nu-\delta_1/\pi, \tag{19}$$

where ν has the same meaning as in (12).

One can obtain a similar restriction on the asymptotic behavior of $F(x)^{-1}$ by introducing the function $\Delta_2(x)=\delta(x)-\delta_2-\eta$ and writing

$$\Psi(z)^{-1}=(1-z)^{(\delta_2+\eta)/\pi}\phi_2(z)\exp\left(-\frac{z}{\pi}\int_1^{\bar{x}}\frac{\Delta_2(x)}{x-z}\frac{dx}{x}\right), \tag{20}$$

where

$$\phi_2(z)=\exp\left(-\frac{z}{\pi}\int_{\bar{x}}^{\infty}\frac{\Delta_2(x)}{x-z}\frac{dx}{x}\right).$$

This is an analytic function with properties analogous to those of $\phi_1(z)$. Following the argument developed before, we conclude that $[\phi_2(x)/x]\in L(1,\infty)$ and therefore

$$F(x)^{-1}x^p\in L(1,\infty);\; p<-1+\nu-\delta_2/\pi. \tag{21}$$

We summarize these results in the following theorem:

Theorem 1. If for $x>\bar{x}$, the phase of the scattering amplitude satisfies the inequalities $\delta_1\leq\delta(x)\leq\delta_2$, then for large $|z|$, the function $F(z)$ is bounded by Eq. (12) uniformly in the angle θ, and along the positive real axis (19) and (21) hold. The bounds (12) are the best possible, since for $F(z)=C(1-z)^{1-\delta/\pi}$ we have $\delta_1=\delta_2=\delta$ and both bounds are saturated. One can draw some immediate consequences of this theorem:

Corollary 1. Condition (19) ensures that only one subtraction is required in the forward scattering dispersion relation for $F(z)$ and therefore that $F(z)$ is a Herglotz function. This result was first given by Khuri and Kinoshita.[3]

Corollary 2. It follows from (18) and the continuity of $\phi_1(x)$ that given an arbitrary positive constant C there is at least an infinite sequence of neighborhoods $\{x_n\}$, $(x_n\to\infty)$ on which $|\phi_1(x)|<C$. Therefore on this set of intervals $F(x)$ is bounded by

$$|F(x)|<C_1x^{\nu-(\delta_1-\eta)/\pi}. \tag{22}$$

This result is stronger than that of Ref. 3. Also there is a similar sequence on which

$$C_2x^{\nu-(\delta_2+\eta)/\pi}<|F(x)|. \tag{23}$$

Corollary 3. If for $x>\bar{x}$ the real part of the scattering amplitude does not change sign and the scattering amplitude is required to satisfy Froissart's bound[1]

$$|F(z)|<C|z|^{1/2}(\ln|z|)^2, \tag{24}$$

then one of the following applies:

(α) $F(x)$ has only one zero ($\nu=0$); then the total cross section falls off at infinity in such a way that $\sigma(\omega)$ is $L(1,\infty)$ in the variable $\omega=x^{1/2}$. We recall the relation

$$\sigma=(4\pi/k\sqrt{s})\,\mathrm{Im}F.$$

(β) The total cross section is a slowly varying function of x [in the sense that $x\sigma'(x)/\sigma(x)\to 0$] and $\mathrm{Re}F(x)/\mathrm{Im}F(x)\to 0$ ($\delta_1\to\delta_2\to\pi/2$).

(γ) The real part of the scattering amplitude stays negative for large x.

Corollary 4. If ($\mathrm{Re}F/\mathrm{Im}F$) tends to a limit $\cot\delta$ ($\delta_1\to\delta_2\to\delta$) then the asymptotic behavior of $F(z)$ is given by $|z|^{\nu-\delta/\pi}$ times a slowly varying function of z.

Although (19) and (21) put some restrictions on the high-energy behavior of $F(x)$ along the positive real axis, they do not provide us with actual bounds. The problem of finding lower and upper bounds for $F(x)$ on the real axis is clearly not one in the proper realm of analytic functions since one is here on the boundary of the domain of analyticity. Thus, it should be possible to obtain the best bounds under condition (11) by examining, for instance, the behavior of

$$\mathrm{Re}\varphi(x)=\frac{x}{\pi}P\int_1^{\infty}\frac{\delta(x')}{x'-x}\frac{dx'}{x'}. \tag{25}$$

Here, however, we shall not attempt to derive the best possible bounds. We shall simply give some bounds which can easily be deduced from (25) or from results already derived in this section. We start by remarking that the function $\phi_1(z)^{\pi/(\delta_2-\delta_1+2\eta)}$ [see (15)] still satisfies condition (13) and therefore this function has the properties (a), (b), (c) given for $\phi_1(z)$, hence also (18). But if we write $\phi_1(x)=[x\psi_1(x)]^{(\delta_2-\delta_1+2\eta)/\pi}$, then $\psi_1(x)\in L(1,\infty)$ which means that it is bounded by a constant "almost everywhere," in the sense that for $x>\bar{x}$ the measure of the set of intervals on which $|\psi_1(x)|>N$ is $m_{\bar{x}}(N)\leq C(\bar{x})/N$ where $C(\bar{x})$ is independent of N and tends to zero as $\bar{x}\to\infty$.

A similar result may be deduced for $\phi_2(x)$. Therefore we obtain for $F(x)$ the following bounds on the positive real axis:

$$C_2x^{\nu-(2\delta_2-\delta_1+\eta)/\pi}<|F(x)|<C_1x^{\nu-(2\delta_1-\delta_2-\eta)/\pi}, \tag{26}$$

which hold "almost everywhere," η being an arbitrarily small positive number. If we make the additional assumption that the phase $\delta(x)$ satisfies a Hölder condition,

$$|\delta(x+h)-\delta(x)|\leq Ah^{\mu}, \tag{27}$$

uniformly in x, for all h in the interval $0\leq h\leq 2a$ and μ a positive numer, then one can show from (25) that the bounds (26) hold everywhere with $\eta=0$. Indeed, we have[7]

$$-2\delta_2 \ln x < xP\int_1^{x-a} \frac{\delta(x')}{x'-x}\frac{dx'}{x'} < -2\delta_1 \ln x,$$

$$\delta_1 \ln x < xP\int_{x+a}^{\infty} \frac{\delta(x')}{x'-x}\frac{dx'}{x'} < \delta_2 \ln x,$$

$$\left| xP\int_{x-a}^{x+a} \frac{\delta(x')}{x'-x}\frac{dx'}{x'} \right| = \left| \delta(x)\ln\frac{x-a}{x+a} + \int_{x-a}^{x+a} \frac{\delta(x')-\delta(x)}{x'-x}\frac{dx'}{x'} \right| < \left| +\frac{2a}{x}\delta(x) + 2A\frac{a^\mu}{\mu} \right|,$$

wherefrom the bounds (26) follow with $\eta=0$. For small enough fluctuation of the phase, the Greenberg-Low upper bound as well as the lower bound obtained by Jin and Martin[5] are improved. For instance, if at high energies $\mathrm{Re}F(x)<0$ ($\delta_1 \geq \pi/2$), Eq. (26) with $\eta=0$ is better than the Greenberg-Low bound. If, on the other hand, the fluctuation is large these bounds become rather weak and for $\delta_2 \geq 2\delta_1$ there will be no improvement over the Greenberg-Low bound.

4. THE ABSORPTIVE HIGH-ENERGY SCATTERING

We shall now consider in more detail the case $\delta_1 \to \delta_2 \to \pi/2$, in which the forward scattering is dominated by the absorptive amplitude. Let us write $\delta(x)=\frac{1}{2}\pi+\Delta(x)$. If the total cross section tends to a constant limit as $x\to\infty$, then $\Delta(x)\to 0$ and

$$I(x)=\frac{1}{\pi}\int_1^x \Delta(x')\frac{dx'}{x'} \tag{28}$$

is convergent. This is mathematically equivalent to Pomeranchuk's theorem on the equality of the limits of total cross sections in crossed channels. However, if we demand that the total cross section is to stay bounded and finite, then it is not necessary that $\Delta(x)\to 0$, but (28) must be bounded. If, on the other hand, (28) is not bounded, we write $\Delta(x)=\alpha(x)/\ln x$ and first suppose that for $x>\bar{x}$, α_1 and α_2 are the g.l.b. and l.u.b. of $\alpha(x)$. One can then follow exactly the same procedure of the preceding section to obtain upper and lower bounds for $F(z)$. These bounds depend on the asymptotic behavior of the real part of

$$\mathcal{L}(z)=\frac{z}{\pi}\int_{\bar{x}}^{\infty} \frac{(\ln x')^{-1}}{x'-z}\frac{dx'}{x'},$$

which is easily deduced by remarking that the discontinuity of the analytic function $l(z)=\ln\ln[1+(1-z)^{1/2}]$ is asymptotically given by

$$\mathrm{Im}\,l(x)=-\tfrac{1}{2}\pi(\ln x)^{-1},$$

so that $\mathrm{Re}\mathcal{L}(z)\simeq-(2/\pi)\ln\ln|z|$. Thus we have the following result:

Theorem 2. If for $x>\bar{x}$ the phase of the scattering amplitude satisfies the inequality

$$\alpha_1 \leq [\delta(x)-\tfrac{1}{2}\pi]\ln x \leq \alpha_2, \tag{29}$$

then for large $z=re^{i\theta}$, the following bounds hold uniformly in the angle:

$$C_2 r^{\nu-1/2}(\ln r)^{-2\alpha_2/\pi}(\sin\tfrac{1}{2}\theta)^{2(\alpha_2-\alpha_1)/\pi\ \ln r} < |F(z)| < C_1 r^{\nu-1/2}(\ln r)^{-2\alpha_1/\pi}(\sin\tfrac{1}{2}\theta)^{2(\alpha_1-\alpha_2)/\pi\ \ln r}, \tag{30}$$

and along the positive real axis we have

$$F(x)/x^{1+\nu-1/2}(\ln x)^p \in L(1,\infty);\quad p>1-2\alpha_1/\pi \tag{31}$$

and

$$F(x)^{-1}x^{-1+\nu-1/2}(\ln x)^p \in L(1,\infty);\quad p<-1-2\alpha_2/\pi. \tag{32}$$

It follows from (30) that if we impose on $F(z)$ the Froissart bound (23), then either $\nu=0$ or $\alpha_2 \geq -\pi$. Also if $[\mathrm{Re}F(x)/\mathrm{Im}F(x)]\ln x$ tends to a limit as $x\to\infty$, this must be $\alpha \geq -\pi$.

Again here, as in Sec. 3, the bounds (27) are the best possible. On the positive real axis we have a similar situation, as before. The following bounds hold "almost everywhere" for large x:

$$C_2 x^{\nu-1/2}(\ln x)^{2(\alpha_1-2\alpha_2-\eta)/\pi} < |F(x)| < C_1 x^{\nu-1/2}(\ln x)^{2(\alpha_2-2\alpha_1+\eta)/\pi}. \tag{33}$$

Under the assumption of a Hölder condition (27) on the phase these bounds hold everywhere and with $\eta=0$. An interesting consequence is that when $(\mathrm{Re}F/\mathrm{Im}F)\times\ln x\to\alpha$ then, *unless* $\alpha=0$, the total cross section is either not bounded ($\alpha<0$, $\nu=1$) or tends to zero faster than $C/(\ln x)^{2(\alpha-\eta)/\pi}$, ($\alpha>\eta>0$).[8]

Finally we shall prove the following important result: *Theorem 3.* If for $x>\bar{x}$ the total cross section decreases steadily to the limit $\sigma(\infty)$ but not faster than const/$(x^{1/2}\ln x)$, then $\mathrm{Re}F(x)$ becomes negative for sufficiently large x, and tends to $-\infty$.

Let us introduce the variable $\omega=\sqrt{z}$ and the analytic function

$$\phi(\omega)=F(\omega^2)-2m\mu[\sigma(\infty)/4\pi](1-\omega^2)^{1/2}, \tag{34}$$

with two cuts $(-1,-\infty)$ and $(1,\infty)$ in the ω plane. By hypothesis we have for $|\omega|>\bar{\omega}$,

$$a_1/\ln|\omega| < \mathrm{Im}\phi(\omega) < a_2|\omega|, \tag{35}$$

where a_1 and a_2 are positive constants. Let us next write the dispersion relation (for $\omega>\bar{\omega}$):

$$\mathrm{Re}\phi(\omega)=\phi(0)+\frac{\omega}{\pi}P\int_1^{\infty}\left(\frac{1}{\omega'-\omega}-\frac{1}{\omega'+\omega}\right)\mathrm{Im}\phi(\omega')\frac{d\omega'}{\omega'}. \tag{36}$$

We shall divide the interval of integration in the follow-

[7] G. Frye and R. L. Warnock, Phys. Rev. 130, 478 (1963).

[8] P. Olesen, Phys. Letters 13, 175 (1964); 14, 66 (1965).

ing way:

$$\mathrm{Re}\phi(\omega)=\phi(0)+\frac{2\omega^2}{\pi}\int_1^{\bar\omega}\frac{\mathrm{Im}\phi(\omega')}{\omega'^2-\omega^2}\frac{d\omega'}{\omega'}+\frac{\omega}{\pi}P\int_{\bar\omega}^{\omega^2}\left(\frac{1}{\omega'-\omega}-\frac{1}{\omega'+\omega}\right)\mathrm{Im}\phi(\omega')\frac{d\omega'}{\omega'}+\frac{2\omega^2}{\pi}\int_{\omega^2}^{\infty}\frac{\mathrm{Im}\phi(\omega')}{\omega'^2-\omega^2}\frac{d\omega'}{\omega'}. \quad (37)$$

Now for sufficiently large $\omega>\bar\omega$, by hypothesis $\mathrm{Im}\phi(\omega)$ is a decreasing function of ω. Therefore one increases the right-hand side of (36) by replacing $\mathrm{Im}\phi(\omega')$ by $\mathrm{Im}\phi(\omega)$ in the principal value integral and by its upper bound (34) in the last integral. Then

$$\mathrm{Re}\phi(\omega)<\phi(0)+\frac{1}{\pi}\ln\left|\frac{\omega-1}{\omega-\bar\omega}\frac{\bar\omega}{\omega}\right|\mathrm{Im}\phi(\omega)-\frac{\omega}{\pi}\int_{\bar\omega}^{\omega^2}\frac{\mathrm{Im}\phi(\omega')}{\omega'+\omega}\frac{d\omega'}{\omega'}+a_2\frac{\omega}{\pi}\ln\left|\frac{\omega+1}{\omega-1}\right|, \quad (38)$$

or

$$\mathrm{Re}\phi(\omega)<0(1)-\frac{\omega}{\pi}\int_{\bar\omega}^{\omega^2}\frac{\mathrm{Im}\phi(\omega')}{\omega'+\omega}\frac{d\omega'}{\omega'}. \quad (39)$$

But $\mathrm{Im}\phi(\omega)>a_1/\ln\omega$; hence the right-hand side of (39) becomes negative for large ω and tends to $-\infty$ as $\omega\to\infty$. The theorem is thus proved.

5. NONSYMMETRIC AMPLITUDES

So far we have been dealing with amplitudes which are symmetric under crossing of the Mandelstam variables s and u such as in $p\pi^0$ scattering or with symmetric combinations like $(F_{p\pi^+}+F_{p\pi^-})$ in which the amplitude has only one cut in the variable $z=\omega^2$. However, the methods developed in the preceding sections apply equally to the case of nonsymmetric amplitudes like $F_{p\pi^+}$ or $F_{p\pi^-}$ which are analytic functions of ω in the two-cut plane. The phase representation in this case takes on the form

$$F(\omega)=\frac{A}{\omega^2-\omega_0{}^2}\prod_{i=1}^{n}(\omega-\omega_i)\times\exp\left[\frac{\omega}{\pi}\left(\int_1^\infty\frac{\delta^+(\omega')}{\omega'-\omega}\frac{d\omega'}{\omega'}-\int_1^\infty\frac{\delta^-(\omega')}{\omega'+\omega}\frac{d\omega'}{\omega'}\right)\right], \quad (40)$$

where $\delta^+(\omega)$ and $\delta^-(\omega)$ are the phases in the direct and crossed channels, respectively. The bounds (12) of Sec. 2 hold with δ_1 and δ_2 replaced by half the sum $(\delta_{1,2}{}^++\delta_{1,2}{}^-)$ of the bounds on the phases of the two channels. The number of zeros and poles in the variable ω is twice the number in the variable z so that the difference between zeros and poles is $2\nu=-2,\ -1,\ 0,\ 1,\ 2$ where the values $(-2,0)$ or $(0,2)$ correspond to the cases in which the scattering lengths in the two channels are either both negative or both positive, respectively. The values $(-1,1)$ would correspond to cases in which the scattering lengths have opposite sign, which obviously has no counterpart for symmetric amplitudes. According to the convention that $\delta=0$ at threshold, the phases will then have opposite signs in the two channels. The analogue of Corollary 3(γ) of Theorem 1 is that at least in one channel the amplitude stays negative for large ω. On the other hand, if the total cross sections tend to a finite constant, either $(\cot\delta^++\cot\delta^-)\to 0$ or it oscillates indefinitely.

As for the bounds on the real axis one can see that they can be improved if the conditions formulated in Sec. 3 for the variable $x=\omega^2$ are assumed to apply to ω. Then one can easily deduce from (40) that for $\omega>\bar\omega$ the following bounds hold on the real axis except possibly on a set of measure $m_{\bar\omega}\to 0$ in the variable ω:

$$C_2|\omega|^{\lambda_2^\pm-\eta}<|F^\pm(\omega)|<C_1|\omega|^{\lambda_1^\pm+\eta}, \quad (41)$$

where $F^\pm(\omega)=F(\pm\omega)$ and

$$\lambda_{2,1}{}^\pm=2\nu-(\delta_{2,1}{}^++\delta_{2,1}{}^-)+(\delta_{1,2}{}^\pm-\delta_{2,1}{}^\pm). \quad (42)$$

Also if we impose the Hölder condition

$$|\delta^\pm(\omega+h)-\delta^\pm(\omega)|\leq Ah^\mu \quad (43)$$

to hold uniformly in ω, for every h in the interval $0\leq h\leq 2a$ and positive μ, then the bounds (41) hold everywhere with $\eta=0$. Condition (43) is clearly stronger than (27) since $|z_2-z_1|=|\omega_2+\omega_1||\omega_2-\omega_1|$. However, ω is essentially the energy of the incoming particle in the rest frame of the target or the square of the energy in the center of mass system; therefore, (43) is physically a reasonable assumption whereas to impose (27) instead would be somewhat artificial.

In the symmetric case we would have obtained the bounds $C_2x^{\nu-(3\delta_2-\delta_1)/2\pi}$ and $C_1x^{\nu-(3\delta_1-\delta_2)/2\pi}$. The upper bound improves the Greenberg-Low bound provided that $\delta_2<3\delta_1$.

Similar considerations could be made with respect to the results of Sec. 4. If the Froissart bound is assumed for $F(\omega)$ one needs, in general, two subtractions in the forward dispersion relation. But if the phases of both channels satisfy the inequalities (29) with $x=\omega$, and if $(\alpha_1{}^++\alpha_1{}^-)>\pi/2$, then one subtraction will be sufficient. Also the bounds (33) can be improved under the stronger Hölder condition (43) and a result analogous to (41) would obtain.[9]

Finally if the sum of the total cross sections satisfies the conditions of Theorem 3, then the result of the theorem clearly applies at least to one channel, namely that its amplitude becomes negative at large energies and tends to $-\infty$ as $\omega\to\infty$ (or $\omega\to-\infty$).

6. CONCLUDING REMARKS

We have proved the equivalence between the phase representation and the dispersion relation for the for-

[9] If $\alpha^+(\omega)$ and $\alpha^-(\omega)$ tend to the limits α^+ and α^-, then, for the total cross sections, $\sigma^\pm$ to be bounded and finite one must have $\alpha^++\alpha^-=0$. Hence, either $\mathrm{Re}F$ has opposite signs in the two channels as $|\omega|\to\infty$ or $(\mathrm{Re}F/\mathrm{Im}F)\ln|\omega|\to 0$.

ward scattering amplitude with the sole restriction that the number of zeros be finite. The result also applies to the scattering amplitude in the unphysical region $0<t<4\mu^2$. In the physical region $t<0$, the absorptive part is not positive definite and the restriction on the number of zeros is not physically so plausible. However, if one makes some assumption such as, for instance, Regge behavior at high energies, then the number of zeros will be finite and the phase representation is valid.

It is quite apparent from our analysis that the phase representation is a very powerful tool in the study of asymptotic behavior even when the phase is allowed to oscillate, or ReF to change sign, indefinitely. In complex directions the bounds obtained, both upper and lower bounds, are the best possible under the assumptions made. Our results for the behavior on the real axis are qualitatively better than those of Khuri and Kinoshita.[3] In particular we have a rigorous proof that if (ReF/ImF) is bounded at high energies, then one needs at most two subtractions in the dispersion relation for $F(\omega)$ as a function of ω, or one in the case of a symmetric amplitude $F(x)$ as a function of $x=\omega^2$.

The analysis also shows that there is a tendency for Re$F(\omega)$ to become negative at least in one channel. For instance, one cannot have an asymptotic behavior such that (ReF/ImF)$>\epsilon$ without violating the Froissart bound. Second, if (ReF/ImF)$\to 0$, the scattering being predominantly absorptive, the total cross section will not be bounded unless (ReF/ImF) $\ln|\omega|<\epsilon$. Third, if the total cross section decreases steadily to a constant limit as suggested by the experimental results, again the real part must become negative. These conclusions apply at least to one channel; the first two are related to the "crossing" property of field theory, whereas the third one is simply a consequence of dispersion relations. In terms of an effective potential one would say that its real part tends to become repulsive at small distances. These results are in agreement with the analysis of Söding[10] based on a Regge-pole picture of the asymptotic behavior. They are also consistent with recent experimental determinations of Re$F(\omega)$ in which negative values were found.[11,12] One would like to have a better experimental determination of the high-energy behavior of total cross sections as well as of Re$F(\omega)$.

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APPENDIX

We shall prove here that if a function $F(z)$ satisfies conditions (i), (ii-a), (iii), (iv-a), of Sec. 2, then it admits a representation (8).

Let us introduce the function

$$R(z)=\mathrm{Re}\,\ln[G(z)/G(0)]\,, \tag{A1}$$

where $G(z)$ has the same definition in terms of $F(z)$ as in Sec. 2. Then it follows from (ii-a) that for large $|x|$, $R(x)$ is bounded by

$$|R(x)|<C|x|^{1/(1+\beta)}. \tag{A2}$$

Therefore the function $(R(x)/x)$ belongs to $L_p(-\infty,\infty)$ with $p=1+1/\beta'$, $(0<\beta'<\beta)$, and its Hilbert transform

$$\frac{\bar{\delta}(x)}{x}=-\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{R(x')}{x'}\frac{dx'}{x'-x} \tag{A3}$$

exists almost everywhere and also belongs to $L_p(-\infty,\infty)$.[13] The reciprocal formula

$$\frac{R(x)}{x}=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\delta}(x')}{x'}\frac{dx'}{x'-x} \tag{A4}$$

also holds almost everywhere and $|G(x)|e^{i\bar{\delta}(x)}$ is the boundary value of the analytic function

$$\bar{G}(z)=G(0)\exp\left(\frac{z}{\pi}\int_{-\infty}^{\infty}\frac{\bar{\delta}(x')}{x'-z}\frac{dx'}{x'}\right). \tag{A5}$$

We shall now show that $\bar{G}(z)$ and $\bar{G}(z)^{-1}$ are $o(\exp|z|^{1/(1+\beta')})$. On the real axis the result is obviously true. For complex $z=re^{i\theta}$, $(\theta\neq 0$ or $\pi)$ we have

$$\begin{aligned}\left|z\int_{-\infty}^{\infty}\frac{\bar{\delta}(x')}{x'-z}\frac{dx'}{x'}\right| &\leq\int_{-\infty}^{\infty}\left|\frac{z}{x'-z}\right|\left|\frac{\bar{\delta}(x')}{x'}\right|dx'\leq\left\{\int_{-\infty}^{\infty}\left|\frac{\bar{\delta}(x')}{x'}\right|^{p}dx'\right\}^{1/p}\\ &\quad\times\left[\int_{-\infty}^{\infty}\left(\frac{r^2}{u^2+r^2\sin^2\theta}\right)^{(1+\beta')/2}du\right]^{1/(1+\beta')}\\ &=\frac{Cr^{1/(1+\beta')}}{(\sin\theta)^{\beta'/(1+\beta')}},\end{aligned} \tag{A6}$$

where we have used Hölder's inequality. It follows from this result and condition (ii-a) that the function $\chi(z)=\bar{G}(z)/G(z)$ and its inverse are $o(\exp|z|^{1/(1+\beta')})$. But $\chi(z)$ is analytic in the upper half-plane, has no zeros and is of modulus one on the real axis; therefore by the maximum modulus principle $\chi(z)=e^{i\eta}$ where η is a real constant. Then writing $\bar{\delta}(x)-\eta=\delta(x)$, the representation (8) follows from (A5), with $\delta(x)$ readily identified as the phase of $F(x+i\epsilon)$.

[13] E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford University Press, London, 1948), p. 132.