Trouble with Relativistic $SU(6)$*

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Recently, several authors have proposed a relativistic generalization of the $SU(6)$ group, based on $U(6)\otimes U(6)$. The smallest group containing this group and the Poincaré group is found. This group has the catastrophic feature that all of its faithful unitary representations contain an infinite number of states for fixed four-momentum; thus there are an infinite number of particles in every supermultiplet. It is conjectured that similar difficulties afflict every group that contains an internal symmetry group and the Poincaré group in such a way that these groups do not commute. The conjecture is proven for a large class of Lie groups (semidirect products of semisimple groups and Abelian groups) under a restrictive assumption (that the translations are contained in the Abelian group).

I. INTRODUCTION

RECENTLY, there has appeared a series of papers[1] which apply the group $SU(6)$ to elementary particles physics. This group is not an internal symmetry group in the usual sense, but a group which is conceived as mixing internal and spin degrees of freedom, in a manner similar to that of the $SU(4)$ group which appears in Wigner's theory of nuclear supermultiplets. This approach has led to some remarkable successes, but it is not without difficulties. Prominent among these is one which rests upon the fact that the formulation of the theory involves a separation of total angular momentum into its spin and orbital parts. This separation is not Lorentz-invariant; thus, neither is the theory (at least in appearance). Three alternative views on this difficulty have been advanced:

(1) The true theory of the strong interactions does not possess $SU(6)$ symmetry; the symmetry appears only in the non-relativistic limit.[2] The hydrogen atom provides an example of a dynamical system where something very similar occurs. The trouble with this idea is that all attempts to implement it involve a model of the observed baryons and mesons as bound states of fundamental unitary triplets ("quarks"). Because there exist no such triplets below 3 BeV, the binding energy of the nucleon must be more than $\frac{8}{9}$ of the rest energy of its components; this is an ultrarelativistic situation, not a nonrelativistic one.

(2) The true theory of the strong interactions does not possess $SU(6)$ symmetry; $SU(6)$ is the dynamical symmetry group of a particular problem—for example,[3] the baryon-resonance system in the static limit. This viewpoint is extremely attractive, but, at least in its current state of development, it severely reduces the realm claimed for $SU(6)$. For instance, it offers no reason to believe that $SU(6)$ has any application to purely mesonic systems.

(3) The true theory of the strong interactions possesses $SU(6)$ symmetry in some badly broken, but still relativistic, approximation. In the non-relativistic limit, the $SU(6)$ group acts in the familiar way on spin variables, but, for higher energies, its behavior does not have such a simple interpretation. If we adopt this view, we have to answer such questions as, "What are the Lorentz transformation properties of $SU(6)$?" or, equivalently, "What is the structure of the smallest group containing $SU(6)$ and the Poincaré group?"[4]

It is our purpose here to investigate this last alternative and explain why we find it unsatisfactory.

In Sec. II we examine a theory recently proposed by several authors.[5] In this theory $SU(6)$ is imbedded in a group isomorphic to $U(6)\otimes U(6)$ whose generators have known Lorentz transformation properties. We calculate the smallest group containing the Poincare group and this group and find its irreducible unitary representations. These representations turn out to have a catastrophic property: each of them contains an infinite number of states with the same four-momentum. Thus this theory predicts an infinite number of different types of elementary particles, all in the same supermultiplet, all with the same mass.

In Sec. III we examine another group that contains both $SU(6)$ and the Poincaré group. The irreducible unitary representations of this group contain only a finite number of states for any given four-momentum, but they each contain a continuum of masses. Thus, once again, we have an infinity of particles in the same supermultiplet, but in this case it is a mass infinity,

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[1] F. Gürsey and L. Radicatti, Phys. Rev. Letters **13**, 173 (1964); A. Pais, *ibid.* **13**, 175 (1964); B. Sakita, Phys. Rev. **136**, B1756 (1964).

[2] B. Sakita (Ref. 1); P. Freund and B. Lee, Phys. Rev. Letters **13**, 592 (1964).

[3] R. Capps, Phys. Rev. Letters **14**, 31 (1965); J. G. Belinfante and R. Cutkosky, *ibid.*, p. 33 (1965).

[4] M. Gell-Mann suggests that a group might be useful for classifying particles even if it has no connection whatsoever with the approximate symmetries of the world (private communication). For example, useful information can be obtained from the assumption that the operators of the group turn single-particle states at rest into single-particle states at rest. There is no need to assume that they transform any other states in any simple way, as would be implied by the third view above. This would be a fourth viewpoint.

[5] R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964). Similar groups have also been discussed by K. Bardakci, J. Cornwall, P. Freund, and B. Lee, *ibid.* **13**, 698 (1964); by S. Okubo and R. E. Marshak, *ibid.* **13**, 818 (1964); and by R. Delbourgo, A. Salam, and J. Strathdee (unpublished).

rather than a spin infinity. We conjecture that this infinite-particle catastrophe afflicts any theory except those which are constructed by combining an internal symmetry group and the Poincaré group in a certain trivial way. If this conjecture is correct, it is impossible to formulate a satisfactory relativistic $SU(6)$-invariant theory.

In Sec. IV we prove the conjecture for a large class of Lie groups (semi-direct products of semi-simple groups and Abelian groups) under a restrictive assumption (that the translations are contained in the Abelian group).

Sec. V contains some speculations and our conclusions.

II. $U(6)\otimes U(6)$ AND ITS GENERALIZATIONS

Let us consider two unitary triplets of two-component Weyl fields, u_+ and u_-. The very strong interactions of these fields must be invariant under the action of the group $U(3)$ and also under the action of the connected group of inhomogeneous Lorentz transformations, the Poincaré group, P.

The action of $U(3)$ is generated by the infinitesimal transformations

$$\delta u_\pm = i\lambda_a u_\pm\,, \tag{1}$$

where λ_a are a set of nine independent 3×3 Hermitian matrices. That of the Poincaré group is generated by infinitesimal transformations of the three forms,

$$\delta u_\pm = \tfrac{1}{2}i\sigma_i u_\pm + \epsilon_{ijk}x^j\partial^k u_\pm\,, \tag{2a}$$

$$\delta u_\pm = \pm\tfrac{1}{2}\sigma_i u_\pm + x^0\partial^i u_\pm - x^i\partial^0 u_\pm\,, \tag{2b}$$

and

$$\delta u_\pm = \partial_\mu u_\pm\,. \tag{2c}$$

These generate rotations, pure Lorentz transformations, and translations, respectively. We call those terms in (2) that involve derivatives the space parts of the transformations; the remainder we call the spin parts.

Several authors[5] have suggested that, in some badly broken[6] but still useful approximation, the strong interactions are invariant not only under the transformations (2) and (3), but also under the transformations

$$\delta u_+ = i\lambda_a\sigma_\mu u_+,\quad \delta u_- = 0\,, \tag{3}$$

and

$$\delta u_+ = 0,\quad \delta u_- = i\lambda_a\sigma_\mu u_-\,, \tag{4}$$

where $\sigma_\mu = (1,\boldsymbol{\sigma})$. The matrices occurring in (3) form a set of 36 independent 6×6 Hermitian matrices, which must span the space of all such matrices, and thus generate the group $U(6)$. When we add the transformations of the form (4), we obtain $U(6)\otimes U(6)$.

This group is not Lorentz-complete. The commutator of a transformation of the form (3) with one of the form (2a) or (2c) is indeed a transformation of the form (3). However, commutation with (2b) yields the transformations

$$\delta u_+ = \lambda_a\sigma_\mu u_+,\quad \delta u_- = 0\,. \tag{5}$$

The missing factor of i is a consequence of the fact that the commutator of an infinitesimal rotation and an infinitesimal Lorentz transformation is a Lorentz transformation, not a rotation. The matrices occurring in (5) form a set of 36 independent *anti-Hermitian* matrices. When adjoined to those in (3), they form a set of generators for the group of all nonsingular complex-valued 6×6 matrices, $GL(6)$. When we apply corresponding arguments to the transformations (4), we obtain $GL(6)\otimes GL(6)$. This group is the Lorentz-completion of the group $U(6)\otimes U(6)$.[7]

Even $GL(6)\otimes GL(6)$ is not the full group generated by the transformations (2) and (3). However, it is easy to see what this group is. If we multiply a transformation of the form (2) by an appropriately chosen infinitesimal generator of $GL(6)\otimes GL(6)$, we may eliminate the spin part of the transformation. The space parts generate a group isomorphic to P, which we call P'. The transformations of P' evidently commute with those of $GL(6)\otimes GL(6)$; thus the full group is

$$G = GL(6)\otimes GL(6)\otimes P'\,. \tag{6}$$

A general element of G may be written as (g_1,g_2,Λ,a) where g_1 and g_2 are elements of $GL(6)$, Λ is a homogeneous Lorentz transformation, and a is a translation. The true Poincaré group P consists of all elements of G of the form $(\Lambda,\Lambda,\Lambda,a)$.

Invariance under the $U(6)\otimes U(6)$ group, together with invariance under P, implies invariance under this 154-parameter group. It should be emphasized that this result is critically dependent on the explicit realizations of these groups as transformations on Weyl fields; this alone enables us to calculate the commutators of the two types of transformations. If we consider these structures as abstract groups only, no such result holds.[8]

We would now like to examine some properties of the approximate world in which G is an exact symmetry of the fundamental interactions. In particular, we would like to find the irreducible unitary representations of G,

[6] It should be emphasized that none of the authors cited in Ref. 5 claims to be able to construct a Lagrangian field theory for which this group is an exact symmetry. Bardakci *et al.* write down an interaction Lagrangian which is invariant under these transformations, but the free Lagrangian breaks the symmetry. Perturbation expansions are done with the free Lagrangian as a perturbation. Likewise, Feynman *et al.* construct operators which effect the transformations (3) and (4) on the fields at any fixed time, but these operators at different times need not be equal.

Nevertheless, it is difficult to see how these theories can be used for classifying hadronic states, and how they can be connected with the $SU(6)$ theories of Ref. 1, if the group in question is not an approximate symmetry of the strong interactions.

I would like to thank Professor G. Zweig and Professor B. Lee for discussions of their work.

[7] This was first shown to me by R. Sawyer. It was also known at an early date to S. L. Glashow (private communications).

[8] $U(6)\otimes PU(7)$, where $PU(7)$ is the group defined at the beginning of the next section, is a counter example. This is a 99-parameter group, and so cannot contain G.

for the one-particle states must transform like the basis for a sum of these; thus knowledge of these tells us the possible supermultiplets of elementary particles.[9]

The necessary analysis is a trivial extension of the classic work of Wigner[10] on the Poincaré group. Let P_μ be the four generators of space translations. Then

$$m^2 = P_\mu P^\mu \tag{7}$$

commutes with all the generators of the group and is a constant for all the irreducible representations. Thus, all the particles in a supermultiplet have the same mass. We are, of course, particularly interested in the cases where m^2 is greater than or equal to zero. Let us consider the first case. The P_μ all commute and may therefore all be simultaneously diagonalized. In any irreducible representation all momentum four-vectors on the mass hyperboloid must occur. Let us consider those states for which P_μ has the value $(m,0,0,0)$, and let us define the little group of G as that subgroup which leaves this set invariant. Then, as Wigner has shown, the irreducible unitary representations of G are characterized by the irreducible unitary representations of the little group.

In our case, the little group is $GL(6)\otimes GL(6)\otimes SO(3)$. Now $GL(6)$ is noncompact; furthermore, the only compact factor groups of $GL(6)$ are Abelian. Thus, all the unitary representations of $GL(6)$ are either one-dimensional or infinite-dimensional. If all the elementary particles transform according to the one-dimensional representations of $GL(6)$, there is no point in putting $GL(6)$ in the theory in the first place—$GL(1)$ would have done as well. If they do not transform in this trivial way, there are an infinite number of elementary particles in every supermultiplet.[11]

A common criterion for proposed higher symmetry schemes is this: The more unobserved particles a theory places in a supermultiplet, the less plausible it is. By this standard, the scheme discussed here is the least plausible yet proposed.

[9] There is a well-known method, shorter than this one, for finding at least some of the possible supermultiplets; this is to look for the finite-dimensional representations of the homogeneous group. (For the Poincaré group, this method gives all of the positive-mass representations and some of the zero-mass ones.) These representations tell us the possible fields directly; thus, once we have identified the positive- and negative-frequency parts of the fields with annihilation and creation operators, we know the possible states. Unfortunately, this identification depends on the canonical commutators; since the transformations of $GL(6)$ do not preserve these, the short method cannot be justified for our problem. In fact, it is easy to see that for the group G discussed above it gives patently false results.

The reader should be warned that almost all statements in the literature on the content of G supermultiplets have been obtained by this method, and are incorrect.

[10] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[11] The same analysis applies to the mass-zero representations. Here the little group is $GL(6)\otimes GL(6)\otimes E(2)$, where $E(2)$ is the Euclidean group on two variables.

III. ANOTHER EXAMPLE AND A CONJECTURE

Let us define $PU(n)$ as the connected part of the group of all linear transformations on n complex variables which leave unchanged the Hermitian quadratic function of pairs of points,

$$(x-y)^*{}_\rho g^{\lambda\rho}(x-y)_\lambda\,,$$

where ρ and λ run from 1 to n, and g is a Hermitian matrix with signature $(+-\cdots-)$. $PU(n)$ is the generalization of the Poincaré group to unitary transformations. It has n^2+2n generators; n^2 of these generate homogeneous transformations and $2n$ generate translations. It will be convenient to gather these last into a complex n-vector $\mathcal{P}_\lambda$.

$PU(7)$ evidently contains both $SU(6)$ (as the homogeneous transformations of determinant one on the last six variables) and P (as the real transformations on the first four variables). Thus it is a possible candidate for the underlying symmetry group of a relativistically invariant theory that contains $SU(6)$. Just as in the last section, let us find the irreducible unitary representations. $M^2=\mathcal{P}_\lambda{}^\dagger\mathcal{P}^\lambda$ commutes with all the generators of the group and is a constant for all the irreducible representations. The $\mathcal{P}_\lambda$ and the $\mathcal{P}_\lambda{}^\dagger$ all commute and may be simultaneously diagonalized. In any irreducible representation all complex seven-vectors on the hyperboloid of constant M^2 must occur.

But this is already catastrophic! For the true mass, the mass we measure experimentally, is not M^2 but

$$m^2 = P_\mu P^\mu (\mu = 0, 1, 2, 3) \tag{8}$$

$$= \tfrac{1}{4}\sum_{\lambda=1}^{4}(\mathcal{P}_\lambda+\mathcal{P}_\lambda{}^\dagger)(\mathcal{P}^\lambda+\mathcal{P}^{\lambda\dagger})\,, \tag{9}$$

and this assumes a continuum of values for any irreducible representation, running from $-\infty$ to $+\infty$.[12]

We emphasize that the disaster here is of the same general kind as that of Sec. II, though different in detail. There we had an infinite number of states for any given four-momentum—a "spin infinity"—but only one mass in a supermultiplet. Here we have only a finite number of states for any given mass (the little group is $U(6)$, a compact group), but an infinite number of masses occurs in any supermultiplet—a "mass infinity." In both cases, however, an infinite number of elementary particles occurs in any supermultiplet, and, furthermore, in such a way that an infinite number of elementary particles occur in a finite-mass range. It is this latter feature that makes the situation especially unpleasant: We would probably be willing to accept a theory with an infinite number of particles, as long as they were spread out in mass in such a way that experi-

[12] The same difficulty afflicts the $SL(6)$ model of Michel and Sakita (unpublished), and the Lie-group contraction to flat space of the De Sitter space theory of Roman and Aghassi [Phys. Letters **14**, 68 (1965)]. In these cases, for some representations the continuum runs from a finite lower bound up to infinity.

ments conducted at limited energy could only detect a finite number of them. (Similar phenomena occur in certain nuclear models.[13])

These considerations lead us to propose the following definitions:

Definition. Let G be a group that contains the Poincaré group and let D be a unitary representation of G. We say "D is particle-finite" if D, when restricted to P, decomposes into the direct sum of positive-mass representations of P, in such a way that, if M is any positive number, there occurs only a finite number of representations of P with mass less than M.

Definition. Let G be a group that contains P. We say "G is particle-finite" if there exists at least one locally faithful unitary particle-finite representation of G.

The local faithfulness in the above definition is merely a technical restriction. We want the elementary particles to transform according to a particle-finite representation of G. If this representation is not locally faithful, we can find a factor group of G that is represented faithfully, and use that as our fundamental symmetry group.

In the language of these definitions, the trouble with the two groups discussed above is that they are not particle-finite.

It is easy to construct a large family of particle-finite groups. Unfortunately, they are almost completely uninteresting from the view-point of physics. This family is defined below.

Definition. Let $D_a{}^b(\Lambda)$ be any real matrix representation of the homogeneous Lorentz group. Let us add to the generators of the Poincaré group a set of operators Q_a, whose commutators are given by

$$[Q_a,P_\mu]=0\,, \tag{10a}$$

$$[Q_a,Q_b]=0\,, \tag{10b}$$

and

$$\Lambda^{-1}Q_a\Lambda=D_a{}^b(\Lambda)Q_b\,. \tag{10c}$$

The enlarged set of generators generates an extension of the Poincaré group. We call this a trivial extension.[14]

Trivial extensions of the Poincaré group are evidently particle-finite. In the particle-finite representations, the eigenvalues of the Q's are simply tensor functions of the eigenvalues of the P's. Each particle-finite representation of the extended group contains only one representation of P. It is for these reasons that we use the pejorative adjective "trivial."

Conjecture. Every connected particle-finite Lie group is locally isomorphic to the direct product of a compact Lie group and a trivial extension of the Poincaré group.

I have been unable to prove this conjecture, but I have also been unable to find a counter example. I have been able to prove a more restricted form of the conjecture; this proof is presented in the next section.

If the conjecture is correct, any attempt to formulate a relativistic $SU(6)$ theory in the sense discussed in the introduction, indeed, any attempt to combine space-time and internal symmetries in any but the most trivial way, is doomed to failure.

IV. PROOF OF A THEOREM

In this section we will state and prove a restricted version of the conjecture of Sec. III. To simplify some of the equations, we use the symbols $\cong$ for isomorphism, $\doteq$ for local isomorphism, and $\times$ for semi-direct product.

Let G be the group referred to in the conjecture. Then we will prove the conjecture under the additional conditions:

$$(1)\quad G\doteq S\times A\,,$$

where S is semi-simple and A is Abelian; and

$$(2)\quad A\supseteq T,$$

where T is the group of space-time translations.[15]

These two conditions are severe restrictions. We adopt them because they enable us to analyze the unitary representation sof G, using the method invented by Wigner for the Poincaré group.[10]

The remainder of this section consists of the proof. Let P_μ be the generators of T, and let Q_a be the other generators of A. Let us suppose that there exists a faithful, unitary, particle-finite representation of G, which we call D. We use the same method as before, and begin by diagonalizing the generators of A. There can be no linear combination of these operators whose eigenvalues are everywhere zero, for if this were the case, D would not be faithful. The elements of G induce, through inner automorphisms, linear transformations on the generators of A, and therefore on their eigenvalues, which we call p_μ and q_a. The same is true for any subgroup of G. Let L be the group of homogeneous Lorentz transformations; then we may divide the P's and the Q's into sets which transform according to various irreducible representations of L. The P_μ transform like a vector, and the Q's like various irreducible tensors. If D is to be particle finite, the q's must be tensor functions of p_μ. These functions are determined, within a scale factor, by the transformation properties

[13] S. Goshen (Goldstein) and H. Lipkin, Ann. Phys. (N.Y.) **6**, 301 (1959). I am indebted to Professor Lipkin for removing my considerable confusion on this point. [*Note added in proof.* A beautiful theorem recently proven by O'Raifeartaigh [Phys. Rev. Letters **14**, 575 (1965)] shows that this phenomenon cannot occur in relativistic theories. If D is any irreducible representation of any connected Lie group containing P, for which the representation of P is unitary, then the mass spectrum of D is either a continuum or a single point. There is never any band structure.]

[14] To my knowledge, these objects were discovered by P. Federbush (private communication).

[15] The theorem is also true if we consider ray representations as well as true representations. Ray representations introduce multiples of the identity operator on the right-hand side of the commutation relations which define the Lie algebra of the representation. However, we may always enlarge the group by adding the identity to its generators; the ray representation then becomes a true representation of the enlarged group. If the original group satisfies our conditions, so does the enlarged group, and the remainder of the proof is as above.

of the Q's. Under the action of S, these scale factors must change continuously. If they run through a continuum of values, D is not particle finite. Therefore they are invariant. By the same reasoning, $P_\mu P^\mu$ is invariant. From now on we will restrict ourselves to the manifold of states for which these invariants have constant values. Let the action of S turn $(P_\mu,Q_a,p_\lambda,q_b)$ into $(P'_\mu,Q'_a,p'_\lambda,q'_b)$. P_μ' is a linear function of P_λ and Q_a; however, when we write q_a as a function of p_μ, we find p'_μ as a polynomial function of p_μ. We may reduce the degree of this polynomial by using the fact that $P_\mu P^\mu$ is a constant; let us suppose the polynomial has been reduced to its minimum degree d. Let $\hat{Q}$ be that one of the Q's which transforms like a tensor of maximum rank, and let this rank be r. Since $\hat{Q}'$ is a linear function of the Q's and P's, $\hat{q}'$ is a polynomial function of p_μ of degree $\leq r$. On the other hand, since $\hat{q}'$ is a known polynomial function of p'_μ of degree r, it is a polynomial function of p_μ of degree rd. The only way these two statements can be consistent is if $d=1$. Thus p'_μ is a linear function of p_μ. Since $p_\mu p^\mu$ is an invariant, this transformation defines a mapping of S into the set of all linear transformations leaving this quadratic form invariant, that is to say, into L.

It is clear that this mapping is a homomorphism. Since $G\supseteq P$, the homomorphism must be onto. Let K be its kernel. Then

$$S/K\cong L.$$

But S is a semisimple connected Lie group; all of its normal subgroups are locally direct factors, and thus

$$S\doteq K\otimes L.$$

By the definition of K, $[K,T]=0$. If $[K,Q]\neq 0$, the eigenvalues of the Q's must run through a continuum of values, for fixed four-momentum. Since D is particle finite, this cannot be. Therefore,

$$[K,A]=0$$

and

$$G\doteq K\otimes(L\times A).$$

The little group of G has K as a direct factor. If K is not compact, the unitary faithful representations of the little group are infinite dimensional, and this cannot be. Thus K must be compact. It is easy to show that $L\times A$ must be a trivial extension of P, and hence the theorem is proved.[16]

[16] Michel and Sakita (Ref. 10) have established independently somewhat more restricted results, using similar methods. The conditions of the Michel-Sakita theorem exclude the trivial extensions of P, and their conclusions apply only to the factor group G/T, not to the full group. I am indebted to Dr. Sakita for an enlightening discussion of this work.

V. SPECULATIONS AND CONCLUSIONS

We conclude with a sequence of speculative remarks, arranged in order of decreasing optimism.[17]

(1) Perhaps the conjecture of Sec. III is false. In this case, particle-finiteness suggests itself as a valuable criterion for proposed higher symmetry schemes. Since every connected Lie group may be written as a semi-direct product of a semi-simple group and a solvable group, the theorem of Sec. IV shows that to construct a particle-finite group, we must—speaking very loosely—either put the translations in a semi-simple group or put them in a non-Abelian solvable group. It is easy to imbed P in a semi-simple group; the familiar representation of P as a set of 5×5 matrices displays it as a subgroup of $GL(5)$. But this construction (and all similar ones I have investigated) involves including an element with the effect of a dilatation, and thus insuring that the group is not particle-finite.

(2) If the conjecture is true, perhaps we can escape its consequences by looking for objects more general than unitary representations. Certainly the Poincaré group and the $SU(3)$ group must both be represented by unitary transformations, if we are not to sacrifice the understanding of these invariances we already possess, but there is no real reason why this should be the case for the other elements of G. Then these transformations would not represent invariances of the system in the usual sense, but might still give us information about selection rules and the structure of supermultiplets. Straightforward analysis shows that this relaxation of our conditions does not help matters for the group discussed in Sec. II, but the general case remains to be investigated.

(3) In the same vein, perhaps we will be forced to objects more general than Lie groups. Infinite-parameter continuous groups are sufficient for our purposes. For example, the group of all unitary transformations on the positive-energy states of a single spinless particle contains P and is particle finite, but it is not a semi-direct product. A more interesting example, both because its structure constants may be specified in closed terms and because it leads to a nontrivial mass spectrum, may be constructed in the following way: Let T_i be a set of three operators that obey the angular-momentum commutation rules. Let us consider the infinite-dimensional Lie algebra spanned by the generators of the homogeneous Lorentz transformations, plus all operators of the form of polynomials in the P_μ multiplied by the T_i. If we identify the generators of translations with the operators $P_\mu T_x$, we have an embedding of the Lie algebra of P inside this algebra. In the particle-finite representations, the masses of the particles either run from zero, through the positive

[17] The speculations in this section are a product of conversations at the Second Coral Gables Conference on Symmetry Principles at High Energies. I profited especially from discussions with Professor R. Hermann, Professor B. Lee, Professor L. Michel, and Professor E. C. G. Sudarshan.

integers, to some upper bound, or run from $\frac{1}{2}$, through the half-odd integers, to some upper bound. It is clear that we can build parallel structures using an arbitrary Lie group instead of $SO(3)$, and thus obtain considerably more complicated mass spectra.

(4) Nevertheless, although these are all intriguing possibilities, they remain at the moment unfulfilled hopes. The situation is dark; all we can say with confidence is that the construction of a relativistic $SU(6)$ theory along the lines described in Sec. I is a more difficult task than previously has been imagined.

ACKNOWLEDGMENTS

Most of the differences between the first version of this paper and the current one are the results of conversations at the Second Coral Gables Conference on Symmetry Principles at High Energies. Individual debts have been acknowledged in the notes; here I would like to thank the University of Miami and Professor B. Kurşonoglu for their hospitality. I would also like to thank Professor Paul Federbush for several extraordinarily helpful suggestions made during the progress of this work.

APPENDIX ADDED IN PROOF: ON SUPPLEMENTARY CONDITIONS

In the body of this paper, we have argued that a relativistic $SU(6)$ symmetry inevitably leads to the appearance of a plethora of undesired states. There is, of course, a standard remedy for such a condition; this is the introduction of a supplementary condition. A subspace of the set of all states is defined as the space of physical states; the orthogonal subspace is defined as the space of unphysical states. The physical S matrix is defined as the original S matrix restricted to the space of physical states. This automatically removes the undesired states; the danger is that it may also remove unitarity. If the original S matrix has nonvanishing matrix elements between physical and unphysical states, the squares of the transition probabilities, restricted to physical states, will no longer sum to one.[18]

In this Appendix we discuss two suggested supplementary-condition theories that violate unitarity.

1. Fulton and Wess[19] have considered a symmetry group in which the four-momenta are part of a set of 36 commuting operators. It has been suggested that the 32 extra operators ("the unphysical momenta") be set equal to zero for initial and final states in any scattering process. This removes the unwanted states for the one-particle subspace; however, in elastic scattering, the particles can exchange unphysical momentum, and thus, particles beginning in physical states may scatter to unphysical states.

2. Let Φ^a be a set of fields transforming according to some (perhaps reducible) representation of the homogeneous Lorentz group, and let us consider a Lagrangian of the form

$$\mathcal{L}=\partial_\mu\Phi^a\partial^\mu\Phi_a+\mathcal{L}',$$

where $\mathcal{L}'$ is invariant but involves no derivatives. The theory defined by this Lagrangian is invariant under a group of transformations of the form $L\otimes P'$, where L is the homogeneous Lorentz group acting on spin indices only, and P' is the group of purely spatial Lorentz transformations. It is clear that by choosing the interaction Lagrangian properly, such a theory may be made invariant under a larger group of the form $G\times P'$, where G is a group acting on spin indices that may contain $SU(6)$.

The difficulty with this theory is that the quadratic form $\Phi^a\Phi_a$ is not positive-definite; thus, the quanta of some field-components are associated with states of negative norm. This difficulty can be removed by a supplementary condition. The physical space must consist only of states of positive norm and must be Lorentz-invariant, but otherwise it may be chosen arbitrarily.[20] Since the original theory involves states of negative norm, it is not as evident as in the previous example that unitarity is necessarily lost, but specific calculation shows this to be the case for all the theories of this type which have been proposed.[21]

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[18] Supplementary conditions may be imposed in other ways, which do preserve unitarity. For example, if we define K by $S=(1+iK)/(1-iK)$, we may apply a supplementary condition to K rather than to S. The unitarity of S is equivalent to the Hermiticity of K; supplementary conditions, as defined above, preserve Hermiticity, so unitarity cannot be lost. However, in this case, crossing symmetry, which is automatically preserved in the S-matrix-restriction procedure, is endangered.

[19] T. Fulton and J. Wess (unpublished). The group is the same as that of Ref. 10.

[20] Many theories of this sort have been proposed, although not in the language used here. Some examples are: A. Salam, Proceedings of the Second Coral Gables Conference on Symmetry Principles at High Energies (to be published); M. Bég and A. Pais, Phys. Rev. Letters **14**, 267 (1965); B. Sakita and K. Wali, *ibid.* **14**, 404 (1965). The choice of the physical subspace corresponds to "choosing a boost" in the language of Bég and Pais.

[21] M. Bég and A. Pais, Phys. Rev. Letters **14**, 509 (1965); R. Blankenbecler, M. L. Goldberger, K. Johnson, and S. B. Treiman, *ibid.* **14**, 518 (1965). In fact, even if we loosen our requirements by allowing the original S matrix to be nonunitary, the restricted S matrix still cannot be made unitary.