where K denotes the channel momenta, and the F's are the Jost matrices. The Regge poles are determined from the relation

$$\det F(\lambda, -K)=0.$$

The above relation implies that there is a constant (row) vector a such that at a Regge pole

$$aF(\lambda, -K)=0.$$

Using the same technique as in the single-channel case,[6] namely calculating the Wronskian of the wave function and its first derivative with respect to λ, we obtain

$$y\dot{F}^{\mathrm{T}}(\lambda, -K)[F^{\mathrm{T}}(\lambda,K)]^{-1}Ky^{\mathrm{T}} = -i\lambda\int_0^\infty dr\, r^{-2} y f^{\mathrm{T}}(\lambda, -K, r) f(\lambda, -K, r) y^{\mathrm{T}},$$

where "T" means transpose, and the dot means differentiation with respect to λ; $y=aF(\lambda,K)K^{-1}$ is a column vector and f is the matrix of the irregular wave functions defined by

$$f_{ij}(\lambda, -K, r) \to \delta_{ij}e^{-ik_jr}, \quad \text{as} \quad r\to\infty,$$

k_j being the channel momentum in the j channel. At a Regge pole $l=\alpha$, and yf^{T} is a well-behaved function at the origin. It vanishes like $r^{\alpha+1}$ as $r\to 0$. One can easily check that

$$e^{i\pi(\lambda-\frac{1}{2})}y\frac{\partial}{\partial l}(S^{\mathrm{T}})^{-1}Ky^{\mathrm{T}} = y\dot{F}^{\mathrm{T}}(\lambda, -K)[F^{\mathrm{T}}(\lambda,K)]^{-1}Ky^{\mathrm{T}}.$$

Now let

$$S_{ij}=(g_{ij}/l-\alpha)+r_{ij}.$$

Then

$$\frac{\partial}{\partial l}(S)_{ij}{}^{-1}=\frac{z_{ij}}{\mathrm{Tr}(gz)},$$

where z is the matrix of the cofactors of r and "Tr" means trace. Thus

$$e^{i\pi(\lambda-\frac{1}{2})}\frac{yz^{\mathrm{T}}Ky^{\mathrm{T}}}{\mathrm{Tr}(gz)} = -i\lambda\int_0^\infty dr\, r^{-2}\hat{f}^{\mathrm{T}}\hat{f},$$

where $\hat{f}=fy^{\mathrm{T}}$ is a column vector. If we denote the residue of the symmetric T matrix by β then $g=2iK\beta$ and we get

$$\mathrm{Tr}(\beta e^{-i\pi\alpha}u)=\left(2\lambda\int_0^\infty dr\, r^{-2}\hat{f}^{\mathrm{T}}\hat{f}\right)^{-1}, \qquad \text{(A20)}$$

where

$$u=zK/yz^{\mathrm{T}}Ky^{\mathrm{T}}. \qquad \text{(A21)}$$

Note that the matrix elements u_{ij} are not necessarily positive.

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Normalization Condition and Normal and Abnormal Solutions of the Bethe-Salpeter Equation*

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The normalization constants for normal and abnormal solutions of the Bethe-Salpeter equation are explicitly calculated in the Wick-Cutkosky model. In the case of the vanishing total four-momentum, it is shown that the abnormal solutions with odd κ have a negative norm, where κ is the Wick-Cutkosky quantum number, and that the corresponding scattering Green's function contains all the normal and abnormal solutions as the residues of poles. It is demonstrated explicitly for $\kappa=0, 1, 2, 3$ that the first conclusion remains true also in the case of an infinitesimally positive mass. As for the case of massless bound states, its special character is emphasized, and solid harmonics are constructed corresponding to the "little group" for a massless particle. Non-Cutkosky integral equations are obtained for the weight functions of the integral representation for the Bethe-Salpeter amplitude.

1. INTRODUCTION

IN 1954, Wick[1] and Cutkosky[2] obtained a complete set of the solutions of the Bethe-Salpeter (B-S) equation for bound states of two scalar particles exchanging massless scalar particles in the ladder approximation. They discovered that in addition to normal solutions there exist abnormal solutions which have no counterparts in the nonrelativistic potential theory. The appearance of these extra solutions is intimately related to the additional freedom in a covariant two-body problem, i.e., "relative time" or "relative energy," which leads to the introduction of a new quantum number κ (normal solutions correspond to $\kappa=0$).

Scarf and Umezawa[3] tried to exclude the abnormal

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solutions by the non-normalizability due to the degeneracy of their eigenvalues at zero binding energy, but this criterion is not adequate because the degeneracy does not occur if the exchanged particles are not massless.[4] Ohnuki, Takao, and Umezawa[4] showed that the B-S equation has abnormal solutions even in a static model and that they do not correspond to the eigenstates of the original Hamiltonian. If the latter conclusion is true also in the relativistic case, it forbids one to interpret κ as a physically meaningful quantum number such as the strangeness.[5] Mugibayashi[6] showed that abnormal solutions appear even in the exact B-S equation for the static model, but they do not appear if one considers two first-order equations from which his exact B-S equation follows.

In order to distinguish abnormal solutions from normal ones mathematically, Ida and Maki[7] and the present author[8,9] investigated the analytic properties of the B-S amplitude for two scalar particles by using integral representations, but no appreciable difference between them was found. Recently, the present author[10–12] has explored whether or not abnormal solutions are related to the high-energy behavior of the scattering Green's function in the crossed channel, as was suggested by the Regge-pole theory. It has been concluded there that in the asymptotic expansion in powers of a certain invariant, no abnormal solutions appear in the forward scattering, but those with even κ only seem to appear in the nonforward scattering.

Now, the purpose of the present paper is to investigate the normalizability of normal and abnormal solutions. The normalization condition for the B-S amplitude was proposed by Nishijima,[13] Mandelstam,[14] and Klein and Zemach.[15] Their prescription was based on the assumption of the existence of a conserved current. Recently, Cutkosky and Leon[16] have proposed an elegant derivation of the normalization condition without assuming a conserved current. In the present paper, we shall use the normalization formula given by Cutkosky and Leon with a slight but nontrivial modification. We shall not discuss another normalization condition, proposed by Sato,[17] which is linear with respect to the B-S amplitude.

The next section is devoted to the general consideration of the solid harmonics of Wigner's "little group,[18]" with respect to the total four-momentum of the system, because some confusion seems to prevail concerning the solutions for massless bound states. In Sec. 3, we present a derivation of the normalization condition in a slightly different way from that of Cutkosky and Leon. In Secs. 4–6, our consideration solely concerns the Wick-Cutkosky model. In Sec. 4, we consider the case of the vanishing total four-momentum. We explicitly find a complete set of solutions and their normalization constants. It is demonstrated that all the solutions appear in the scattering Green's function as the residues of poles. In Sec. 5, the normalization condition is analyzed for the bound states with an infinitesimal but nonzero mass. Section 6 deals with the special situation in the case of massless bound states. Our results are summarized in the final section.

2. SOLID HARMONICS OF LITTLE GROUP

A bound-state amplitude should have a definite transformation property under the inhomogeneous Lorentz group. Let x_μ and $2k_\mu$ be the center-of-mass coordinate and the total four-momentum of the system, respectively. Then the B-S amplitude behaves like $e^{i(2k)x}$ as an irreducible representation of the translation group. After separating this factor, the B-S amplitude for the internal freedom should be transformed according to Wigner's little group[18] with respect to k_μ, which is the set of all the homogeneous Lorentz transformations under which k_μ remains invariant. Its group structure depends on the property of k_μ.

(a) $k^2>0$ (time-like). The little group $\mathfrak{G}$ is isomorphic to the three-dimensional rotational group.

(b) $k^2=0$ but $k_\mu\neq 0$ (light-like). $\mathfrak{G}$ is equivalent to the two-dimensional Euclidean group.

(c) $k_\mu=0$. $\mathfrak{G}$ becomes identical with the homogeneous Lorentz group itself.

(d) $k^2<0$ (space-like). $\mathfrak{G}$ is isomorphic to the $(2+1)$-dimensional homogeneous Lorentz group.

Now, we introduce "solid harmonics of little group" by the following definition. An lth-order solid harmonic of the little group $\mathfrak{G}$ with respect to k_μ is a homogeneous lth-order polynomial $X_l(p)$ of p_0, p_1, p_2, p_3 such that

$$(\partial/\partial p)^2 X_l(p)=0\,, \tag{2.1}$$

$$k_\mu(\partial/\partial p_\mu)X_l(p)=0\,. \tag{2.2}$$

Then it is evident that the totality of the lth-order solid harmonics of $\mathfrak{G}$ spans a vector space invariant under the homogeneous Lorentz transformations of p_μ belonging to $\mathfrak{G}$.

In the case $k^2>0$, we can take the rest frame ($\mathbf{k}=0$, $k_0\neq 0$). Then our solid harmonics reduce to the ordinary solid harmonics. We shall take

$$\mathcal{Y}_{lm}(\mathbf{p})\equiv|\mathbf{p}|^l Y_{lm}(\theta,\varphi)\,,\quad (|m|\leq l) \tag{2.3}$$

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[14] S. Mandelstam, Proc. Roy. Soc. (London) **233**, 248 (1955).
[15] A. Klein and C. Zemach, Phys. Rev. **108**, 126 (1957).
[16] R. E. Cutkosky and M. Leon, Phys. Rev. **135**, B1445 (1964).
[17] I. Sato, J. Math. Phys. **4**, 24 (1963).
[18] E. P. Wigner, Ann. Math. **40**, 149 (1939).

as the basic solid harmonics, where θ and φ are the polar angles of $\mathbf{p}$.

The Lth-order solid harmonics $\mathfrak{z}_{Llm}(p_0,\mathbf{p})$, in the case $k_\mu=0$, can be constructed from the four-dimensional spherical harmonics $H_{Llm}(\alpha,\theta,\varphi)$:

$$\mathfrak{z}_{Llm}(p_0,\mathbf{p})\equiv\mathcal{H}_{Llm}(-ip_0,\mathbf{p})\,, \tag{2.4}$$

$$\mathcal{H}_{Llm}(p_4,\mathbf{p})\equiv r^L H_{Llm}(\alpha,\theta,\varphi)\,, \tag{2.5}$$

with $r^2\equiv p_4^2+\mathbf{p}^2$ and $\cos\alpha\equiv p_4/r$, ($0\leq\alpha<\pi$ for real p_4), and

$$H_{Llm}(\alpha,\theta,\varphi)\equiv A_{Ll}(\sin\alpha)^l C_{L-l}^{l+1}(\cos\alpha)Y_{lm}(\theta,\varphi)\,, \quad (|m|\leqslant l\leqslant L)\,, \tag{2.6}$$

where $C_j^\omega(x)$ denotes a Gegenbauer polynomial. The normalization constant A_{Ll} is determined by the requirement

$$\int_0^\pi \sin^2\alpha d\alpha\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\varphi|H_{Llm}(\alpha,\theta,\varphi)|^2=1\,. \tag{2.7}$$

Using the orthogonality relation of $C_j^\omega(x)$,[19]

$$\int_{-1}^1 dz(1-z^2)^{\omega-\frac{1}{2}}C_j^\omega(z)C_k^\omega(z) =\frac{\pi\Gamma(2\omega+j)}{2^{2\omega-1}j!(\omega+j)[\Gamma(\omega)]^2}\delta_{jk}\,, \tag{2.8}$$

we obtain

$$|A_{Ll}|^2=2^{2l+1}(L+1)(L-l)!(l!)^2/\pi(L+l+1)!\,. \tag{2.9}$$

In the case $k^2<0$, we can choose a Lorentz frame such that $k_0=k_1=k_2=0$, $k_3\neq0$. Then the solid harmonics, in this case, are defined by

$$\tilde{\mathcal{Y}}_{lm}(p_0,p_1,p_2)\equiv\mathcal{Y}_{lm}(p_1,p_2,-ip_0)\,. \tag{2.10}$$

Finally, we consider the case $k^2=0$ but $k_\mu\neq0$. In this case, our solid harmonics $\chi_{lm}(p)$ cannot be expressed in terms of spherical harmonics. We take the frame in which $k=(k_0,0,0,k_0)$, ($k_0\neq0$). Then (2.2) and (2.1) reduce to

$$(\partial/\partial p_0-\partial/\partial p_3)\chi_{lm}(p)=0\,, \tag{2.11}$$

$$[(\partial/\partial p_1)^2+(\partial/\partial p_2)^2]\chi_{lm}(p)=0\,. \tag{2.12}$$

Hence

$$\chi_{lm}(p)=\tilde{A}(p_0+p_3)^{l-|m|}(p_1\pm ip_2)^{|m|}\,, \tag{2.13}$$

where the double sign corresponds to the sign of m. The constant $\tilde{A}$ is undetermined, because we do not have the orthogonality relation for $\chi_{lm}(p)$.

3. NORMALIZATION CONDITION

We consider an elastic scattering of two particles. Their initial momenta are $k+q$ and $k-q$, while their final ones are $k+p$ and $k-p$. The total invariant energy is $s\equiv(2k)^2$.

The two-body Green's function $G(p,q;k)$ satisfies the integral equation

$$G=K^{-1}+\lambda K^{-1}IG \tag{3.1}$$

in the operator notation. Here K^{-1} is a product of two one-body propagators, I the irreducible kernel, and λ a parameter, which is identified with the squared coupling constant if one considers the ladder approximation. Assuming the existence of G, we can formally solve (3.1):

$$G=(1-\lambda K^{-1}I)^{-1}K^{-1}\,. \tag{3.2}$$

Differentiation of (3.2) with respect to λ yields

$$\partial G/\partial\lambda=GIG\,. \tag{3.3}$$

Eliminating I from (3.3) by means of (3.1), we have[20]

$$\lambda\partial G/\partial\lambda=-G+GKG\,. \tag{3.4}$$

Now, if there exists a bound state of the two particles, G will have a pole in the complex s plane[14]:

$$G(p,q;k)=\{i\phi_a(p,k)\bar{\phi}_a(q,k)/[s-s_a(\lambda)]\}+\hat{G}(p,q;k)\,, \tag{3.5}$$

if the bound state $|a\rangle$ is assumed to have a positive norm, where $\hat{G}$ is regular at $s=s_a(\lambda)$. In (3.5), $\phi_a(p,k)$ is a B-S amplitude with an eigenvalue $s=s_a(\lambda)$, and the "conjugate" $\bar{\phi}_a$ is defined in such a way that the absorptive part of $\bar{\phi}_a$ is the Hermitian conjugate of that of ϕ_a (multiplied by γ_4 if one considers a Dirac spinor), and its dispersive part is related to its absorptive part in the same way as in ϕ_a. More concretely, if $\phi_a(p,k)$ has an integral representation

$$\phi_a(p,k)=-i\int dx_1\cdots\int dx_N \times\frac{F(x_1,\cdots,x_N;p,k)}{[V(x_1,\cdots,x_N;p,k)-i\epsilon]^m}\,, \tag{3.6}$$

where V is a real function, F being regular in p, and $\epsilon\to0^+$, then the conjugate is given by

$$\bar{\phi}_a(p,k)=-i\int dx_1\cdots\int dx_N \times\frac{[F(x_1,\cdots,x_N;p,k)]^*A}{[V(x_1,\cdots,x_N;p,k)-i\epsilon]^m}\,, \tag{3.7}$$

where A is a possible constant matrix factor. ($A=1$ if F is scalar.)

Substituting (3.5) in (3.4) and comparing the co-

[19] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (National Bureau of Standards, Washington, D. C., 1964), p. 774.

[20] This kind of nonlinear integral equation was first derived by Amati *et al.*, in a special case [D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento **26**, 896 (1962)]. Extension to the general case was made by the present author [N. Nakanishi, Phys. Rev. **133**, B1224 (1964)].

efficients of the term

$$[s-s_a(\lambda)]^{-2} \tag{3.8}$$

in both sides, we are immediately led to

$$i\bar{\phi}_a K\phi_a=\lambda ds_a/d\lambda\,, \tag{3.9}$$

or, in terms of the inverse function, $\lambda=\lambda_a(s)$, of $s=s_a(\lambda)$,

$$i\bar{\phi}_a K\phi_a=(\lambda_a'/\lambda_a)^{-1}\,, \tag{3.10}$$

where

$$\lambda_a'\equiv d\lambda_a/ds\,. \tag{3.11}$$

We have thus obtained the normalization condition for ϕ_a.[21]

Since the B-S equation

$$K\phi_a=\lambda_a I\phi_a\,, \tag{3.12}$$

together with its conjugate, leads to

$$(\lambda_a'/\lambda_a)\bar{\phi}_a K\phi_a=\bar{\phi}_a[\partial K/\partial s-\lambda_a\partial I/\partial s]\phi_a\,, \tag{3.13}$$

(3.10) can be rewritten as

$$i\bar{\phi}_a[\partial K/\partial s-\lambda_a\partial I/\partial s]\phi_a=1\,, \tag{3.14}$$

a result which is equivalent to the normalization condition of Cutkosky and Leon.[16] We shall prefer to use (3.10), however, because of the following reasons:

(a) It is generally easier to calculate (3.10) than to do (3.14).

(b) The Cutkosky-Leon normalization condition (3.14) sometimes gives a false result, as we shall see in the next section.

(c) Our formula (3.10) is akin to the orthogonality relation

$$\bar{\phi}_b K\phi_a=0\,,\quad \text{for}\quad b\neq a\,. \tag{3.15}$$

4. SOLUTIONS FOR $k_\mu=0$

We shall hereafter consider the B-S equation for two scalar particles with a unit mass exchanging massless scalar particles in the ladder approximation. In this section, we explicitly calculate the normalization condition (3.10) for the case $k_\mu=0$.

The B-S equation is

$$(1-v)^2\phi_{NLlm}(p)=\frac{\lambda_N}{\pi^2 i}\int d^4p'\frac{\phi_{NLlm}(p')}{-(p-p')^2-i\epsilon}\,, \tag{4.1}$$

with $v\equiv p^2$. From the consideration presented in Sec. 2, we can write

$$\phi_{NLlm}(p)=B_{NLl}\mathfrak{z}_{Llm}(p)f_{NL}(v)\,, \tag{4.2}$$

where B_{NLl} is a normalization constant, and $f_{NL}(v)$ was explicitly found already[11]:

$$f_{NL}(v)=-i\sum_{j=0}^{N-L-1}g_{NL}{}^j/(1-v-i\epsilon)^{N-j+2}\,, \tag{4.3}$$

with

$$g_{NL}{}^j\equiv(-1)^j(2N-j)!/j!(N-j)!(N-L-j-1)! \tag{4.4}$$

and

$$\lambda_N=N(N+1)\,. \tag{4.5}$$

The quantum numbers N and L are related to the conventional ones κ and n through

$$\begin{aligned}N&=\kappa+n\,,\\ L&=\kappa+l\leq N-1\,.\end{aligned} \tag{4.6}$$

According to (3.7), the conjugate amplitude of (4.2) reads

$$\bar{\phi}_{NLlm}(p)=B_{NLl}{}^*[\mathfrak{z}_{Llm}(p)]^*f_{NL}(v)\,. \tag{4.7}$$

Special care must be taken concerning $[\mathfrak{z}_{Llm}(p)]^*$. Since p_0 is a real quantity, from (2.4) we have

$$\begin{aligned}[\mathfrak{z}_{Llm}(p)]^*&=[\mathcal{H}_{Llm}(-ip_0,\mathbf{p})]^*\\ &=(-1)^{L-l}\mathcal{H}_{Llm}{}^*(-ip_0,\mathbf{p})\,,\end{aligned} \tag{4.8}$$

where we define $F^*(z)\equiv[F(z^*)]^*$.

In order to calculate the normalization integral, it is convenient to rewrite (4.3) in a more compact form. It is easy to see that $f_{NL}(v)$ is expressed in terms of Jacobi's polynomial[22] of the argument $(1-v-i\epsilon)^{-1}$, and, moreover, it can be rewritten in terms of a Gegenbauer polynomial.[23] In this way, we find

$$f_{NL}(v)=-i\frac{(2L+2)!}{(L+1)!}\frac{1}{(1-v-i\epsilon)^{L+3}}\times C_{N-L-1}{}^{L+\frac{3}{2}}\left(\frac{1+v}{1-v-i\epsilon}\right)\,. \tag{4.9}$$

Of course, one may verify (4.9) by substituting (4.2) with (4.9) in (4.1) directly. Since

$$K^{-1}=[-i/(1-v-i\epsilon)]^2\,, \tag{4.10}$$

in our case, the left-hand side of (3.10), with $a\equiv(NLlm)$, now becomes

$$I_{NLl}\equiv-i|B_{NLl}|^2\int d^4p|\mathfrak{z}_{Llm}(p)|^2\times(1-v)^2[f_{NL}(v)]^2\,. \tag{4.11}$$

The p_0 contour can be rotated into the imaginary axis $p_0=ip_4$, (p_4 real). Then noticing (4.8), (2.4)–(2.7), and (4.9), we obtain

$$\begin{aligned}I_{NLl}&=(-1)^{L-l}|B_{NLl}|^2\int_0^\infty dr\,r^{2L+3}(1+r^2)^2\times[f_{NL}(-r^2)]^2\\ &=-(-1)^{L-l}|B_{NLl}|^2\times[(2L+2)!/(L+1)!]^2J_{NL}\,,\end{aligned} \tag{4.12}$$

[21] A result similar to (3.10) is found also in the paper of Cutkosky and Leon.

[22] Reference 19, p. 775.

[23] Reference 19, pp. 777–778.

with

$$J_{NL}\equiv\tfrac{1}{2}\int_0^\infty dx\, x^{L+1}(1+x)^{-2L-4}\times\left[C_{N-L-1}{}^{L+\frac{3}{2}}\left(\frac{1-x}{1+x}\right)\right]^2. \quad (4.13)$$

With $z=(1-x)/(1+x)$, Eq. (4.13) is transformed into the normalization integral (2.8) of the Gegenbauer polynomial, so that

$$J_{NL}=\frac{[(L+1)!]^2(N+L+1)!}{2(2N+1)(N-L-1)![(2L+2)!]^2}. \quad (4.14)$$

Thus (3.10) leads to

$$-(-1)^{L-l}|B_{NLl}|^2\times[(N+L+1)!/2(2N+1)(N-L-1)!]=(\lambda_{N\kappa}'/\lambda_N)^{-1}. \quad (4.15)$$

In the next section, we shall show

$$\lambda_{N\kappa}'/\lambda_N=-\frac{N(N+1)+(N-\kappa)^2-1}{2(2N-1)(2N+3)}. \quad (4.16)$$

The negativeness of $\lambda_{N\kappa}'$ physically corresponds to the fact that the bound-state mass monotonically decreases as λ increases. We are thus led to the conclusion that *the bound states* (if we may call them so) *corresponding to the abnormal solutions with odd* $\kappa=L-l$ *have a negative norm*,[24] since otherwise (4.15) is not self-consistent. Therefore, we have to introduce an indefinite metric η such that

$$\langle a|\eta=(-1)^\kappa\langle a|, \quad (4.17)$$

but, instead, for simplicity we shall employ a non-mathematical notation in which $|B_{NLl}|^2$ can become negative. Thus

$$|B_{NLl}|^2=(-1)^\kappa\frac{4(2N-1)(2N+1)(2N+3)(N-L-1)!}{(N+L+1)![N(N+1)+(N-\kappa)^2-1]}, \quad (4.18)$$

with $\kappa=L-l$. Especially, for the normal solutions ($\kappa=0$), we have

$$|B_{nl}|^2=\frac{4(2n+1)(2n+3)(n-l-1)!}{(n+1)(n+l+1)!}. \quad (4.19)$$

The calculation of the normalization constant can also be made by using (3.14). Since

$$\partial K/\partial s=\tfrac{1}{2}(1-p^2+2p_0^2),\quad \partial I/\partial s=0, \quad (4.20)$$

for $s=4k_0^2=0$, the four-dimensional angular integration is no longer trivial. In addition to (2.8), we have to use a formula

$$\int_{-1}^1 dz(1-z^2)^{\omega-\frac{1}{2}}z^2[C_j{}^\omega(z)]^2\times\left\{\int_{-1}^1 dz(1-z^2)^{\omega-\frac{1}{2}}[C_j{}^\omega(z)]^2\right\}^{-1}$$
$$=\frac{j(2\omega+j)+\omega-1}{2(\omega+j-1)(\omega+j+1)}\quad(\text{for }\omega+j\neq1),$$
$$=\tfrac{1}{4}\quad(\text{for }j=0\text{ and }\omega=1), \quad (4.21)$$

which follows from the recurrence formula of the Gegenbauer polynomial[25] together with the orthogonality (2.8). The calculation of the integral over r also requires us to use (4.21). The final result is

$$|B_{NLl}|^2=(-1)^\kappa\frac{4(2N-1)(2N+1)(2N+3)(N-L-1)!}{(N+L+1)!E_{NLl}}, \quad (4.22)$$

with

$$2E_{NLl}\equiv3N(N+1)-(L+1)^2-2+l(l+1)+N(N+1)l(l+1)/L(L+2)\quad(\text{for }L\neq0), \quad (4.23)$$
$$\equiv\tfrac{7}{2}N(N+1)-3\quad(\text{for }L=l=0).$$

It is not difficult to see that (4.22) with (4.23) coincides with (4.18) only when

$$\kappa=0\ \text{or}\ 1\ \text{and}\ n=l+1\ \text{or}\ l+2. \quad (4.24)$$

In the other cases, the normalization condition of Cutkosky and Leon gives a false result. The reason is that (3.13) is no longer true in our case, because our solutions (4.2) are *not*, in general, the $s\to0$ limits of the solutions in the case $s>0$. Rearrangement of solutions due to degeneracy happens, if there are two or more states having the same N, l, m, and $(-1)^\kappa$ [e.g., ($\kappa=0$, $n=l+3$) and ($\kappa=2$, $n=l+1$)]. The four cases given in (4.24) precisely represent the states for which such rearrangement cannot occur.

We have shown that the abnormal solutions with odd κ represent ghost states. One might expect, therefore, that they would not appear in the corresponding Green's function as the residues of poles. We shall show below that this expectation is not true, namely, *all the normal and abnormal solutions do appear in the Green's function.* Here the negative sign to the residue corresponding to a negative-norm state is automatically taken into account by our notation (4.18).

The Green's function $G(p,q)\equiv G(p,q;0)$ is given by

$$G(p,q)=-(1-v_0-i\epsilon)^{-2}\times[\delta^4(p-q)+(\lambda/\pi^2i)f(t,v,v_0)], \quad (4.25)$$

[24] Hereinafter we shall omit the phrase "the bound state corresponding to" for simplicity.

[25] Reference 19, p. 782.
$2(\omega+j)zC_j{}^\omega(z)=(j+1)C_{j+1}{}^\omega(z)+(2\omega+j-1)C_{j-1}{}^\omega(z)$.

where $f(t,v,v_0)$ satisfies

$$(1-v)^2 f(t,v,v_0)=\frac{1}{-t-i\epsilon}+\frac{\lambda}{\pi^2 i}\times\int d^4p'\frac{f(t',v',v_0)}{-(p-p')^2-i\epsilon}, \quad (4.26)$$

with $t=(p-q)^2$, $t'=(p'-q)^2$, $v=p^2$, $v'=p'^2$, and $v_0=q^2$. The exact solution to (4.26) was found already[11]:

$$f(t,v,v_0)=2\int_0^1 dy\frac{(1-y)\varphi(y,v_0)}{[(1-y)(1-v)+y(-t)-i\epsilon]^3}, \quad (4.27)$$

where

$$\varphi(y,v_0)\equiv F(-\nu,\nu+1;2;-[(1-y)/y(1-v_0-i\epsilon)]), \quad (4.28)$$

with

$$\nu\equiv(\lambda+\tfrac{1}{4})^{1/2}-\tfrac{1}{2}, \quad (4.29)$$

and F denotes a hypergeometric function. When the integrand of (4.27) is more singular than y^{-1}, Hadamard's finite part[26] should be taken.

In terms of the integration variable

$$x\equiv(1-y)/y, \quad (4.30)$$

the integral (4.27) has essentially the same form as that of the S-wave solution of Goldstein's B-S equation.[27] According to the calculation in that case, we have

$$f(t,v,v_0)=2B(-\nu+1,\nu+2)(1-v-i\epsilon)^{-3}(1-v_0-i\epsilon)^{-1}\times F(-\nu+1,\nu+2;2;1+[(t+i\epsilon)/(1-v-i\epsilon)(1-v_0-i\epsilon)]), \quad (4.31)$$

where B stands for Euler's beta function. When ν tends to a positive integer N (or equivalently, $\nu\to -N-1$ because of the symmetry with respect to $\nu=-\frac{1}{2}$), $G(p,q)$ becomes infinite. This infinity corresponds to the bound-state pole at $s=0$ with $\lambda=N(N+1)$.

In order to calculate the residues of the poles, we have to consider the case $s\neq 0$. We can avoid the extremely difficult problem of finding explicitly the Green's function for $s\neq 0$ by calculating the residues of the corresponding Regge poles. The Regge trajectories are given by $\mu_\kappa(s)-j-1$, $(\kappa, j=0, 1, 2,\cdots)$,[28] where $\mu_\kappa(s)$ is determined by the continued partial-wave B-S equation.[8] The Green's function will become infinite when $\mu_\kappa(s)$ is equal to a positive integer n. Let

$$\nu_\kappa(s)\equiv\mu_\kappa(s)+\kappa. \quad (4.32)$$

Then $\nu_\kappa(s)\to\nu$ as $s\to 0$ independently of κ because of the degeneracy at $s=0$. The residue $R_{\kappa n}(s)$ of a pole at $s\neq 0$ can be calculated by means of the residue of the corresponding Regge pole:

$$\begin{aligned}R_{\kappa n}(s)&=\lim_{\mu_\kappa(s)\to n}[d\mu_\kappa(s)/ds]^{-1}\times[\mu_\kappa(s)-n]G(p,q;k)\\&=\lim_{\nu_\kappa(s)\to N}[d\nu_\kappa(s)/ds]^{-1}\times[\nu_\kappa(s)-N]G(p,q;k).\end{aligned} \quad (4.33)$$

When s tends to zero, care must be taken because all of $\nu_\kappa(s)$, $(\kappa=0, 1,\cdots, N-1)$, tend to the same limit ν. We find

$$\sum_{\kappa=0}^{N-1}\nu_{N\kappa}'\lim_{s\to 0}R_{\kappa,\,N-\kappa}(s)=\lim_{\nu\to N}(\nu-N)G(p,q), \quad (4.34)$$

with

$$\nu_{N\kappa}'\equiv d\nu_\kappa(s)/ds\,|_{s=0,\ \nu=N}. \quad (4.35)$$

The continued partial-wave B-S equation, in which λ is a *constant* parameter, gives equations

$$\lambda=\nu_\kappa(s)[\nu_\kappa(s)+1]+F_\kappa(\nu_\kappa(s))s+O(s^2), \quad (4.36)$$

for Regge trajectories, where F_κ is a certain function. When $\nu_\kappa(s)=N$, the right-hand side of (4.36) is nothing but the eigenvalue formula for $\lambda_{N\kappa}(s)$ of the B-S equation, so that

$$F_\kappa(N)=\lambda_{N\kappa}'. \quad (4.37)$$

Therefore, (4.36) leads to

$$\nu_{N\kappa}'=-\lambda_{N\kappa}'/(2N+1). \quad (4.38)$$

From (4.38) and (4.16), we see that all Regge trajectories have a positive slope at $s=0$.

Now, the condition that all the solutions of the B-S equation appear in the Green's function is written as

$$-iR_{\kappa n}(s)=\sum_{l=0}^{n-1}\sum_{m=-l}^{l}\phi_{\kappa nlm}(p,k)\bar{\phi}_{\kappa nlm}(q,k), \quad (4.39)$$

for $s\neq 0$. Hence, for $k_\mu=0$, (4.34) leads to

$$-i\lim_{\nu\to N}(\nu-N)G(p,q)=\sum_{L=0}^{N-1}\sum_{l=0}^{L}\sum_{m=-l}^{l}\nu_{N\kappa}'\phi_{NLlm}(p)\bar{\phi}_{NLlm}(q), \quad (4.40)$$

where the summations have been rearranged in the following way:

$$\sum_{\kappa=0}^{N-1}\sum_{l=0}^{N-\kappa-1}=\sum_{\kappa=0}^{N-1}\sum_{L=\kappa}^{N-1}=\sum_{L=0}^{N-1}\sum_{\kappa=0}^{L}=\sum_{L=0}^{N-1}\sum_{l=0}^{L}. \quad (4.41)$$

The left-hand side of (4.40) is easily evaluated.

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[26] N. Nakanishi, Phys. Rev. **133**, B214 (1964).

[27] N. Nakanishi, Phys. Rev. **137**, B1352 (1965). The calculation is given in Appendix A.

[28] We have pointed out in Ref. 12 that the abnormal solutions with odd κ do not appear in the asymptotic expansion of the Green's function with $v_0=w_0$ in powers of $(-t-1+v_0)$. This does *not* mean, however, that there are no Regge poles with odd κ in the complex l plane in the case $v_0\neq w_0$.

Since[29]

$$C_j^\omega(z)=[\Gamma(2\omega+j)/j!\Gamma(2\omega)]\times F(-j,2\omega+j;\omega+\tfrac{1}{2};\tfrac{1}{2}(1-z)),\quad(4.42)$$

and

$$\lim_{\nu\to N}(\nu-N)\Gamma(-\nu+1)=(-1)^N/(N-1)!,\quad(4.43)$$

(4.25) together with (4.31) yields

$$-i\lim_{\nu\to N}(\nu-N)G(p,q)=-\frac{2N(N+1)}{\pi^2(1-v-i\epsilon)^3(1-v_0-i\epsilon)^3}\times C_{N-1}^{3/2}\left(1+\frac{2t}{(1-v-i\epsilon)(1-v_0-i\epsilon)}\right).\quad(4.44)$$

Our task is, therefore, to evaluate explicitly the right-hand side of (4.40), which is denoted by $M_N(p,q)$. From (4.2), together with (4.15) and (4.38), we have

$$M_N(p,q)=\sum_{L=0}^{N-1}\frac{2N(N+1)(N-L-1)!}{(N+L+1)!}\times Q_L(p,q)f_{NL}(v)f_{NL}(v_0),\quad(4.45)$$

with

$$\begin{aligned}Q_L(p,q)&\equiv\sum_{l=0}^{L}\sum_{m=-l}^{l}(-1)^{L-l}\mathfrak{z}_{Llm}(p)[\mathfrak{z}_{Llm}(q)]^*\\&=\frac{L+1}{2\pi^2}\sum_{l=0}^{L}\frac{2^{2l}(L-l)!(l!)^2(2l+1)}{(L+l+1)!}|\mathbf{p}|^l|\mathbf{q}|^l\\&\quad\times P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})(\mathbf{p}^2-p_0^2)^{\frac{1}{2}(L-l)}(\mathbf{q}^2-q_0^2)^{\frac{1}{2}(L-l)}\\&\quad\times C_{L-l}^{l+1}(-ip_0/(\mathbf{p}^2-p_0^2)^{\frac{1}{2}})\\&\quad\times C_{L-l}^{l+1}(-iq_0/(\mathbf{q}^2-q_0^2)^{\frac{1}{2}}),\end{aligned}\quad(4.46)$$

where $\hat{\mathbf{p}}\equiv\mathbf{p}/|\mathbf{p}|$. We make use of the addition theorem of the Gegenbauer polynomial[30]:

$$\begin{aligned}C_j^\omega&(xy-(x^2-1)^{1/2}(y^2-1)^{1/2}\cos\varphi)\\&=\frac{\Gamma(2\omega-1)}{[\Gamma(\omega)]^2}\sum_{l=0}^{j}(-1)^l\times\frac{4^l(j-l)![\Gamma(\omega+l)]^2(2\omega+2l-1)}{\Gamma(2\omega+j+l)}\\&\quad\times(x^2-1)^{\frac{1}{2}l}(y^2-1)^{\frac{1}{2}l}C_{j-l}^{\omega+l}(x)\\&\quad\times C_{j-l}^{\omega+l}(y)C_l^{\omega-\frac{1}{2}}(\cos\varphi).\end{aligned}\quad(4.47)$$

Putting $\omega=1$ and $j=L$ in (4.47) (note $P_l\equiv C_l^{1/2}$), we find

$$Q_L(p,q)=\tfrac{1}{2}(L+1)\pi^{-2}(-1)^Lv^{\frac{1}{2}L}v_0^{\frac{1}{2}L}C_L^1(pq/v^{1/2}v_0^{1/2}).\quad(4.48)$$

[29] Reference 19, p. 779.
[30] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (The Macmillan Company, New York, 1947), p. 335.

Substitution of (4.48) and (4.9) in (4.45) leads to

$$\begin{aligned}M_N(p,q)=&-\frac{N(N+1)}{\pi^2(1-v-i\epsilon)^3(1-v_0-i\epsilon)^3}\sum_{L=0}^{N-1}(-1)^L\\&\times\frac{(L+1)(N-L-1)![(2L+2)!]^2}{2^{2L}(N+L+1)![(L+1)!]^2}\\&\times\frac{(4v)^{\frac{1}{2}L}(4v_0)^{\frac{1}{2}L}}{(1-v-i\epsilon)^L(1-v_0-i\epsilon)^L}\\&\times C_{N-L-1}^{L+\frac{3}{2}}\left(\frac{1+v}{1-v-i\epsilon}\right)\\&\times C_{N-L-1}^{L+\frac{3}{2}}\left(\frac{1+v_0}{1-v_0-i\epsilon}\right)C_L^1\left(\frac{pq}{v^{1/2}v_0^{1/2}}\right).\end{aligned}\quad(4.49)$$

Using again (4.47) with $\omega=\frac{3}{2}$ and $j=N-1$, we see that $M_N(p,q)$ is *exactly* equal to (4.44). Thus (4.40) has been verified. Our explicit demonstration of the equality (4.40) provides also a beautiful check of the validity of our normalization condition (3.10) and the normalization constant (4.18).

5. SOLUTIONS WITH AN INFINITESIMAL MASS

As was emphasized in Sec. 2, the case $s=0$ needs special consideration, which will be made in the next section. In this section, we shall calculate the normalization integral for the case $s>0$ but infinitesimal. For simplicity, we confine ourselves to considering the bound states with $n=l+1$.

The Cutkosky solutions with $n=l+1$ read

$$\phi_{\kappa nlm}(p,k)=B_{\kappa n}(s)\mathcal{Y}_{lm}(\mathbf{p})\int_{-1}^{1}dz\times\frac{(-i)g_{\kappa n}(z,s)}{[\frac{1}{2}(1+z)(1-v)+\frac{1}{2}(1-z)(1-w)-i\epsilon]^{n+2}},\quad(5.1)$$

where $s=(2k)^2$, $v=(k+p)^2$, $w=(k-p)^2$, and $B_{\kappa n}(s)$ is a normalization constant. The Cutkosky function $g_{\kappa n}(z,s)$ satisfies[2]

$$\{D_n(z)+\lambda_{\kappa n}(s)[1-\tfrac{1}{4}(1-z^2)s]^{-1}\}g_{\kappa n}(z,s)=0,\quad(5.2)$$

with

$$g_{\kappa n}(\pm1,s)=0,\quad(5.3)$$

where

$$D_n(z)\equiv(1-z^2)(d/dz)^2+2(n-1)z(d/dz)-n(n-1).\quad(5.4)$$

If we expand $g_{\kappa n}(z,s)$ and $\lambda_{\kappa n}(s)$ in powers of s,

$$g_{\kappa n}(z,s)=\sum_{r=0}^{\infty} s^r g_{\kappa n}{}^{(r)}(z)\,, \tag{5.5}$$

$$\lambda_{\kappa n}(s)=\sum_{r=0}^{\infty} s^r \lambda_{\kappa n}{}^{(r)}\,, \tag{5.6}$$

then (5.2) with (5.3) leads to

$$[D_n(z)+\lambda_{\kappa n}{}^{(0)}]g_{\kappa n}{}^{(r)}(z)=H_{\kappa n}{}^{(r)}(z)\,, \tag{5.7}$$

with

$$g_{\kappa n}{}^{(r)}(\pm 1)=0\,, \tag{5.8}$$

where

$$H_{\kappa n}{}^{(r)}(z)\equiv -\sum\,[\tfrac{1}{4}(1-z^2)]^{r_1}\lambda_{\kappa n}{}^{(r_2)}g_{\kappa n}{}^{(r_3)}(z)\,. \tag{5.9}$$

The summation in (5.9) goes over all combinations of three nonnegative integers r_1, r_2, r_3 satisfying $r_3\neq r$ and $r_1+r_2+r_3=r$.

The explicit expression for $g_{\kappa n}{}^{(0)}(z)$ has already been presented by Cutkosky[2]:

$$g_{\kappa n}{}^{(0)}(z)=(1-z^2)^n C_\kappa{}^{n+\frac{1}{2}}(z)\,, \tag{5.10}$$

with

$$\lambda_{\kappa n}{}^{(0)}=(\kappa+n)(\kappa+n+1)\,. \tag{5.11}$$

Hence $g_{\kappa n}{}^{(r)}(z)$ can be determined by (5.7) with (5.8) successively. Since the homogeneous part of (5.7) is satisfied by $g_{\kappa n}{}^{(0)}(z)$, the function

$$\varphi_{\kappa n}{}^{(r)}(z)\equiv g_{\kappa n}{}^{(r)}(z)/g_{\kappa n}{}^{(0)}(z) \tag{5.12}$$

satisfies

$$(1-z^2)g_{\kappa n}{}^{(0)}(z)[\varphi_{\kappa n}{}^{(r)}(z)]''+2\{(1-z^2)[g_{\kappa n}{}^{(0)}(z)]' + (n-1)zg_{\kappa n}{}^{(0)}(z)\}[\varphi_{\kappa n}{}^{(r)}(z)]'=H_{\kappa n}{}^{(r)}(z)\,. \tag{5.13}$$

Since (5.13) is a linear differential equation of first order for $[\varphi_{\kappa n}{}^{(r)}(z)]'$, it can be solved by the standard method. Thus

$$g_{\kappa n}{}^{(r)}(z)=(1-z^2)^n C_\kappa{}^{n+\frac{1}{2}}(z) \times\int_{-1}^{z} dz' \frac{f_{\kappa n}{}^{(r)}(z')}{(1-z'^2)^{n+1}[C_\kappa{}^{n+\frac{1}{2}}(z')]^2}\,, \tag{5.14}$$

with

$$f_{\kappa n}{}^{(r)}(z)\equiv\int_{-1}^{z} dz' C_\kappa{}^{n+\frac{1}{2}}(z')H_{\kappa n}{}^{(r)}(z')\,. \tag{5.15}$$

Here the lower limit of integration in (5.15) is due to the boundary condition $g_{\kappa n}{}^{(r)}(-1)=0$. The other boundary condition $g_{\kappa n}{}^{(r)}(+1)=0$ leads to[31]

$$f_{\kappa n}{}^{(r)}(1)=\int_{-1}^{1} dz\, C_\kappa{}^{n+\frac{1}{2}}(z)H_{\kappa n}{}^{(r)}(z)=0\,, \tag{5.16}$$

which determines $\lambda_{\kappa n}{}^{(r)}$. For example, for $r=1$ we have

$$\int_{-1}^{1} dz[\lambda_{\kappa n}{}^{(1)}+\tfrac{1}{4}(1-z^2)\lambda_{\kappa n}{}^{(0)}] \times(1-z^2)^n[C_\kappa{}^{n+\frac{1}{2}}(z)]^2=0\,. \tag{5.17}$$

The use of (4.21) yields

$$\lambda_{\kappa n}{}^{(1)}=-\lambda_{\kappa n}{}^{(0)}\frac{(\kappa+n)(\kappa+n+1)+n^2-1}{2(2\kappa+2n-1)(2\kappa+2n+3)}\,, \tag{5.18}$$

a result which was used in Sec. 4. When $\kappa=0$, Eq. (5.18), of course, coincides with our previous result,[12] which was obtained by solving Cutkosky's integral equation directly. The lower limit of integration in (5.14) is completely arbitrary because it is related only to the normalization of $g_{\kappa n}(z,s)$. Since the integrand of (5.14) has real poles for $\kappa\geq 2$, the integration path should lie in the complex z' plane. We conjecture that there are double poles only. If so, $g_{\kappa n}{}^{(r)}(z)$ is a polynomial of z and independent of the choice of the path. Then $g_{\kappa n}{}^{(r)}(z)$, $(r\geq 1)$, has an $(n+1)$th-order zero at $z=\pm 1$, on account of our choice -1 of the lower limit of integration.

Our next task is to evaluate the moments

$$G_{\kappa n}{}^{(r,j)}\equiv\int_{-1}^{1} dz\, z^j g_{\kappa n}{}^{(r)}(z)\,. \tag{5.19}$$

To do this, it is not convenient to employ (5.14). Rather, we integrate (5.7) directly. Integrations by parts lead to

$$\int_{-1}^{1} dz\, z^j[D_n(z)+\lambda_{\kappa n}{}^{(0)}]g_{\kappa n}{}^{(r)}(z)$$

$$(\kappa=-j)(\kappa+2n+j+1)G_{\kappa n}{}^{(r,j)} +j(j-1)G_{\kappa n}{}^{(r,\,j-2)}\,. \tag{5.20}$$

When $\kappa>j$, therefore, $G_{\kappa n}{}^{(r,j)}$ can be expressed as a linear combination of

$$\int_{-1}^{1} dz\, z^{j-2m}H_{\kappa n}{}^{(r)}(z)\,,\quad (0\leq m\leq\tfrac{1}{2}j)\,. \tag{5.21}$$

From this fact, we obtain an important result

$$G_{\kappa n}{}^{(r,j)}=0\,,\quad \text{for } 2r+j<\kappa\,. \tag{5.22}$$

This formula is evidently true for $r=0$, because $H_{\kappa n}{}^{(0)}(z)\equiv 0$. Hence we use mathematical induction with respect to r. Since (5.21) is a linear combination of

$$G_{\kappa n}{}^{(r_3,J)} \quad\text{with}\quad J\equiv(j-2m)+2(r_1-m')\,,\quad (0\leq m'\leq r_1)\,, \tag{5.23}$$

the assumption of induction tells us that (5.21) vanishes because

$$2r_3+J\leq 2r+j<\kappa \tag{5.24}$$

and $r_3\leq r-1$. Thus (5.22) is established.

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[31] By means of (5.16), it is easy to show that $f_{\kappa n}{}^{(r)}(z)$ is an odd function of z, and hence the even-oddness of $g_{\kappa n}{}^{(r)}(z)$ is the same as that of $g_{\kappa n}{}^{(0)}(z)$, as it should be.

In the case $2r+j=\kappa$, the quantity $G_{\kappa n}{}^{(r,j)}$ no longer vanishes, but it is relatively easy to evaluate it because of (5.22). We find

$$2r(2\kappa+2n-2r+1)G_{\kappa n}{}^{(r,\,\kappa-2r)} = -\lambda_{\kappa n}{}^{(0)}\sum_{r'=0}^{r-1}(-\tfrac{1}{4})^{r-r'}G_{\kappa n}{}^{(r',\,\kappa-2r')}. \quad (5.25)$$

In particular,

$$G_{\kappa n}{}^{(1,\,\kappa-2)}=\lambda_{\kappa n}{}^{(0)}G_{\kappa n}/8(2\kappa+2n-1)\,, \quad (5.26)$$

where

$$G_{\kappa n}\equiv G_{\kappa n}{}^{(0,\kappa)}=\frac{2^{\kappa+2n+1}n!(\kappa+n)!(\kappa+2n)!}{(2n)!(2\kappa+2n+1)!}. \quad (5.27)$$

We are now ready to evaluate the normalization integral

$$I_{\kappa n}\equiv -i\int d^4p(1-v)(1-w)\bar{\phi}_{\kappa nlm}(p,k)\phi_{\kappa nlm}(p,k)\,. \quad (5.28)$$

Substituting (5.1) in (5.28) and using the Feynman parametrization, we can easily carry out the momentum integration:

$$I_{\kappa n}=-|B_{\kappa n}(s)|^2\frac{(2n+1)!}{[(n+1)!]^2}\int_0^1 dx\, x^{n+1}(1-x)^{n+1} \times\int_{-1}^1 dz\, g_{\kappa n}(z,s) \times\int_{-1}^1 d\zeta g_{\kappa n}(\zeta,s)R_n(\alpha,1-\alpha,s)\,, \quad (5.29)$$

with

$$\alpha=\tfrac{1}{2}(1+z)x+\tfrac{1}{2}(1+\zeta)(1-x)\,,$$
$$1-\alpha=\tfrac{1}{2}(1-z)x+\tfrac{1}{2}(1-\zeta)(1-x)\,, \quad (5.30)$$

where

$$R_n(\alpha,\beta,s)\equiv -i\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\times\int d^4p\frac{|\mathcal{Y}_{lm}(\mathbf{p})|^2}{[\alpha(1-v)+\beta(1-w)-i\epsilon]^{2n+2}} =\frac{\pi}{2^{2n+1}(2n+1)}\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\times[(\alpha+\beta)^2-\alpha\beta s]^{-n-1}. \quad (5.31)$$

Expanding $R_n(\alpha,\beta,s)$ in powers of s and putting $\alpha+\beta=1$ after the differentiations are performed, we have

$$R_n(\alpha,1-\alpha,s)=[\pi/2^{2n+1}(2n+1)]\sum_{j=0}^{\infty}[(n+j)!/n!j!] \times[(2n+2j+2)(2n+2j+3)\alpha^j(1-\alpha)^j -j(2n+j+2)\alpha^{j-1}(1-\alpha)^{j-1}]s^j. \quad (5.32)$$

On account of (5.30), the part of the maximal degree with respect to z and ζ in the coefficient of s^j in $R_n(\alpha,1-\alpha,s)$ has the form

$$(-1)^j\sum_{m=-j}^{j}a_m(x)z^{j+m}\zeta^{j-m},\quad [a_m(x)>0]. \quad (5.33)$$

Expanding $g_{\kappa n}(z,s)$ and $g_{\kappa n}(\zeta,s)$ in powers of s, we see that the coefficient of s^r is a linear combination of the terms

$$\int_{-1}^1 dz\, g_{\kappa n}{}^{(r_1)}(z)\int_{-1}^1 d\zeta\, g_{\kappa n}{}^{(r_2)}(\zeta)z^{j+m}\zeta^{j-m} \quad (5.34)$$

and terms of lower degrees with respect to z and ζ, where

$$r=r_1+r_2+j\,,$$
$$|m|\leq j. \quad (5.35)$$

Because of (5.22), Eq. (5.34) vanishes, unless

$$2r_1+j+m\geq\kappa\,,$$
$$2r_2+j-m\geq\kappa. \quad (5.36)$$

Hence

$$r=r_1+r_2+j\geq\kappa. \quad (5.37)$$

Therefore, the leading term (i.e., lowest order non-vanishing term) must satisfy

$$r=\kappa\,, \quad (5.38)$$

namely, $I_{\kappa n}/|B_{\kappa n}(s)|^2$ *is of order* s^κ. When (5.38) holds, (5.36) and (5.35) yield

$$m=r_2-r_1\,,$$
$$j\geq|r_2-r_1|\,,$$
$$\kappa\geq r_1+r_2+|r_1-r_2|=2\max(r_1,r_2)\,. \quad (5.39)$$

It is not very cumbersome to evaluate $I_{\kappa n}$ explicitly for $\kappa=0, 1, 2, 3$. When $\kappa\leq 3$, (5.39) leads to $r_1\leq 1$ and $r_2\leq 1$, and hence we encounter $G_{\kappa n}{}^{(1,\,\kappa-2)}$ only, which is given by (5.26). The explicit expressions for $I_{\kappa n}$, ($\kappa=0, 1, 2, 3$), are

$$I_{0n}/|B_{0n}(s)|^2=-[\pi(G_{0n})^2/2^{2n+1}(2n+1)]+O(s)\,,$$
$$I_{1n}/|B_{1n}(s)|^2=+\frac{\pi(n+2)^2(G_{1n})^2}{2^{2n+3}(2n+1)(2n+3)}s+O(s^2)\,,$$
$$I_{2n}/|B_{2n}(s)|^2=-\frac{\pi(n+1)(n+2)^2(n+3)^2(G_{2n})^2}{2^{2n+5}(2n+1)(2n+3)^2(2n+5)}\times s^2+O(s^3)\,,$$
$$I_{3n}/|B_{3n}(s)|^2=+\frac{\pi(n+1)(n+2)^2(n+3)^2(n+4)^2(G_{3n})^2}{2^{2n+7}3(2n+1)(2n+3)(2n+5)^2(2n+7)}\times s^3+O(s^4)\,, \quad (5.40)$$

where $G_{\kappa n}$ is given by (5.27).

The normalization condition (3.10) leads to

$$I_{\kappa n}=\lambda_{\kappa n}'(s)/\lambda_{\kappa n}(s)=\lambda_{\kappa n}^{(1)}/\lambda_{\kappa n}^{(0)}+O(s)\,. \quad (5.41)$$

Since $\lambda_{\kappa n}^{(1)}/\lambda_{\kappa n}^{(0)}$ is negative, as was shown in (5.18), the sign of the norm of $\phi_{\kappa nlm}(p,k)$ is $(-1)^\kappa$ for $\kappa=0$, 1, 2, 3. Thus it is extremely likely that all the abnormal solutions with odd κ have a negative norm.

Finally, we consider the case $s<0$. The only differences from the above are the solid harmonic (2.10) and the sign of s. Thus the sign of the norm of a solution is $(-1)^{l-m}$. This is quite analogous to the result in Sec. 4. Summarizing the three cases $s>0$, $k_\mu=0$, and $s<0$, we can say that *the sign of the norm of any solution is identical with its "p_0 parity"* (positive for an even function of p_0 and negative for an odd one).

6. SOLUTIONS FOR MASSLESS BOUND STATES

We discuss the case $s=0$ but $k_\mu\neq 0$ in this section. Since we cannot take the rest frame of a massless particle, no correct discussion, in this case, has been made so far, at least within the present author's knowledge.

We have found the solid harmonics $\chi_{lm}(p)$ in (2.13). Contrary to the other cases, $\chi_{lm}(p)$ is not self-reproducing in the momentum integration appearing in the B-S equation, except for the case $m=\pm l$. When $m=\pm l$, we see

$$\chi_{l\,\pm l}(p)=\mathcal{Y}_{l\,\pm l}(\mathbf{p})\,, \quad (6.1)$$

provided that

$$\tilde{A}=(1/4\pi)^{\frac{1}{2}}[(2l+1)!]^{\frac{1}{2}}/2^l l!. \quad (6.2)$$

In this case, our solutions are nothing but the $s\to 0$ limits of the Cutkosky solutions apart from the normalization constants. This corresponds to the well-known fact that *a massless particle has only two polarizations independently of its spin* $l(\neq 0)$. The norms of our B-S amplitudes with $n=l+1$ are, therefore, *positive for* $\kappa=0$ and *zero for* $\kappa\geq 1$, according to the result given in Sec. 5 [$B_{\kappa n}(0)$ is undefined for $\kappa\geq 1$[32]].

Now, our main task in this section is to seek for solutions other than the above. For simplicity, we consider the solutions with $n=l+1$ only, but l is no longer a good quantum number, in our case. We introduce a quantum number

$$M\equiv l-|m|=n-|m|-1 \quad (6.3)$$

and consider a B-S amplitude

$$\phi_{\kappa nMm}(p,k)=\sum_{r=0}^{M}(p_3+p_0)^{M-r}(p_1\pm ip_2)^{n-M-1}\int_{-1}^{1}dz \times\frac{(-i)h_{\kappa nM}{}^r(z)}{[\frac{1}{2}(1+z)(1-v)+\frac{1}{2}(1-z)(1-w)-i\epsilon]^{n-r+2}}, \quad (6.4)$$

where the double sign corresponds to the sign of m, and a normalization constant is omitted. When the B-S kernel operates on (6.4), we encounter the integrals

$$\frac{1}{\pi^2 i}\int d^4p'\frac{1}{-(p-p')^2-i\epsilon}\,\frac{(p_3'+p_0')^{M-r}(p_1'\pm ip_2')^{n-M-1}}{[\frac{1}{2}(1+z')(1-v')+\frac{1}{2}(1-z')(1-w')-i\epsilon]^{n-r+2}}$$
$$=\frac{n-r+2}{\pi^2 i}\int_0^1 dx(1-x)^{n-r+1}\int d^4p'\frac{(p_3'+p_0')^{M-r}(p_1'\pm ip_2')^{n-M-1}}{\{-[p'-xp+(1-x)z'k]^2+(1-x)[1-xp^2-2xz'pk]-i\epsilon\}^{n-r+3}}. \quad (6.5)$$

The change of the integration variables into

$$p_\mu''=p_\mu'-xp_\mu+(1-x)z'k_\mu \quad (6.6)$$

leads to

$$p_3'+p_0'=p_3''+p_0''+x(p_3+p_0)-2(1-x)z'k_0. \quad (6.7)$$

The appearance of the last term is the essential difference from the case $s>0$. It is easy to see, by means of rotation $p_0''\to ip_4''$, that the part $p_3''+p_0''$ vanishes when integrated. Thus after carrying out the momentum integration, (6.5) is a linear combination of the terms proportional to $(p_3+p_0)^j$, $(j=0, 1,\cdots, M-r)$. Our calculation is formally quite analogous to Cutkosky's[2] for the general case $(0\leq l\leq n-1)$. Hence, we write here the final result only. The weight functions $h_{\kappa nM}{}^r(z)$ should satisfy the integral equations

$$h_{\kappa nM}{}^r(z)=\frac{\lambda_{\kappa n}}{2(n-r)}\sum_{r'=0}^{r}\frac{(M-r')!(n-r+1)!}{(M-r)!(n-r'+1)!}\times\int_{-1}^{1}dz'(-2z'k_0)^{r-r'}\times[R(z,z')]^{n-r}h_{\kappa nM}{}^{r'}(z')\,, \quad (6.8)$$

with

$$R(z,z')\equiv(1\mp z)/(1\mp z')\,,\quad \text{for } z\gtrless z'. \quad (6.9)$$

For $r=0$, we have

$$h_{\kappa nM}{}^0(z)=\frac{\lambda_{\kappa n}}{2n}\int_{-1}^{1}dz'[R(z,z')]^n h_{\kappa nM}{}^0(z')\,, \quad (6.10)$$

[32] We have shown that $B_{\kappa n}(s)\sim s^{-\frac{1}{2}\kappa}$ as $s\to 0$. If $k_\mu\to 0$, too, then the z integral in (5.1) also tends to zero, and $\phi_{\kappa nlm}(p,k)$ has a definite limit. In the present case ($k_\mu\neq 0$), however, the z integral does not vanish.

which has the same form with the corresponding Cutkosky equation. Hence $h_{\kappa nM}{}^0(z)$ and $\lambda_{\kappa n}$ are given by (5.10) and (5.11), respectively. For $r=1$, $(n>M\geq 1)$, (6.8) becomes

$$h_{\kappa nM}{}^1(z)=\frac{(\kappa+n)(\kappa+n+1)}{2(n-1)}\left[\frac{M}{n+1}\int_{-1}^{1}dz' \times(-2z'k_0)[R(z,z')]^{n-1}h_{\kappa nM}{}^0(z') + \int_{-1}^{1}dz'[R(z,z')]^{n-1}h_{\kappa nM}{}^1(z')\right]. \quad (6.11)$$

We expect that (6.11) will have no nonsingular solution, because its homogeneous part is satisfied by $g_{\kappa+1,\,n-1}{}^{(0)}(z)$. For example, consider

$$\hat{h}_{nM}\equiv\int_{-1}^{1}dz\, zh_{0nM}{}^1(z)\,. \quad (6.12)$$

Since

$$\int_{-1}^{1}dz\, z[R(z,z')]^{n-1}=\frac{2(n-1)}{n(n+1)}z', \quad (6.13)$$

(6.11) with $\kappa=0$ leads to

$$\hat{h}_{nM}=-\frac{2Mk_0}{n+1}\int_{-1}^{1}dz\, z^2(1-z^2)^n+\hat{h}_{nM}. \quad (6.14)$$

Hence $\hat{h}_{nM}$ cannot be finite. The situation is quite analogous to that of the Cutkosky equation for $n=l+3$, $(\kappa=0)$.[10,12] In that case, we have found a double-Regge-pole behavior in the high-energy asymptotic expansion.[12] Correspondingly, it is expected that there exists a double pole at $s=0$ in the Green's function. In general, we conjecture that there will be multiple poles at $s=0$ for both $(n\geq l+3,\ m=\pm l)$ and $(n\geq l+1,\ |m|<l)$. If so, and if the B-S amplitudes are well defined also for such cases, then their normalization integrals should vanish on account of (3.4).

7. SUMMARY AND DISCUSSION

In this paper, we have explicitly calculated the normalization constants of the B-S amplitudes in the cases $k_\mu=0$ and s infinitesimal. The most remarkable result is that the abnormal solutions with odd κ have a *negative* norm. Furthermore, we have shown, in the case $k_\mu=0$, that *all* the normal and abnormal solutions appear in the corresponding scattering Green's function as the residues of poles. It is very likely, because of continuity, that our conclusions remain valid for all $0<s<4$. If so, we are led to the following important results.

(a) Since the appearance of ghost states may be inconsistent with an axiom of the conventional quantum-field theory, an upper bound on the coupling constant should be imposed in order to avoid it.[33] In the present model, we have[1,2]

$$\lambda\equiv(4\pi)^{-1}(g^2/4\pi)<\tfrac{1}{4}. \quad (7.1)$$

(b) Ohnuki, Takao, and Umezawa[4] inferred, by a crude argument, that the scattering Feynman amplitude would, in general, contain no abnormal solutions of the B-S equation as the residues of poles. Our result may provide a counterexample to their statement.

(c) Since it is likely[4,6] that abnormal solutions do not correspond to the eigenstates of the total Hamiltonian, we might have to conclude that the Green's function will contain some *unphysical* poles characteristic to the off-the-mass-shell amplitude.

It is therefore very desirable to check our conjecture by explicitly calculating the normalization integral in the case of an infinitesimal binding energy $s\simeq 4$. Of course, one might suspect that the ladder approximation would be doubtful. That will be the case *quantitatively*, but we believe that its *qualitative* conclusions will be true. For example, our results in the case $k_\mu=0$ are essentially based on the Minkowski metric property of the solid harmonics, so that they will not be affected by the inclusion of higher order kernels.[34]

Our second emphasis is the speciality of the case of massless bound states ($s=0$ but $k_\mu\neq 0$). Only the normal solutions with $m=\pm l$ are the limits of the solutions for $s>0$. We have pointed out that all the abnormal solutions with $m=\pm l$ (and $n=l+1$) have a *zero* norm. For the values of m other than $\pm l$, we have obtained a set of non-Cutkosky integral equations for weight functions, but they seem to have no nonsingular solution in exact analogy to the Cutkosky equations for the massless bound states with $n\geq l+3$. It is extremely likely that, in these cases, the Green's function has a multiple pole at $s=0$. This is quite an interesting problem and subject to future investigation.

Note added in proof. The conjecture that the sign of the norm is $(-1)^\kappa$ for $0<s<4$ has been verified explicitly for the following solutions with $n=l+1$: (1) κ arbitrary, s infinitesimal: (2) $\kappa=0$, $0<s\leq 2$; (3) $\kappa=1$, $0<s\leq 2+(n+2)^{-1}$: (4) $\kappa=0$, $4-s$ infinitesimal. Detailed accounts will appear in a forthcoming paper.

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[33] Our bound is qualitatively different from the bound in the S-matrix theory, because in the latter it is derived under the assumption that the number of stable one-particle states is assigned *a priori* (or in a self-consistent way), so that even if the coupling constant exceeds the bound, we can modify the theory by changing the number of stable states without violating unitarity.

[34] Indeed, this property can be proved even in the exact B-S equation (the exchanged meson may not be massless) if we assume $\lambda_{n\kappa}'<0$, because to see the sign of I_{NLl} we need only the fact that both $[\Delta_F'(v)]^{-1}$ and $f_{NL}(v)$ have spectral representations with real spectral functions [see (4.11)].