Residues of Regge Poles and the Diffraction Peak

BIPIN R. DESAI*
Department of Physics, University of California, Los Angeles, California
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It is shown, on the basis of potential theory, that for a Regge pole with position $\alpha(t)>\frac{1}{2}$ the reduced residue $\beta(t)/\nu^{\alpha(t)}$ behaves essentially as $[\alpha'(t)/R]R^{2\alpha(t)}$ near threshold ($\nu\lesssim 0$), where R is the effective radius interaction. The quantity $\gamma(t)=(m_0^2/\nu)^\alpha\beta(t)$ which occurs in the asymptotic term $\gamma(t)(s/2m_0^2)^\alpha$ can therefore be considered as a slowly varying function of t only if one takes $m_0=R^{-1}$. Assuming that our threshold expression gives a qualitative description in the relativistic case, we note that if $MR>1$, where M is the nucleon mass, then the normalization $m_0=M$ used in high-energy phenomenology will give rise to an exponential falloff of $\gamma(t)$ with $(\text{width})^{-1}\approx 2\alpha'(0)\ln(MR)$. The values of R in the t channel of $\pi\pi$, πN, and NN, estimated from the knowledge of the nearest left-hand branch points or factorization, occur in increasing order, and for each of them $MR>1$. Our results for the Pomeranchuk trajectory roughly reproduce the exponential diffraction shape and indicate that a larger total cross section implies a sharper falloff of the amplitude in the diffraction region. A connection between f^0 and the diffraction width is discussed, as well as the question of zeros in β/ν^α.

I. INTRODUCTION

RECENTLY the properties of the residues of Regge poles have attracted a good deal of interest. It has been known for some time that at zero momentum transfer, the reduced residue of the Pomeranchuk pole (P) should be proportional to the total cross section.[1] It is now realized that it must, in addition, contribute substantially to the diffraction width.[2] The ω residue, it turns out, must have a zero at a small value of the momentum transfer in order to explain the difference between $\bar{p}p$ (K^-p) and pp (K^+p) differential cross sections.[2]

Consider $\pi\pi$ scattering. Let $A_I(s,t)$ be the scattering amplitude, where s is the square of the center-of-mass energy, $-t$ the square of the transfer momentum, and I the isospin index. $A_I(s,t)$ can be represented as a sum of the contributions of the Regge poles in the crossed channel. Writing explicitly the contribution of the leading pole P with position $\alpha(t)$, we obtain

$$A_I(s,t)=-\rho_I\pi(2\alpha+1)\beta(t)P_{\alpha(t)}\left(1+\frac{s}{2\nu}\right)\times\left(\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}\right)+\text{other poles}\,, \quad (1)$$

$$\underset{s\to\infty}{\longrightarrow}-\rho_I\pi(2\alpha+1)\frac{2^\alpha\pi^{1/2}\Gamma(\frac{1}{2}+\alpha)}{\Gamma(1+\alpha)}\beta(t)\times\left(\frac{s}{2\nu}\right)^\alpha\left(\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}\right), \quad (2)$$

where ρ_I is the crossing matrix, $t=4(\nu+m_\pi^2)$, $\alpha(0)=1$, and $\beta(t)$ is the residue. The quantity β/ν^α or the dimensionless quantity $(m_0^2/\nu)^\alpha\beta$, where m_0 is an arbitrary mass, is called the reduced residue. Note that one can write

$$\beta\left(\frac{s}{2\nu}\right)^\alpha=\left(\frac{m_0^2}{\nu}\right)^\alpha\beta\left(\frac{s}{2m_0^2}\right)^\alpha=\gamma(t)\left(\frac{s}{2m_0^2}\right)^\alpha, \quad (3)$$
$$\gamma(t)=(m_0^2/\nu)^\alpha\beta(t).$$

One generally takes m_0 to be a sufficiently large mass, such as the nucleon mass M. However, it is important to realize that the behavior of $\gamma(t)$ depends sensitively on the choice of the scaling factor m_0^2. The quantity $\bar{\gamma}(t)$ introduced in I is given by

$$A(s,t)\underset{s\to\infty}{\longrightarrow}-\bar{\gamma}(t)(s/2M^2)^\alpha((1+e^{-i\pi\alpha})/\sin\pi\alpha)\,,$$
$$\bar{\gamma}(t)=\rho_I\frac{\pi(2\alpha+1)2^{\alpha-1}\pi^{1/2}\Gamma(\frac{1}{2}+\alpha)\gamma(t)}{\Gamma(1+\alpha)}\,, \quad (4)$$

where we take $m_0=M$. The reduced residue $\gamma(t)$ can be written [see (A20)] as[3]

$$\gamma(t)=\gamma(0)e^{\eta(t)}\,, \quad (5)$$

$$\eta(t)=\frac{t}{\pi}\int_{4m_\pi^2}^\infty\frac{dt'}{t'(t'-t)}\eta_I(t')\,. \quad (6)$$

We shall call $\eta'(0)$ the $(\text{width})^{-1}$ of $\gamma(t)$ at $t=0$. In case $\eta'(0)$ turns out to be non-negligible for $m_0=M$ then the form (5) would be quite convenient, since it would show, in a natural way, a sharp exponential falloff indicated by the experiments.[2] From the optical theorem,

$$\text{Im}A_I(s,0)=\frac{(s(s-4m_\pi^2))^{1/2}}{16\pi}\sigma_{\pi\pi}{}^I(s)\,, \quad (7)$$

where $\sigma_{\pi\pi}{}^I(s)$ is the total cross section. A comparison of

* Supported in part by the National Science Foundation.
[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[2] T. O. Binford and B. R. Desai, preceding paper, Phys. Rev. **138**, B1167 (1965); hereafter referred to as I. References to the earlier experimental and theoretical work are given.
[3] B. R. Desai, Phys. Rev. **135**, B180 (1964).

(2) and (7) shows that $\gamma(0)$ is proportional to the total cross section at infinity.

In Sec. II we derive an expression for β/ν^α on the basis of potential theory. It is shown that as long as $\alpha>\frac{1}{2}$, the quantity β/ν^α for $\nu\lesssim 0$ is essentially $\approx(\alpha'/R)R^{2\alpha}$, where R is the radius of interaction in the t channel. Thus if we take the scaling factor $m_0{}^2$ to be equal to R^{-2}, then $\gamma(t)$ can be considered as a slowly varying function of t. Alternatively if one takes $m_0=M$, the normalization commonly used in high-energy phenomenology, then $\gamma(t)$ is essentially $\sim(M^2R^2)^\alpha$ and its behavior then depends sensitively on the value of R. If $MR>1$, then it will fall off exponentially in the diffraction region. The asymptotic behavior of the amplitude is $\sim(sR^2)^\alpha$. This has two important consequences in the relativistic case, for P in particular, where $\alpha\approx 1$ near threshold. It shows that the shape of the diffraction pattern depends on the radius R, which can be different for different reactions. Secondly, it shows that a larger total cross section implies a larger R and therefore a sharper falloff in the amplitude for negative t. Applications to $\pi\pi$ as well as πN scattering, etc., where multichannel effects are important (with the $\pi\pi$ intermediate state dominating), are discussed. Typically, the t-channel forces arise from ρ exchange in $\pi\pi$ scattering, N exchange in πN scattering, etc. Our estimates show that the order of increasing R is $\pi\pi$, πN, NN, and for each of them $MR>1$. In all these cases the diffraction widths are roughly reproduced.

In Sec. III a relation between the f^0 parameters, the total cross section, and the diffraction width is established and it is shown that the value of the $\pi\pi$ width is consistent with the value deduced from factorization.[4] It is also shown that a larger value of the total cross section implies a sharper falloff. This work is along the line of Ref. 3, but in view of the recent numerical results of I, it is *not* necessary to assume that the phase of the reduced residue is 2π at f^0. A small phase consistent with the potential-theory results is sufficient to give a sharp enough falloff.

In Sec. IV we discuss the question of the zeros in β/ν^α for $\nu<0$. It is shown, on the basis of potential theory, that short-range repulsion together with long-range attraction can give rise to such zeros. For the single-channel as well as multichannel case simple zeros can occur only when the pole recedes into the left half-plane ($l<-\frac{1}{2}$). These results are derived for spin-zero particles only. In the relativistic case, however, it is possible for $\alpha(\infty)$ to lie in the right half-plane, between $l=-\frac{1}{2}$ and $l=1$.[5] In that case the zeros may occur when the pole is in the right half-plane. As mentioned earlier, the existence of such zeros may be of interest for the particular case of ω exchange. We also show that the residues of the higher rank trajectories, in general, develop zeros; the higher the rank, the larger the number of zeros.

[4] M. Gell-Mann, Phys. Rev. Letters 8, 263 (1962); V. N. Gribov and I. Ya. Pomeranchuk, *ibid.* 8, 343 (1962).
[5] G. F. Chew and C. E. Jones, Phys. Rev. 135, B208 (1964).

In the Appendix we summarize some useful results about the residues obtained from potential scattering or from the low-energy effective-range theory.

II. THRESHOLD EXPRESSION FOR β/ν^α ON THE BASIS OF POTENTIAL THEORY

From (A1) and (A2) it follows that for $\mathrm{Im}\alpha=0$, and $\nu=k^2<0$

$$\beta=e^{i\pi\alpha}\frac{d\alpha}{d\nu}\left(\int_0^\infty dr\, f^2(\alpha,-k,r)\right)^{-1},\quad k=i|k|\,,\quad (8)$$

where f satisfies the boundary condition[6]

$$f(\alpha,-k,r)\underset{r\to\infty}{\longrightarrow} e^{ikr}. \quad (9)$$

Let $r=R_0$ be the distance beyond which the potential vanishes. We can then write

$$\int_0^\infty dr\, f^2=\int_0^{R_0} dr\, f^2+\int_{R_0}^\infty dr\, f^2. \quad (10)$$

On a Regge pole the wave function f behaves as $r^{\alpha+1}$ at the origin. Hence the integrand in the first term above is a well-behaved function of r and k. For $r>R_0$, the potential vanishes, and therefore f in that region is the free wave function[6]

$$f(\alpha,-k,r)=e^{(i\pi/2)(\alpha+1)}(\tfrac{1}{2}\pi kr)^{1/2}H_{\alpha+(1/2)}{}^{(1)}(kr)\,,\quad (11)$$

where $H_{\alpha+(1/2)}{}^{(1)}$ is the Hankel function of the first kind.[7] It depends on k and r only through the product $kr=x$. At infinity it vanishes, consistent with the boundary condition (9), and for $0\leqslant|x/2|<1$,

$$f\approx e^{i\pi\alpha/2}\Gamma(\alpha+\tfrac{3}{2})/\pi^{1/2}(\alpha+\tfrac{1}{2})(x/2)^{-\alpha}.$$

Thus when $x_0=kR_0$ is small in absolute value and $\alpha>\frac{1}{2}$, the second term in (10) is dominant. In that case

$$\int_0^\infty dr\, f^2\approx\int_{R_0}^\infty dr\, f^2\approx e^{i\pi\alpha}\frac{\Gamma^2(\alpha+\frac{3}{2})}{\pi(\alpha+\frac{1}{2})^2}\frac{2^{2\alpha}}{k}\int_{x_0} dx\, x^{-2\alpha}$$

$$=e^{i\pi\alpha}\frac{\Gamma^2(\alpha+\frac{3}{2})}{\pi(\alpha+\frac{1}{2})^2}\frac{2^{2\alpha}}{k}\frac{(kR_0)^{-2\alpha+1}}{2\alpha-1} \quad (12)$$

and

$$\beta/\nu^\alpha=\frac{\pi(2\alpha-1)}{\Gamma^2(\alpha+\frac{1}{2})}\frac{1}{R_0}\frac{d\alpha}{d\nu}\left(\frac{R_0{}^2}{2}\right)^\alpha. \quad (13)$$

Because (13) is an expression valid near threshold, we may expect it to give a reasonable description in the relativistic case also. The magnitude of R_0 then should presumably be somewhere between the pion and nucleon Compton wavelengths.

[6] R. G. Newton, J. Math. Phys. 3, 867 (1962).
[7] $H_\lambda{}^{(1)}(z)=(i/\sin\pi\lambda)[e^{-i\pi\lambda}J_\lambda(z)-J_{-\lambda}(z)]=J_\lambda(z)+iN_\lambda(z)$, where N_λ is the Neumann function and $J_\lambda(z)=((z/2)^\lambda/\Gamma(1+\lambda))\times[1-(1+\lambda)^{-1}(z/2)^2+\cdots]$.

Substituting (13) in (2) we obtain for the P pole

$$A_I(s,t) \underset{s\to\infty}{\longrightarrow} -\rho_I \pi^{3/2} \frac{(2\alpha+1)(2\alpha-1)}{\Gamma(\alpha+1)\Gamma(\alpha+\frac{1}{2})} \times \frac{2\alpha'(t)}{R_0}\left(\frac{sR_0^2}{4}\right)^{\alpha}\left(\frac{1+e^{-i\pi\alpha}}{\sin\pi\alpha}\right). \quad (14)$$

This result is valid for $0 \leqslant \frac{1}{4}|\nu R_0^2| < 1$ and $\alpha(t) > \frac{1}{2}$. The result for $\alpha(t) \leqslant \frac{1}{2}$ is discussed in the Appendix. It is shown that in that case β/ν^α *increases* as ν decreases from threshold.[8] If we assume R_0 to be small enough to allow us to use (14) near $t=0$, then for $sR_0^2/4 \gg 1$ the term $(sR_0^2/4)^\alpha$ gives rise to a sharp exponential falloff with $(\text{width})^{-1} \approx \alpha'(0)\ \ln(sR_0^2/4)$.[9] An application of (14) to $\pi\pi$ scattering is discussed later in this section.

Expression (13) is valid for the single-channel case. For the multichannel problem, consider the situation where the channels are not very close together. Of course, we have in mind πN, KN, NN, etc., scattering where, in the t channel, the lowest threshold is $\pi\pi$, which is well below the $K\bar{K}$ and $N\bar{N}$ thresholds. The amplitudes involved will each have their nearest right-hand branch point at the threshold of the first channel. The reduced partial-wave amplitude for scattering from channel 1 to channel n for a given l is

$$A_{1n,l}(\nu) = B_{1n,l}(\nu) - \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu} B_{1n,l}(\nu')\, \mathrm{Im}D_{11,l}(\nu') \Big/ D_{11,l}(\nu), \quad (15)$$

where $\mathrm{Im}D_{11,l} = -\rho_{1l}N_{11,l}$, the quantities N_{11} and D_{11} being the usual numerator and denominator functions of A_{11}, and B has the left-hand cut of A. The reduced residue at $l=\alpha$ is

$$\frac{\beta_{1n}}{(\nu_1\nu_n)^{\alpha/2}} = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu} B_{1n,\alpha}(\nu')\rho_{1\alpha}N_{11,\alpha}(\nu') \Big/ \frac{\partial}{\partial l} D_{11,l}(\nu)\Big|_{l=\alpha} \quad (16)$$

or

$$\frac{\beta_{1n}}{(\nu_1\nu_n)^{\alpha/2}} = \frac{\beta_{11}}{\nu_1^\alpha}\int_0^\infty \frac{d\nu'}{\nu'-\nu} B_{1n,\alpha}(\nu')\rho_{1\alpha}N_{11,\alpha}(\nu') \Big/ \int_0^\infty \frac{d\nu'}{\nu'-\nu} B_{11,\alpha}(\nu')\rho_{1\alpha}N_{11,\alpha}(\nu'). \quad (17)$$

For β_{11}/ν_1^α one can safely assume (13) to be true. However, for the other channels $\beta_{1n}/(\nu_1\nu_n)^{\alpha/2}$ depends on the relative behavior of B_{1n} and B_{11}. An indication of this behavior can be obtained if we consider the Born amplitudes

$$B_{1n,\alpha} = \frac{-g_{1n}}{2(\nu_1\nu_n)^{(\alpha+1)/2}} Q_\alpha\left(\frac{\mu_{1n}^2+\nu_1+\nu_n}{2(\nu_1\nu_n)^{1/2}}\right), \quad (18)$$

where g_{1n} and μ_{1n}^{-1} are the usual strength and range parameters. The ν's are the channel momenta.

Clearly, if the strength and the range of B_{1n} are large compared to B_{11}, then so is $\beta_{1n}/(\nu_1\nu_n)^{\alpha/2}$ compared to β_{11}/ν_1^α. Roughly speaking, for small $\nu_1(\lesssim 0)$,

$$\frac{\beta_{1n}}{(\nu_1\nu_n)^{\alpha/2}} \approx \frac{\beta_{11}}{\nu_1^\alpha}\frac{g_{1n}}{g_{11}}\left(\frac{|\nu_{L1}|}{|\nu_{L1}'\nu_{Ln}'|^{1/2}}\right)^{\alpha+1}, \quad (19a)$$

$$\sim R^{2\alpha}, \quad (19b)$$

where R is the effective radius,[10] and ν_L and ν_L' denote the positions of the nearest left-hand branch points of B_{11} and B_{1n}, respectively. These positions, of course, depend inversely on the range of interaction. The α dependence above is determined by the distance of the left-hand branch point to the first threshold as well as by the separation between the thresholds. Comparing (19) with (13) we see that for a fixed separation between the thresholds, the closer the branch point of B_{1n} to $\nu_1=0$, the larger the effective radius of $\beta_{1n}/(\nu_1\nu_n)^{\alpha/2}$. Once β_{11} and β_{1n} are known, all the remaining residues can be obtained from the factorization theorem. Thus if β_{1n} has a larger effective radius than β_{11}, then the radius for β_{nn} will be even larger.

Consider some applications of (13) and (19) to the relativistic case. As mentioned earlier, these expressions should give a qualitative description of the relativistic phenomena. For $\pi\pi$ scattering we use (13). The scattering here presumably proceeds through a single ρ exchange. Roughly speaking the range of interaction should be inversely proportional to the distance of the left-hand branch point from the origin. The ρ exchange gives rise to a branch point at $\nu=-m_\rho^2/4$ and therefore we expect $R_0^2 \approx (m_\rho^2/4)^{-1}$. This value of R_0 yields the total cross section $\sigma_{\pi\pi}$ at infinity [see (7)] to be

$$\sigma_{\pi\pi} = 20\ \text{mb},$$

where $\alpha'(0)$ is taken to be 0.4 $(\text{BeV}/c)^{-2}$ on the basis of the numerical fits of I. According to the factorization theorem,[4] $\sigma_{\pi\pi}$ should be about 15 mb. The $(\text{width})^{-1}$ of $\bar{\gamma}$, from (4) and (14), is found to be [in units of $(\text{BeV}/c)^{-2}$]

$$\bar{\Gamma}_{\pi\pi} = [d\ln\bar{\gamma}(t)/dt]|_{t=0} = 1.4 + 2.5\alpha''(0).$$

The value of $\alpha''(0)$ is not known from experiments as yet, but we expect it to be positive and quite small

[8] The difference in the behavior above and below $\alpha=\frac{1}{2}$ arises because of the centrifugal barrier which tends to push the dominant part of the wave function away from the origin for higher partial waves. The most sensitive dependence of β/ν^α is on the range of the potential. However, it also depends on the strength of the potential through the α-dependent terms in (13).

[9] In general, α' is a decreasing function, and therefore it gives an additional positive contribution to the diffraction width.

[10] Note that it is the range of forces in the t channel that appears here.

[$\lesssim\alpha'(0)$ in magnitude].[11] The above result should be compared with the value of 1.27 $(\mathrm{BeV}/c)^{-2}$ obtained in I through factorization.

For πN scattering, the t channel involves $\pi\pi\to N\bar{N}$ where the Born amplitude corresponds to a single nucleon exchange. The left-hand branch point in this case is at $-m_\pi^4/4M^2$ where M is the nucleon mass. Therefore in (19),

$$\nu_{L1}'=-m_\pi^4/4M^2$$

and

$$\nu_{Ln}'=-m_\pi^4/4M^2-M^2+m_\pi^2\approx -M^2.$$

We have already mentioned above that $\nu_{L1}=-m_\rho^2/4$. The quantity

$$(|\nu_{L1}|/|\nu_{L1}'\nu_{Ln}'|^{1/2})^\alpha=(m_\rho^2/2m_\pi^2)^\alpha \tag{20}$$

then is greater than unity. Therefore, the effective radius in πN appears to be larger than in $\pi\pi$. In a more accurate treatment of the πN problem, which would include spins, etc., there may be deviations from expression (19). However, the above feature, namely, a larger radius in πN, should survive, since it is based on rather general grounds, depending primarily on the distance of the nearest branch point. Expressions (19) and (20) should reasonably well approximate the actual diffraction shape even though the total cross section would depend on the details of the interaction and not just on the radius. Therefore, it would be interesting to see the diffraction shape predicted by (19). The $(\text{width})^{-1}$ of $\bar{\gamma}$ in units of $(\mathrm{BeV}/c)^{-2}$ is then given by

$$\bar{\Gamma}_{\pi N}=[d\ln\bar{\gamma}(t)/dt]|_{t=0}=2.7+2.5\alpha''(0).$$

From I the experimental value for this quantity is 3.47 $(\mathrm{BeV}/c)^{-2}$. For NN scattering the width can be obtained from factorization. We find

$$\bar{\Gamma}_{NN}=4.0+2.5\alpha''(0),$$

The experimental value is 5.67 $(\mathrm{BeV}/c)^{-2}$.

The πN problem within the relativistic framework is being investigated.

Even though one may not expect (13) and (19) to give quantitatively accurate results, they nevertheless contain information which, we believe, will be carried over into the relativistic case. For one thing it is clear that the most sensitive dependence of β/ν^α comes from the term $R^{2\alpha}$, where R is the effective radius (in either the single- or the multichannel case). If one takes the scaling factor m_0^2 to be equal to R^{-2} [see (3)] then the resulting reduced residue $\gamma(t)$ is a slowly varying or, more precisely, a nonexponential function of t. On the other hand, if one takes the normalization $m_0=M$, the nucleon mass used in high-energy phenomenology, then $\gamma(t)$ and $\bar{\gamma}(t)$ are $\sim(M^2R^2)^\alpha$ [see (4)] and fall off exponentially if $MR>1$. The value of R will be different for different reactions. A larger value of R implies a sharper falloff. Our previous estimates showed, for instance, that the value of R for πN is greater than for $\pi\pi$, and for both cases $MR>1$. In the earlier work on the Regge hypothesis, such a dependence on the radius was not realized, and neither was the fact that R can vary from one reaction to another. This led some to approximate $\bar{\gamma}(t)$ by a constant. Our results show that $\bar{\gamma}$ may in fact fall off exponentially for negative t.

Whatever value we choose for m_0, the contribution of P to the scattering amplitude behaves as $(sR^2)^\alpha$. A larger total cross section implies a larger R and, therefore, a sharper falloff in the diffraction region. This result is borne out quite well by the experiments. Some further amusing consequences follow from (13) and (19) that would be interesting to check experimentally once high enough energy becomes available. It appears that the diffraction shapes of two different reactions are the same at energies that are inversely proportional to their total cross sections. Also, as we noted earlier, (14) is true only for $0\leqslant\frac{1}{4}|\nu R_0^2|<1$. Thus the range within which the diffraction pattern is exponential decreases with increasing total cross section.

III. CONNECTION BETWEEN f^0, THE TOTAL CROSS SECTION, AND THE DIFFRACTION WIDTH

It was pointed out in Ref. 3 that useful information about the width of $\gamma(t)$ at $t=0$ can be obtained if one notes that f^0 lies on the P trajectory. Unitarity tells us that at the position of f^0 the residue of P should be proportional to the width of f^0. Since the width is a real quantity, the imaginary part of the residue must be quite small. Therefore, the phase of the residue at f^0 must be $2n\pi+\theta$, where θ is a small number and $n=0, 1, 2, \cdots$. Two assumptions regarding the experimental values of the Regge parameters were made in Ref. 3, which, as we shall point out in the next paragraph, have turned out to be erroneous. It was assumed on the basis of the fits of Foley *et al.*[12] on πp scattering (with P alone) that $\alpha'\lesssim 0.2$ $(\mathrm{BeV}/c)^{-2}$ and therefore is negligible as far as the evaluation of the diffraction width was concerned. Secondly, the $(\text{width})^{-1}$ of $\bar{\gamma}^2(t)$ was taken to be the same as the experimental value of the $(\text{width})^{-1}$ ($\sim$10 $(\mathrm{BeV}/c)^{-2}$) of $d\sigma/dt$ of πN and NN scattering. Since α' was considered negligible the $(\text{width})^{-1}$ of $\gamma(t)$ would be the same as that of $\bar{\gamma}(t)$ and, therefore, $\eta'(0)$ would be about 5 $(\mathrm{BeV}/c)^{-2}$. A phase representation of the type (5) with $m_0=M$ was written with the following expression for η_I:

$$\eta_I(t)=\epsilon\pi((t-4m_\pi^2/t+c))^{\alpha_0+1/2},\quad t>4m_\pi^2, \tag{21}$$

where $\alpha_0(\approx 1)$ is the threshold value of α. This form was taken in order to give the correct threshold power $\alpha_0+\frac{1}{2}$ and satisfy the behavior at infinity suggested by the

[11] G. F. Chew and V. L. Teplitz, Phys. Rev. **136**, B1154 (1964), have obtained reasonable agreement with the $\pi\pi$ total cross section and the diffraction width within the strip approximation. Their $q_{\bar{s}}$ should be roughly inversely proportional to our R_0. They take the magnitude of $\alpha''(0)$ to be equal to that of $\alpha'(0)$.

[12] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **10**, 376, 543 (1963).

strip approximation.[13] A knowledge of $\gamma(0)$ ($\sim\sigma$), $\gamma_R(t_f)$ ($\sim$width of f^0), and $\eta_I(t_f)$ would enable one to determine the two parameters ϵ and c from which the value of $\eta'(0)$ can be determined. As pointed out earlier, η_I at f^0 could be either a very small number or essentially 2π, 4π, etc. It was shown in Ref. 3 that $\eta'(0)=5$ was impossible to achieve unless $\eta_I(t_f)$ was about 2π, the point being that to obtain a large value of $\eta'(0)$ requires a sharp increase in η_I above threshold which is provided only if the value of η_I at f^0 is large. The choice 2π, however, as pointed out if Ref. 3, is in contradiction with the potential-theory results. According to potential scattering the phase at a resonance should remain small instead of being 2π.

In view of the results in I, the assumptions about α' and $\eta'(0)$ have to be modified. In I it is found that α' is about 0.4 $(\text{BeV}/c)^{-2}$ and therefore non-negligible. Secondly, even though one may find (in the region below 20 BeV/c) the widths of $d\sigma/dt$ for $\pi\pi$, πN, and NN to be essentially the same, this does not necessarily mean that the widths of $\gamma(t)$ of P at $t=0$ are also the same. In fact, in I it is shown that the (width)$^{-1}$ of $\bar{\gamma}^2$ at $t=0$ for $\pi\pi$ should, on the basis of factorization, be only about 2.54 instead of the previously assumed value of 10 $(\text{BeV}/c)^{-2}$. Taking α' to be 0.4 it is found that the (width)$^{-1}$ of γ^2 with $m_0=M$ is even smaller. It turns out that $\eta'(0)$ should only be about 0.87, a factor of about 6 smaller than the previously assumed value.

We shall show below that the value

$$\eta'(0)=0.87\ (\text{BeV}/c)^{-2}$$

is consistent with the phase $\eta_I(t_f)$ at f^0 being small and, therefore, consistent with the prediction of potential theory. Now

$$\eta_I(t_f)=\alpha_I \ln(M^2/\nu_f)+\theta(t_f)\,, \qquad (22)$$

$$\gamma_R(t_f)=\gamma(0)e^{\eta_R(t_f)}\cos(\eta_I(t_f))\,, \qquad (23)$$

where, as mentioned earlier, θ, the phase of the residue, is assumed to be small. If, for a given angular momentum, l and for all energies a single Regge pole were to satisfy unitarity exactly then θ would be zero identically. However, this is highly unlikely to occur, since secondary trajectories do play an important role.[14] The magnitude of θ, in general, would depend on the contribution of the secondary trajectories. In our case of the P pole it would depend on the behavior of P' near f^0, but one should expect θ to be between zero ($\beta_I=0$) and $\pi/4$ ($\beta_I=\beta_R$). In Table I we give the values of $\eta'(0)$ with $\sigma_{\pi\pi}=15$ mb for the above two cases. We observe that the experimentally predicted value is somewhere between the two extremes. This provides us with some justification for saying that the predicted width of $\gamma(t)$ at $t=0$ is consistent with the potential-theory results and with the f^0 parameters.

If by $g(t)$ we denote the ratio $\gamma(t)/\gamma(0)$ then we find numerically that for fixed η_I a smaller value of $g_R(t_f)$ implies a larger $\eta'(0)$. This means that the larger the value of the total cross section, the sharper the falloff of $\gamma(t)$ at $t=0$. As mentioned earlier, such a relationship is observed in the high-energy experiments. The fact that $d\sigma/dt$ should fall off more sharply as the total cross section increases is a well-known consequence of s-channel unitarity for large s.[15] Here it appears to be the consequence of t-channel unitarity. Our result also means that a smaller width of f^0 implies a sharper fall off.

TABLE I. The values of $\eta'(0)$ in $(\text{BeV}/c)^{-2}$ for different values of the phase θ at f^0. For comparison the experimentally predicted values are given.

	$\theta=0$	$\theta=\pi/4$	Exp
$\pi\pi$	0.55	1.64	1.27
πN	0.71	2.83	3.47
NN	0.87	4.02	5.67

We have used the expression (5) for πN scattering neglecting the nucleon spin and with η_I given by (21), (22) and γ_R by (23). This is done primarily to illustrate the effect of increasing the total cross section. Our results for πN should not be considered, in any sense, as quantitative estimates. For πN scattering $\sigma_{\pi N}\approx 25$ mb. Here the t channel involves $\pi\pi\to N\bar{N}$, but the cut starts from $t=4m_\pi^2$. We assumed $f^0N\bar{N}$ coupling to be the same as $f^0\pi\pi$ coupling. Since $\gamma(0)$ in πN is larger than in $\pi\pi$, we obtained a larger value of $\eta'(0)$ (see Table I). However, it is not so large as to accomodate the experimentally observed value. This might well mean that $f^0N\bar{N}$ coupling is smaller than $f^0\pi\pi$, which will have the effect of increasing $\eta'(0)$ further. The values for NN scattering obtained from factorization are also given in Table I.

IV. THE ZEROS OF THE RESIDUES IN POTENTIAL SCATTERING

In the Appendix it is shown that if at threshold $\alpha>-\frac{1}{2}$, then β/ν^α is positive there. For a general superposition of Yukawa potentials

$$rV(r)=\int_{t_0}^{\infty} dt'\rho(t')\exp(-t'^{(1/2)}r)$$

with

$$\rho(t)\underset{t\to\infty}{\longrightarrow}\rho_0 t^{-(1/2)-\epsilon}. \qquad (24)$$

It is known from (A7) and (A8) that for the leading pole

$$\alpha(\infty)=-\tfrac{1}{2}-\epsilon\,,\quad \tfrac{1}{2}>\epsilon>0\,,$$
$$=-1\,,\quad \epsilon>\tfrac{1}{2}$$

and

$$\beta/\nu^\alpha\to -C_0\,,\quad \tfrac{1}{2}>\epsilon>0\,, \qquad (25a)$$

$$\to -\frac{1}{2}\int_{t_0}^{\infty} dt'\rho(t')\,,\quad \epsilon>\tfrac{1}{2}\,, \qquad (25b)$$

[13] G. F. Chew, Phys. Rev. **129**, 2363 (1963).
[14] A. Ahmadzadeh, Phys. Rev. **133**, B1074 (1964).
[15] R. Glauber, *Lectures in Physics* (Interscience Publishers, Inc., New York, 1958), p. 315.

where C_0 has the same sign as ρ_0. Clearly the value of β/ν^α at infinity depends more on the strength than on the range of the potential.

In what follows we shall take $\nu \leqslant 0$. Consider the situation where one has a long-range attraction followed by a strong short-range repulsion. It is known that if the attraction range is sufficiently long then[6] the threshold value of α must be $> -\frac{1}{2}$ and consequently at threshold β/ν^α must be positive. If, in addition, the short-range repulsion is sufficiently strong, i.e., if ρ is large and positive at infinity, then (25a) and (25b) can be negative. Thus in this case β/ν^α is positive at threshold and negative at infinity. Hence, if we ignore the possibility of cuts, it must have at least one zero in between. We know from (A1) that

$$\beta e^{-i\pi\alpha}=\left[(2\alpha+1)\int_0^\infty dr\, r^{-2}f^2(\alpha,-k,r)\right]^{-1},\qquad \alpha>-\tfrac{1}{2},\quad (26)$$

where $f(\alpha,-k,r)$ is real. Therefore, $\beta e^{-i\pi\alpha}$ and β/ν^α are positive if $\alpha>-\frac{1}{2}$. Hence the zero cannot occur as long as the pole is in the right half-plane. However, zeros can occur when the pole is in the left half-plane. The position where the residue vanishes can be determined from the relation[6]

$$e^{i\pi l}f(l,-k)f(-l-1,k)+e^{-i\pi l}f(l,k)f(-l-1,-k)=2\cos\pi l,\quad (27)$$

where the S matrix is given by

$$S=f(l,k)/f(l,-k).$$

At a Regge pole, $l=\alpha$ and $f(l,-k)=0$. It is easy to see from (27) that the residue can vanish either at negative half-integral values of α or at the fixed poles of $f(l,\pm k)$.[6] The fixed poles of f are known to be at negative integral α and, at $\alpha=-n/2-\epsilon$ where $n=1,2\cdots$.

For the multichannel case, in place of (26), we have the relation (see A20)

$$\mathrm{Tr}(\beta e^{-i\pi\alpha}u)=\left[(2\alpha+1)\int_0^\infty dr\, r^{-2}f^T f\right]^{-1},\quad (28)$$

where the u's are not necessarily positive. It can be shown, as in the single-channel case, that the residues β_{ij} can be negative if the potential V_{ij} has a strong short-range repulsion. Thus the residues in this case can also change sign.

At first sight it appears that since the u's in (28) are not necessarily positive, the residues can change sign even when $\alpha>-\frac{1}{2}$. However, from factorization, it is known that

$$\beta_{ij}^2=\beta_{ii}\beta_{jj}.$$

Therefore, as long as we insist on simple zeros and no left-hand branch points, a zero in one of the residues implies a zero in *all* the residues at the same position. Such a circumstance would be in contradiction with (28). Hence, zeros cannot occur for $\alpha>-\frac{1}{2}$.

Our results are obtained on the basis of potential theory. In the relativistic case, however, it is possible for $\alpha(\infty)$ to be in the right half-plane, between $l=-\frac{1}{2}$ and $l=1$. In that case the zeros may also occur in the right half-plane.

For the ω case, therefore, it is quite possible for zeros to exist if there is a strong short-range repulsion.

We also note that the secondary trajectories, in general, have zeros even though the leading trajectories may not. It is shown in Appendix (v) that for a single attractive Yukawa potential $ge^{-\mu r}$ with $g<0$,

$$\beta/\nu^\alpha\to -\tfrac{1}{2}g\nu^{n-1}.$$

A phase representation of $\gamma(\nu)$ of the type (A16) can be written which satisfies the threshold behavior and has a vanishing phase at infinity. However, in order to satisfy the above condition, (A16) must be multiplied by a polynomial of $n-1$ degree. Hence β/ν^α (and β) of the nth-rank trajectory has $n-1$ zeros. This can be generalized to a superposition of Yukawa potentials and it can be shown that whenever β/ν^α goes to infinity as ν^p it implies the existence of at least p zeros.

Note added in proof. It may be of interest to note that the relation $\beta/\nu^\alpha\sim(R^2/4)^{\alpha(t)}$ can be made plausible also from the following consideration. A partial-wave amplitude $A_l(t)$ can be reasonably well given by the Khuri-representation

$$A_l(t)\approx\beta(t)e^{-(l-\alpha)\xi}/l-\alpha(t),$$

where

$$\xi=\ln[1+(\mu^2/2\nu)+(\mu^2/\nu+\mu^4/4\nu^2)^{1/2}],$$

and μ^{-1} is the range parameter. In analogy with the Breit-Wigner form we expect

$$\beta(t)e^{-(l-\alpha)\xi}\approx c\nu^l$$

near threshold, where c is a slowly varying (non-exponential) function of ν. From this it immediately follows that for $|\nu/\mu^2|<1$

$$\beta/\nu^\alpha\sim(R^2/4)^{\alpha(t)}$$

where the usual identification between R and the nearest left-hand singularity, namely $R^2=(\mu^2/4)^{-1}$, is made.

APPENDIX

Here we summarize some of the useful results about the residues. The results given are based on potential scattering or on low-energy effective-range theory. Most of them are contained either explicitly or implicitly in the published literature. Whenever necessary, explicit derivations are given.

(i) An expression for the residue valid for $\nu=k^2<0$ and $\alpha>-\frac{1}{2}(\mathrm{Im}\alpha=0)$ is the following[6]:

$$\beta e^{-i\pi\alpha}=\left[(2\alpha+1)\int_0^\infty dr\, r^{-2}f^2(\alpha,-k,r)\right]^{-1},\quad (A1)$$

where f is the so-called irregular wave function satisfying the boundary condition

$$f(\alpha,-k,r)\to_{r\to\infty}e^{ikr}.$$

It is real when k is pure imaginary (i.e., $\nu<0$). Also note that in the same region of ν and α

$$\frac{d\alpha}{d\nu}=\frac{1}{2\alpha+1}\int_0^\infty dr\, f^2(\alpha,-k,r)\Big/\int_0^\infty dr\, r^{-2}f^2(\alpha,-k,r). \quad \text{(A2)}$$

(ii) *Threshold behavior.* Let us write $(A/\nu^l)^{-1}$ in the modified form[16] of the effective-range expansion[17]

$$\left(\frac{A}{\nu^l}\right)^{-1}=R(\nu,\lambda)-\sum_{n=0}\frac{\nu^n}{\pi(\lambda-n)}+\frac{(-\nu)^\lambda}{\sin\pi\lambda},$$

where A is the scattering amplitude, $\lambda=l+\frac{1}{2}$, and $-\nu=|\nu|e^{-i\pi}$. The second term above is necessary in order to ensure correct behavior at integral λ.[16] We expand R in a power series in ν and λ at a Regge pole $\lambda=\lambda(\nu)$ $[l=\alpha(\nu)]$:

$$R(\nu,\lambda)=\nu R_\nu+(\lambda-\lambda_0)R_\lambda+(\nu^2/2!)R_{\nu\nu}+\nu(\lambda-\lambda_0)R_{\nu\lambda}+((\lambda-\lambda_0)^2/2!)R_{\lambda\lambda}+\cdots,$$

where λ_0 (>0) is the threshold position of the pole.[17] The effective-range expansion of $\lambda(\nu)$ and $\beta(\nu)$ can be obobtained from this. It can be easily checked that for small ν

$$\mathrm{Im}\lambda(\nu)=\nu^{\lambda_0}/R_\lambda.$$

It is known that $\mathrm{Im}\lambda(\nu)>0$ for $\nu\gtrsim 0$; therefore, R_λ is positive. Furthermore, β/ν^α at $\nu=0$ is equal to R_λ^{-1}, and is, therefore, positive.

Keeping the first two dominant terms we obtain for the reduced residue

$$\beta/\nu^\alpha=(1/R_\lambda)[1-(1/R_\lambda)(-\nu)^{\lambda_0}\ln(-\nu)],\quad 0<\lambda_0<1, \quad \text{(A3a)}$$

$$=(1/R_\lambda)[1-(\nu\ln^2(-\nu)/\pi R_\lambda)],\quad \lambda_0=1, \quad \text{(A3b)}$$

$$=(1/R_\lambda)[1-((R_{\nu\lambda}/R_\lambda)-(R_\nu R_{\lambda\lambda}/R_\lambda^2))\nu],\quad \lambda_0>1. \quad \text{(A3c)}$$

Thus we see that for $0<\lambda_0\leqslant 1$, the quantity β/ν^α *increases* as ν decreases from zero. The same is true for $(m_0^2/\nu)^\alpha\beta(\nu)$, where m_0 is an arbitrary mass. For $\lambda_0>1$ a definite statement about the behavior of β/ν^α cannot be made on the basis of (A3c). However, the case where the potential vanishes outside a radius R_0 is discussed in Sec. II.

(iii) *Behavior at infinity.* Let us consider a superposition of Yukawa potentials

$$rV(r)=\int_{t_0}^\infty dt'\rho(t')\exp(-t'^{(1/2)}r). \quad \text{(A4)}$$

It is known that if the first and second moments of the potential exist then the Mandelstam representation holds.[18] This implies the behavior

$$\rho(t)\underset{t\to\infty}{\longrightarrow}\rho_0 t^{-(1/2)-\epsilon}+\cdots,\quad \epsilon>0. \quad \text{(A5)}$$

As $\nu\to\infty$, the end point of a Regge trajectory, $\alpha(\infty)$, is known to be at a pole of the Born amplitude.[6] The Born amplitude

$$A_B(l,\nu)=-\frac{1}{2\nu}\int_{t_0}^\infty dt'\rho(t')Q_l\left(1+\frac{t'}{2\nu}\right)=-\int_{z_0}^\infty dz\,\rho(2\nu z)Q_l(1+z). \quad \text{(A6)}$$

As $\nu\to\infty$, we then get the following result for the *leading* trajectory. If $0<\epsilon<\frac{1}{2}$, then

$$A_B(l,\nu)\to -C_0\nu^{-\frac{1}{2}-\epsilon}/(l+\tfrac{1}{2}+\epsilon),\quad \alpha(\infty)=-\tfrac{1}{2}-\epsilon,$$

and

$$\beta/\nu^\alpha\to -C_0, \quad \text{(A7)}$$

where C_0 has the same sign as ρ_0. If $\epsilon>\frac{1}{2}$, then

$$A_B(l,\nu)\to -\frac{1}{2\nu}\int_{t_0}^\infty\frac{dt'\rho(t')}{l+1},\quad \alpha(\infty)=-1,$$

and

$$\beta/\nu^\alpha\to -\frac{1}{2}\int_{t_0}^\infty dt'\rho(t'). \quad \text{(A8)}$$

When $\rho(t)$ is a δ function, we obtain the case of a single Yukawa potential.

(iv) *The phase of the residue at infinity.* For simplicity, consider a single Yukawa potential $ge^{-\mu r}/r$, where the Born amplitude is

$$A_B(l,\nu)=(-g/2\nu)Q_l(1+(\mu^2/2\nu)). \quad \text{(A9)}$$

In (iii) we have already obtained the behavior at infinity of the real part of the residue. To obtain the imaginary part, we have to impose unitarity. In order to do this, we write the amplitude $A(l,\nu)$ in the usual N/D formalism where $N(l,\nu)=A_B(l,\nu)$. This prescription will satisfy unitarity correctly at high energies. If one further notes that as $\nu\to\infty$ and $l\to -1$ (the leading pole position),

$$(-\sin\pi l/\pi)Q_l(1+(\mu^2/2\nu))\to 1+(l+1)\tfrac{1}{2}\ln(4\nu/\mu^2);$$

[16] B. R. Desai and B. Sakita, Phys. Rev. **136**, B226 (1964). See also M. Ross and Y. N. Srivastava (to be published).

[17] A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962).

[18] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

then one gets

$$\beta(\nu) \xrightarrow[\nu\to+\infty]{} (-g/2\nu)[1-i(g/2\sqrt{\nu})\ln(4\nu/\mu^2)],$$
$$\alpha(\nu) \to -1-i(g/2\sqrt{\nu}), \qquad \text{(A10)}$$

and

$$\beta/\nu^\alpha \to (-g/2)+i(g^2/4\sqrt{\nu})\ln(4/\mu^2). \qquad \text{(A11)}$$

Thus if $\theta(\nu)$ is the phase of β, then at infinity

$$\tan\theta = -(g/2\sqrt{\nu})\ln(4\nu/\mu^2) \to 0. \qquad \text{(A12)}$$

Hence $\theta(\infty)=n\pi$, $n=0, 1, 2, \cdots$. This result can be easily generalized to the superposition of Yukawa potential. The important thing to note is that the sign of the short-range coupling plays an important role. Thus if the potential is everywhere attractive then g is negative and $\theta(\infty)=0$. On the other hand if there is a strong short-range repulsion then presumably $\theta(\infty)=\pi$.

Finally it can be easily checked that the higher rank trajectories satisfy the *same* phase relation (A12).

(v) *Phase representations of* $\beta e^{-i\pi\alpha}$ *and* β/ν^α. Consider first the quantity $\beta e^{-i\pi\alpha}$. It has a right-hand cut starting at $\nu=0$ and is real along the negative real axis.[6,19] Let $\theta(\nu)$ be the phase of β. From (ii) it can be deduced that

$$\theta(\nu) \xrightarrow[\nu\to0]{} \nu^{\alpha_0+(1/2)}\ln\nu,$$

where α_0 is the threshold value of $\alpha(\nu)$. In (iv) it is shown that θ in general approaches a multiple of π at infinity. Consider for a moment the case where the potential is attractive throughout. In that case the phase at infinity vanishes. One can write the following phase representation for $\beta e^{-i\pi\alpha}$:

$$\beta e^{-i\pi\alpha} = -(C/\nu)e^{\phi(\nu)}, \qquad \text{(A13)}$$

where

$$\phi(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\{\theta(\nu')-\pi[\alpha_R(\nu')+1]\} \qquad \text{(A14)}$$

and C is a positive constant. It can be easily verified that

$$\beta \xrightarrow[\nu\to0]{} C'\nu^{\alpha_0} \qquad \text{(A15)}$$

and

$$\beta \xrightarrow[\nu\to\infty]{} C/\nu,$$

where C' has the same sign as C. Therefore, it exhibits the correct threshold and infinity behaviors. We expect θ to be a smoothly varying function, but not the quantity $\alpha_R(\nu)+1$, because

$$\alpha_R(\nu)+1 = \frac{P}{\pi}\int_0^\infty \frac{d\nu'\alpha_I(\nu')}{\nu'-\nu}$$

is a principal-part integral. It decreases very rapidly in the region where α_I is near its maximum. This rapid decrease will cause wild oscillations in the real and imaginary parts of $\beta e^{-i\pi\alpha}$ for $\nu>0$. Such oscillations are, of course, not surprising, as they are necessary to ensure the vanishing of $\beta e^{-i\pi\alpha}$ at threshold in the manner indicated by (A15). Again this result can be generalized to a superposition of Yukawa potentials.

Following (A13) we can write for the reduced residue

$$\gamma(\nu) = (m_0^2/\nu)^\alpha\beta(\nu) = \gamma(0)e^{\eta(\nu)}, \qquad \text{(A16)}$$

$$\eta(\nu) = \frac{\nu}{\pi}\int_0^\infty \frac{d\nu'}{\nu'(\nu'-\nu)}\eta_I(\nu'), \qquad \text{(A17)}$$

where an explicit subtraction at $\nu=0$ is made. Also[20]

$$\eta_I(\nu) \xrightarrow[\nu\to0]{} \nu^{\alpha_0+(1/2)}\ln\nu.$$

The phase at infinity here also goes to a multiple of π (see (A11)). The representation (A16) [as well as (A13)] is arbitrary up to a polynomial. A polynomial of nth degree would imply n zeros of $\gamma(\nu)$. If it turns out that for a given potential, $\gamma(\nu)$ of the leading trajectory has n zeros, then η_I must approach $n\pi$ at infinity in order to ensure the constancy of $\gamma(\nu)$ at infinity [see (A7) and (A8)].

(vi) *Higher rank trajectories.* For superposition of Yukawa potentials (A4), the behavior at infinity of β for higher rank trajectories depends on the nature of the higher terms in the expansion (A5) of the weight function $\rho(t)$. If the higher powers in (A5) are separated by exactly one unit, then it can be easily shown that for the nth rank trajectory

$$\beta/\nu^\alpha \to \nu^{n-1}. \qquad \text{(A18)}$$

For a single Yukawa potential $ge^{-\mu r}/r$, $\alpha_n(\infty)=-n$, we have

$$\beta(\nu) \to -g/2,$$
$$\beta/\nu^\alpha \to -g/2\nu^{n-1}. \qquad \text{(A19)}$$

Thus the reduced residues, in general, go to infinity. Moreover, if the potential is attractive ($g<0$), then as in (A12) we can show that the phase of $\gamma(\nu)$ is zero at infinity. In the case that the higher powers in the expression (A5) are not separated by one unit then β/ν^α for some of the higher rank trajectories will *not* go to infinity but will rather approach a constant.

(vii) *Multichannel case.* The formalism for the multichannel case has been developed by various authors.[21] We follow the notation and normalization of Ref. 22 appropriately generalized to complex l. The S matrix can be written as

$$S(\lambda,K) = e^{i\pi(\lambda-\frac{1}{2})}F^{-1}(\lambda,-K)F(\lambda,K), \quad \lambda=l+\tfrac{1}{2},$$

[19] J. R. Taylor, Phys. Rev. **127**, 632 (1962).

[20] As long as $\alpha_0>\frac{1}{2}$, the logarithmic factor should not be important. This assumption is implicit in the expression (21).

[21] L. Favella and M. T. Reineri, Nuovo Cimento **23**, 616 (1962). J. M. Charap and E. J. Squires, Ann. Phys. (N. Y.) **20**, 145 (1962); **21**, 8 (1963); **25**, 143 (1963).

[22] R. G. Newton, J. Math. Phys. **2**, 188 (1961).

where K denotes the channel momenta, and the F's are the Jost matrices. The Regge poles are determined from the relation

$$\det F(\lambda, -K)=0.$$

The above relation implies that there is a constant (row) vector a such that at a Regge pole

$$aF(\lambda, -K)=0.$$

Using the same technique as in the single-channel case,[6] namely calculating the Wronskian of the wave function and its first derivative with respect to λ, we obtain

$$y\dot{F}^{\mathrm{T}}(\lambda, -K)[F^{\mathrm{T}}(\lambda,K)]^{-1}Ky^{\mathrm{T}}$$
$$=-i\lambda\int_0^\infty dr\, r^{-2}yf^{\mathrm{T}}(\lambda, -K, r)f(\lambda, -K, r)y^{\mathrm{T}},$$

where "T" means transpose, and the dot means differentiation with respect to λ; $y=aF(\lambda,K)K^{-1}$ is a column vector and f is the matrix of the irregular wave functions defined by

$$f_{ij}(\lambda, -K, r)\rightarrow \delta_{ij}e^{-ik_jr}, \quad \text{as} \quad r\rightarrow\infty\,,$$

k_j being the channel momentum in the j channel. At a Regge pole $l=\alpha$, and yf^{T} is a well-behaved function at the origin. It vanishes like $r^{\alpha+1}$ as $r\rightarrow 0$. One can easily check that

$$e^{i\pi(\lambda-\frac{1}{2})}y\frac{\partial}{\partial l}(S^{\mathrm{T}})^{-1}Ky^{\mathrm{T}}=y\dot{F}^{\mathrm{T}}(\lambda, -K)[F^{\mathrm{T}}(\lambda,K)]^{-1}Ky^{\mathrm{T}}.$$

Now let

$$S_{ij}=(g_{ij}/l-\alpha)+r_{ij}.$$

Then

$$\frac{\partial}{\partial l}(S)_{ij}^{-1}=\frac{z_{ij}}{\mathrm{Tr}(gz)},$$

where z is the matrix of the cofactors of r and "Tr" means trace. Thus

$$e^{i\pi(\lambda-\frac{1}{2})}\frac{yz^{\mathrm{T}}Ky^{\mathrm{T}}}{\mathrm{Tr}(gz)}=-i\lambda\int_0^\infty dr\, r^{-2}\hat{f}^{\mathrm{T}}\hat{f},$$

where $\hat{f}=fy^{\mathrm{T}}$ is a column vector. If we denote the residue of the symmetric T matrix by β then $g=2iK\beta$ and we get

$$\mathrm{Tr}(\beta e^{-i\pi\alpha}u)=\left(2\lambda\int_0^\infty dr\, r^{-2}\hat{f}^{\mathrm{T}}\hat{f}\right)^{-1}, \tag{A20}$$

where

$$u=zK/yz^{\mathrm{T}}Ky^{\mathrm{T}}. \tag{A21}$$

Note that the matrix elements u_{ij} are not necessarily positive.

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Normalization Condition and Normal and Abnormal Solutions of the Bethe-Salpeter Equation*

NOBORU NAKANISHI
Brookhaven National Laboratory, Upton, New York
(Received 22 January 1965)

The normalization constants for normal and abnormal solutions of the Bethe-Salpeter equation are explicitly calculated in the Wick-Cutkosky model. In the case of the vanishing total four-momentum, it is shown that the abnormal solutions with odd κ have a negative norm, where κ is the Wick-Cutkosky quantum number, and that the corresponding scattering Green's function contains all the normal and abnormal solutions as the residues of poles. It is demonstrated explicitly for $\kappa=0$, 1, 2, 3 that the first conclusion remains true also in the case of an infinitesimally positive mass. As for the case of massless bound states, its special character is emphasized, and solid harmonics are constructed corresponding to the "little group" for a massless particle. Non-Cutkosky integral equations are obtained for the weight functions of the integral representation for the Bethe-Salpeter amplitude.

1. INTRODUCTION

In 1954, Wick[1] and Cutkosky[2] obtained a complete set of the solutions of the Bethe-Salpeter (B-S) equation for bound states of two scalar particles exchanging massless scalar particles in the ladder approximation. They discovered that in addition to normal solutions there exist abnormal solutions which have no counterparts in the nonrelativistic potential theory. The appearance of these extra solutions is intimately related to the additional freedom in a covariant two-body problem, i.e., "relative time" or "relative energy," which leads to the introduction of a new quantum number κ (normal solutions correspond to $\kappa=0$).

Scarf and Umezawa[3] tried to exclude the abnormal

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[1] G. C. Wick, Phys. Rev. **96**, 1124 (1954).
[2] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).
[3] F. L. Scarf and H. Umezawa, Phys. Rev. **109**, 1848 (1958).