$a_1{}^\dagger$, $i=2, 3, \cdots N$, in some order, acting on Φ_0,

$$a_1\Phi_N{}^{(j)}=\sum_{Q'}\alpha_{Q'}{}^{(j)}Q'\Phi_{N-1}, \tag{A7}$$

where the numbers $\alpha_{Q'}{}^{(j)}$ remain to be determined. Using Eqs. (A6) and (A7),

$$a_1\Psi_N{}^{(j)}=\sum_{Q'}\sum_{P'}\delta_{Q'}\alpha_{P'}{}^{(j)}Q'P'\Phi_{N-1}$$

$$=\sum_{P'}\delta_{P'}\alpha_{P'}{}^{(j)}\Psi_{N-1}{}^{(a)},$$

where we have used the property of the group sum $\sum_{Q'}$ and the definition of $\Psi_{N-1}{}^{(a)}$, Eq. (A1), to reach the second line. We determine the $\alpha_{Q'}{}^{(j)}$ using the para-Bose rules:

$$a_1\Phi_N{}^{(1)}=[a_1,a_1{}^\dagger]_+\Phi_{N-1}-a_1{}^\dagger a_1\Phi_{N-1},$$
$$=p\Phi_{N-1}$$

for $j=1$. For $j>1$,

$$a_1\Phi_N{}^{(j)}=[a_1,a_2{}^\dagger]_+a_3{}^\dagger\cdots a_j{}^\dagger a_1{}^\dagger a_{j+1}{}^\dagger\cdots a_N{}^\dagger\Phi_0$$
$$-a_2{}^\dagger a_1a_3{}^\dagger\cdots a_j{}^\dagger a_1{}^\dagger a_{j+1}{}^\dagger\cdots a_N{}^\dagger\Phi_0$$
$$=2\sum_{l=2}^{j-1}(-)^lQ'(l,l+1,\cdots j)\Phi_{N-1}$$
$$+(-)^j(2-p)\Phi_{N-1},$$

where

$$Q'(l,l+1,\cdots j)a_2{}^\dagger\cdots a_{l-1}{}^\dagger a_l{}^\dagger a_{l+1}{}^\dagger\cdots a_j{}^\dagger a_{j+1}{}^\dagger\cdots a_N{}^\dagger\Phi_0$$
$$=a_2{}^\dagger\cdots a_{l-1}{}^\dagger a_{l+1}{}^\dagger\cdots a_j{}^\dagger a_l{}^\dagger a_{j+1}{}^\dagger\cdots a_N{}^\dagger\Phi_0.$$

This completes the calculation of $\alpha_{Q'}{}^{(j)}$:

$$\alpha_{Q'}{}^{(j)}=(-)^j(2-p),\quad Q'=1$$
$$=2(-)^l,\quad Q'=Q'(l,l+1,\cdots j)$$
$$=0,\quad \text{otherwise}.$$

Then the sum of interest is

$$\sum_{Q'}\delta_{Q'}\alpha_{Q'}{}^{(j)}=\sum_{l=2}^{j-1}(-)^{j-l}2(-)^l+(-)^j(2-p)$$
$$=(-)^j(2(j-1)-p),$$

which completes the demonstration of Eq. (A4), and the result Eq. (A5).

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High-Energy Elastic Scattering at Low Momentum Transfers

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The $K^\pm p$, $\pi^\pm p$, pp, and $\bar{p}p$ data in the laboratory-energy region between 7 and 20 BeV and momentum transfer squared, $-t$, less than 0.5 $(\text{BeV}/c)^2$ are analyzed in terms of the P, P', and ω Regge poles. A linear approximation to the trajectories is made with slopes α' assumed to be equal. The reduced residues of P and P' are taken to be of the form $(1-b_it)^{-\epsilon_i}$, $i=P, P'(b_i>0)$. In order to explain the difference between the antiparticle (K^-p and $\bar{p}p$) and particle (K^+p and pp) differential cross sections, the ω residue should have a zero at a negative value of t. Hence, the reduced residue for ω is taken to be of the form $(1+t/t_0)(1-b_\omega t)^{-\epsilon_\omega}$, where t_0 is the position of the zero. We choose $\epsilon_P=\epsilon_{P'}=2.5$ and $\epsilon_\omega=3.5$ in order to conform to the high-momentum-transfer behavior ($d\sigma/dt\sim t^{-6}$) observed in pp scattering. The $t=0$ values of the residues and the trajectory intercepts are known from other considerations. Covering the above range of energy and momentum transfer, we thus have five parameters for each of the antiparticle-particle sets, $K^\pm p$ and $pp-\bar{p}p$, and three parameters for $\pi^\pm p$, of which α' and (from factorization) the t_0's should be the same between the different sets. The α' values turn out to be the same $(=0.41(\text{BeV}/c)^{-2})$ for each set, while the t_0 values are reasonably close: 0.061 $(\text{BeV}/c)^2$ for $K^\pm p$ and 0.074 $(\text{BeV}/c)^2$ for $\bar{p}p-pp$. It is found that the residues of P contribute substantially to the diffraction widths. A crude estimate of the contribution of branch cuts indicates that they will not be important compared to P in the above region of energy and momentum transfer.

I. INTRODUCTION

RECENT experiments in the region of 7–20 BeV have shown certain characteristic differences between $K^\pm p$, $\pi^\pm p$, pp, and $\bar{p}p$ scattering.[1–4] For instance, it is found that the pp diffraction pattern shows a considerable amount of shrinkage, and $\pi^\pm p$ shows very little, while the shrinkage in K^+p is inter-

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[1] A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters **9**, 108 (1962).

[2] S. Brandt, V. T. Cocconi, D. R. O. Morrison, A. Wroblewski, P. Fleury, G. Kayas, F. Mueller, and C. Pelletier, Phys. Rev. Letters **10**, 413 (1963).

[3] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **10**, 376, 543 (1963).

[4] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **11**, 425, 503 (1963).

mediate between the two. The antiparticle differential cross sections K^-p, π^-p, and $\bar{p}p$ have higher values in the forward direction than the corresponding particle cross sections but fall off more sharply. In the case of K^-p and $\bar{p}p$ the differential cross sections go below the particle values at very small momentum transfers. The diffraction patterns of K^-p and $\bar{p}p$ do not appear to shrink but show a tendency to expand as the incident energy is increased. The only strong feature shared by all of them is that their cross sections fall off exponentially near the forward direction. Even then, it is found that at 10 BeV the $(\text{width})^{-1}$ of K^+p is $6(\text{BeV}/c)^{-2}$ while that for $\bar{p}p$ is almost twice as great.[4]

In view of the vast differences mentioned above, it is clear that at least up to 20 BeV, a single universal function of energy squared (s) and momentum transfer squared ($-t$) cannot possibly describe all the different reactions. If one takes the Regge-pole hypothesis seriously, then this means that a single pole cannot describe all the different experimental data and, as will be clear below, the reduced residues cannot be considered as slowly varying functions of $-t$.

In the pole approximation for large s and small $-t$, the crossed-channel Regge poles with positions $\alpha_i(t)$ dominate. The differential cross section $d\sigma/dt$ and the total cross section σ_T are then given by[5–8]

$$d\sigma/dt=|\textstyle\sum_i \xi_i\bar{\gamma}_i(t)(s/2m_0{}^2)^{[\alpha_i(t)-1]}|^2\,, \tag{1}$$

$$\sigma_T=(4\sqrt{\pi})\textstyle\sum_i \Upsilon_i\bar{\gamma}_i(t)(s/2m_0{}^2)^{[\alpha_i(0)-1]}\,, \tag{2}$$

where ξ_i is the signature factor given by

$$\xi_i=-(1+\Upsilon_i e^{-i\pi\alpha_i})/\sin\pi\alpha_i\,, \tag{3}$$

and Υ_i is $+$ or $-$ depending on whether one has even or odd signature. For antiparticle cross sections the odd-signature trajectories such as ρ and ω occur with opposite sign. $\bar{\gamma}_i(t)$ is the reduced residue times certain kinematical factors,[9] and m_0 is the scaling factor usually taken as M, the nucleon mass. The leading pole is assumed to be the Pomeranchuk pole with $\alpha_P(0)=1$ and even signature. All other poles such as P', ω, ρ, ϕ, etc., have $\alpha_i(0)<1$. In the physical region of the s channel, $\alpha'(t)$ is positive, t is negative, and $\bar{\gamma}(t)$ is real. Separating out the Pomeranchuk part we have for small $-t$ and $m_0=M$,

$$d\sigma/dt=|\xi_P\bar{\gamma}_P(0)\exp[t\alpha'_P(0)\ln(s/2M^2)] +\textstyle\sum_i\xi_i\bar{\gamma}_i(t)g_i(s)\exp[t\alpha'_i(0)\ln(s/2M^2)]|^2\,, \tag{4}$$

$$\sigma_T=(4\sqrt{\pi})[\gamma_P(0)+\textstyle\sum_i \Upsilon_i\bar{\gamma}_i(0)g_i(s)]\,, \tag{5}$$

where $g_i(s)=\exp[(\alpha_i(0)-1)\ln(s/2M^2)]$.

If one were to neglect the second term in (4) then the first term by itself would give an identical behavior as a function of s for the different reactions. In particular, a plot of $\ln(d\sigma/dt)$ versus $\ln(s/2M^2)$ for fixed t would give the same slope, namely $2\alpha_P'(0)t$. In other words, the amount of shrinkage would be the same for all the different reactions, a result in contradiction with the experimental data. If, in addition, one approximates $\bar{\gamma}_P(t)$ by a constant in the diffraction region, then one obtains an identical behavior as a function of t as well. In particular, this would mean that the $(\text{width})^{-1}$ of the diffraction pattern has the magnitude $2\alpha_P'(0)\times\ln(s/2M^2)$ and is the same for all the different reactions, a result again in contradiction with the experiments. Thus if the pole hypothesis is to succeed then the second term in (4) must play an important role. Furthermore, the $\bar{\gamma}$'s must have a sensitive t dependence and must play a crucial part in determining the diffraction width. These points were first noted by Desai[10] in connection with the $\pi^\pm p$ and pp data. The pp data were fitted in terms of P, P', and ω, while the πp data were fitted in terms of P alone. For πp scattering, the fits to $\ln(d\sigma/dt)$ versus $\ln(s/2M^2)$ by Foley *et al.*[3] on the basis of P had shown that $\alpha_P'(0)$ should be less than 0.2 $(\text{BeV}/c)^{-2}$. With this as the upper limit it was found that the exponent in the first term of (4) was negligible in the energy range of 10–20 BeV and for $-t$ up to 0.5 $(\text{BeV}/c)^2$. Therefore, to explain the exponential falloff for small $-t$, the function $\bar{\gamma}_P(t)$ was approximated by $e^{t/a}$ instead of a constant. Thus by the assumption of a small slope $\alpha_P'(0)$ and a sharp falloff for $\bar{\gamma}_P(t)$, the absence of shrinkage as well as the exponential falloff in πp was explained. For the P pole, a should roughly be proportional to the width of the diffraction pattern. Unlike $\alpha_P(t)$, which is the same for different reactions, $\bar{\gamma}_P(t)$ would be different for different reactions and would, therefore, partially explain the differences in the widths. The presence of shrinkage in pp was then shown to be due to the strong energy dependence of the second term in (4). It is known that both $g_{P'}(s)$ and $g_\omega(s)$ are proportional to the difference, $\sigma_{\bar{p}p}-\sigma_{pp}$, of the total cross sections.[6,7] This difference is found experimentally to be quite large at 10 BeV but it decreases very rapidly[11] as a function of energy in the region of 10–20 BeV. For the residues of P' and ω, the function $\bar{\gamma}_{P'}(t)$ was taken to be equal to $\bar{\gamma}_\omega(t)$. At $t=0$ this assumption automatically guarantees[7,8] the constancy of σ_{pp} above 10 BeV. For small negative t, the $\bar{\gamma}$'s were approximated by $e^{t/b}$. With an appropriate value of b, a good fit to the experimental data was obtained. It was further shown in Ref. 10 that because $(\sigma_{K^-p}-\sigma_{K^+p})$ is not as strongly energy-dependent, the

[5] From now on we shall take BeV as the unit of mass.
[6] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[7] B. M. Udgaonkar, Phys. Rev. Letters **8**, 142 (1962); F. Hadjioannou, R. J. N. Phillips, and W. Rarita, *ibid.* **9**, 183 (1962).
[8] S. D. Drell, *Proceedings of the International Conference on High-Energy Nuclear Physics, Geneva, 1962* (CERN Scientific Information Service, Geneva, Switzerland, 1962), p. 897.
[9] If β is the residue and if we denote $(M^2/\nu)^\alpha\beta$ by γ, where M is the nucleon mass, then

$$\bar{\gamma}=\rho(2^{\alpha-1}\pi^{1/2}(2\alpha+1)\Gamma(\tfrac{1}{2}+\alpha)/\Gamma(1+\alpha))\gamma,$$

where ρ is the appropriate crossing matrix.

[10] B. R. Desai, Phys. Rev. Letters **11**, 59 (1963).
[11] S. J. Lindenbaum, W. A. Love, J. A. Niederer, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **7**, 185 (1961).

shrinkage in K^+p should be less than in pp, a prediction confirmed by recent data.[4] Subsequently, two further calculations were done, essentially along the same line as Ref. 10, one for $\pi^\pm p$ by Ahmadzadeh and Sakmar[12] and the other for pp-$\bar{p}p$ by Rarita and Teplitz.[13] In both works an exponential form for the $\bar{\gamma}$'s was taken but the value of $\alpha_P'(0)$, was not fixed *a priori* on the basis of the analysis of Foley *et al.*[3] Ahmadzadeh and Sakmar added the contribution of P' in πp scattering to that of Ref. 10, the magnitude of P' already being known from the work of Igi.[14] For the magnitude of $\alpha_P'(0)$ they took the result of their earlier calculation based on the dispersion relations satisfied by α.[15] They made a subtraction at infinity, however, and took $\alpha(\infty)$ to be -1, a result true only for a single Yukawa potential. It is known that for a general superposition of Yukawa potentials, the trajectory end point can be arbitrarily close to $l=-\frac{1}{2}$,[16,17] and in the relativistic case it may even be to the right of $l=-\frac{1}{2}$.[18] Therefore, it is quite conceivable that their results would change if a more realistic value of $\alpha(\infty)$ were taken. Rarita and Teplitz took the three-pole P, P', and ω approximation in their fits to the pp-$\bar{p}p$ data. They noted that since the ω term occurs with opposite signs in particle and antiparticle scattering, the sign change in $(d\sigma_{\bar{p}p}/dt-d\sigma_{pp}/dt)$ for negative t, mentioned earlier, can be explained if $\bar{\gamma}_\omega(t)$ has a form $(1+ge^{a_1t})-ge^{a_2t}$ with $a_2<a_1$, rather than a single exponential. This form would enable $\bar{\gamma}_\omega(t)$ to change sign for negative t.

The following considerations prompt us to make a further phenomenological fit to the elastic-scattering data.

(i) Since the work of Refs. 10, 12, and 13, considerably more data have become available at very small momentum transfers.[4] Previously, the experiments[1–3] were, in general, confined to $-t>0.2$ $(\text{BeV}/c)^2$. The diffraction widths, however, are supposed to be about 0.1 $(\text{BeV}/c)^2$ $(\sim 4m_\pi^2)$, and therefore the previous fits were made to the data outside the diffraction region. Since the pole hypothesis is supposed to work best for small $-t$, the new data within the diffraction region should determine its accuracy better.

(ii) Because of the good statistics that is now available, we have confined our fits to the data with $-t<0.5$ $(\text{BeV}/c)^2$. This way we can with good confidence (a) make a linear approximation to $\alpha(t)$, (b) avoid the difficulty of encountering helicity-flip amplitudes which are likely to be important for $-t$ of the order of nucleon mass $(-t\approx 1)$, and (c) neglect the cuts which may become important at high momentum transfers.[19]

(iii) In addition to the $\pi^\pm p$, pp, and $\bar{p}p$ data we also include the recently published $K^\pm p$ data.[4] This should give a more conclusive test of the pole hypothesis and pin down the Regge parameters more accurately.

(iv) It was pointed out in Ref. 20 that instead of a pure exponential approximation $e^{t/a}$ for the $\bar{\gamma}$'s, a much better approximation (valid for negative t) would be $(1-bt)^{-\epsilon}$. Since $\bar{\gamma}$ is supposed to satisfy simple analyticity properties, it would not be correct to approximate it by a function that has an essential singularity. On the other hand, $(1-bt)^{-\epsilon}$ is in the spirit of the usual pole approximation, with $t=b^{-1}$ being roughly the position where the spectral function of $\bar{\gamma}(t)$ is peaked.[21] It behaves as $e^{\epsilon bt}$ for small $-t$, while for large $-t$ it falls off as $(-t)^{-\epsilon}$. Serber has pointed out that for fixed s, the quantity $d\sigma/dt$ for pp scattering behaves as $|t|^{-5}$ for large $-t$.[22] Since $d\sigma/dt$ is proportional to $\bar{\gamma}^2$, and $\alpha(t)$ for large $-t$ presumably goes to a constant, the value of ϵ should be ≈ 2.5. With this as the estimate of ϵ, and b about 2, the expression $(1-bt)^{-\epsilon}$ would give the characteristic exponential falloff with a width of the right order of magnitude.[23]

(v) In Ref. 13 it was remarked that the sign change in the ω residue is hard to understand in view of the fact that it satisfies dispersion relations with only a right-hand cut. In the following paper it is shown, on the basis of potential theory, that if there is a strong short-range repulsion in addition to a long-range attraction, then the residues can change sign.[17] The zero in the residue would occur when the trajectory is in the left half-plane, at least for the spin-zero (single- or multichannel) nonrelativistic case.[17] Thus in analogy to potential scattering we assume that $\bar{\gamma}$ has a simple zero for negative t. In the relativistic case which we are considering, however, we leave open the question whether the sign change occurs when the ω trajectory is in the right or the left half-plane.

We shall fit the $K^\pm p$, pp, and $\bar{p}p$ data in terms of P, P', and ω. There are a few poles which we have not included, such as ρ, ϕ, and the recently discovered B, A, and X^0 mesons.[24] The contribution of ρ is known to be small from other considerations.[8] The spins of some of the other resonances are not yet determined, while the remaining ones are known to have spin less than or

[12] A. Amadzadehh and I. A. Sakmar, Phys. Rev. Letters **11**, 439 (1963).
[13] W. Rarita and V. L. Teplitz, Phys. Rev. Letters **12**, 206 (1964).
[14] K. Igi, Phys. Rev. **130**, 820 (1963).
[15] A. Ahmadzadeh and I. A. Sakmar, Phys. Letters **5**, 145 (1963). The value of α' was found to be 0.34.
[16] R. G. Newton, J. Math. Phys. **3**, 867 (1962).
[17] See B. R. Desai, following paper, Phys. Rev. **138**, B1174 (1965).
[18] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).
[19] S. Mandelstam, Nuovo Cimento **30**, 1127 and 1148 (1963).
[20] B. R. Desai, Phys. Rev. **135**, B180 (1964).
[21] $\bar{\gamma}$ has only the right-hand cut.
[22] R. Serber, Phys. Rev. Letters **10**, 357 (1963).
[23] We do not consider the form, in terms of transverse momentum, suggested by D. S. Narayan and K. V. L. Sarma, Phys. Letters **5**, 365 (1963), and J. Orear, Phys. Rev. Letters **12**, 112 (1964) since they correspond to an essential singularity in the scattering amplitude.
[24] For a summary of the resonances see A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Roos, Rev. Mod. Phys. **36**, 977 (1964). On the effect of trajectories other than P,P', and ω see A. Pignotti, Phys. Rev. **134**, B630 (1964), and A. Ahmadzadeh, *ibid.* **134**, B633 (1964).

equal to the spin of ω. However, their masses are larger than the ω mass. Therefore, it is reasonable to expect that their intercepts at $t=0$ are smaller than the ω intercept and consequently for large energies their contribution in (4) is expected to be small. One can, in principle, include all these poles. However, in so doing one is faced with far too many parameters whose values would be hard to determine unambiguously from the present data. The higher the energy is, the better our three-pole approximation should work.

II. NUMERICAL RESULTS

We shall use the expression (4) for our fits to the elastic-scattering data. We shall take the conventional values

$$\alpha_P(0)=1\,,\quad \alpha_{P'}(0)=\alpha_\omega(0)=0.5$$

and

$$\alpha_P'(0)=\alpha_{P'}'(0)=\alpha_\omega'(0)=\alpha'. \tag{6}$$

The values of $\bar{\gamma}_P(0)$, $\bar{\gamma}_{P'}(0)$, and $\bar{\gamma}_\omega(0)$ are known for each case from other considerations such as the total-cross-section data.[7,8,14] For $K^\pm p$ and pp-$\bar{p}p$ we take $\bar{\gamma}_{P'}(0)=\bar{\gamma}_\omega(0)$. As mentioned earlier we shall take

$$\bar{\gamma}_i(t)=\bar{\gamma}_i(0)(1-b_it)^{-\epsilon_i},\quad i=P,\,P' \tag{7}$$

and for ω, in order to ensure a zero for negative t, we take

$$\bar{\gamma}_\omega(t)=\bar{\gamma}_\omega(0)(1+t/t_0)(1-b_\omega t)^{-\epsilon_\omega}. \tag{8}$$

The Serber result[22] means that ϵ_P and $\epsilon_{P'}$ should be about 2.5, while ϵ_ω should be about 3.5 because of the extra linear factor in (8). We shall take

$$\epsilon_P=\epsilon_{P'}=2.5\quad\text{and}\quad\epsilon_\omega=3.5. \tag{9}$$

For $\pi^\pm p$ we have three parameters α', b_P, and $b_{P'}$. For $K^\pm p$ and pp-$\bar{p}p$ we have five parameters α', b_P, $b_{P'}$, b_ω, and t_0. Among these parameters α', of course, must be the same in each case. From factorization theorem[25]

$$\bar{\gamma}_{KN}{}^2=\bar{\gamma}_{KK}\bar{\gamma}_{NN}$$

it is clear that $\bar{\gamma}_\omega$ for KN inherits the zeros of KK and NN channels. Thus one should expect the t_0's to be the same for $K^\pm p$ and pp-$\bar{p}p$. For the purpose of the present fits, however, we shall not take the t_0 for pp-$\bar{p}p$ and $K^\pm p$ to be the same. For each particle-antiparticle set we shall keep both α' and t_0 arbitrary. A measure of

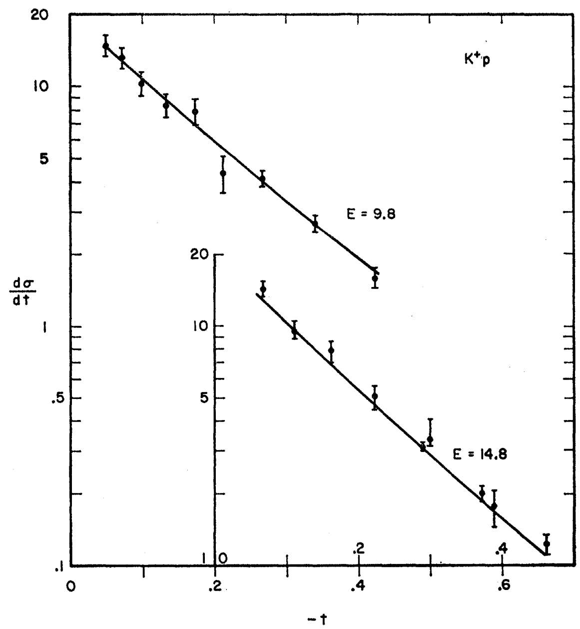

FIG. 1. Fits to the K^+p data at lab energies of 9.8 and 14.8 BeV. Differential cross section $d\sigma/dt$ in mb/(BeV/c)^2 is plotted versus momentum transfer squared $-t$ in (BeV/c)^2 up to $-t$ about 0.5. The data are from Ref. 4. The two plots are separated in order to avoid considerable overlap.

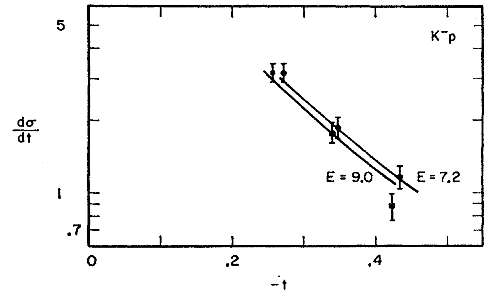

FIG. 2. Fits to the K^-p data at lab energies of 7.2 and 9.0 BeV. Differential cross section $d\sigma/dt$ in mb/(BeV/c)^2 is plotted versus momentum transfer squared $-t$ in (BeV/c)^2 up to $-t$ about 0.5. The data are from Ref. 4.

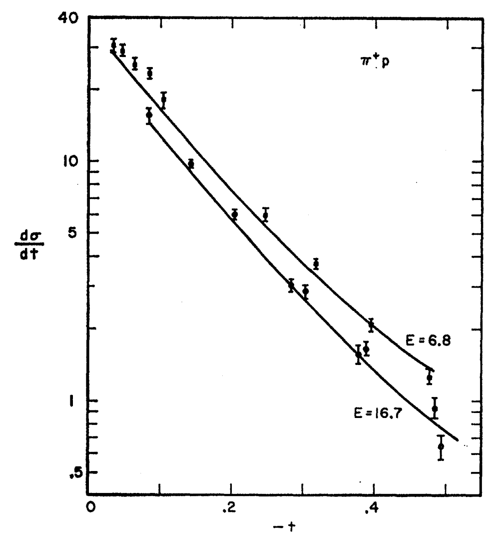

FIG. 3. Fits to the π^+p data at lab energies of 6.8 and 16.7 BeV. Differential cross section $d\sigma/dt$ in mb/(BeV/c)^2 is plotted versus momentum transfer squared $-t$ in (BeV/c)^2 up to $-t$ about 0.5. The data are from Ref. 4.

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[25] M. Gell-Mann, Phys. Rev. Letters **8**, 263 (1962); V. N. Gribov and I. Pomeranchuk, *ibid.* **8**, 343 (1962).

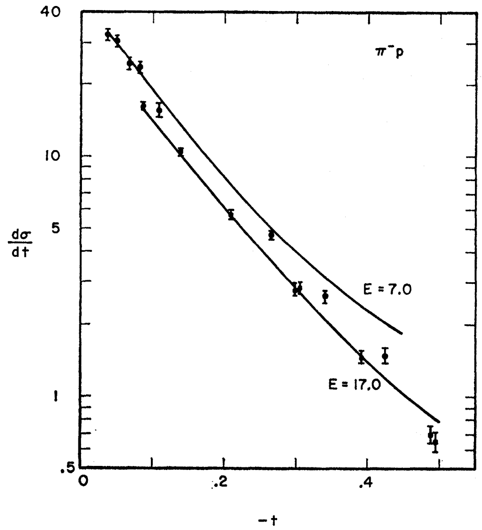

FIG. 4. Fits to the π^-p data at lab energies of 7.0 and 17.0 BeV. Differential cross section $d\sigma/dt$ in mb/(BeV/c)^2 is plotted versus momentum transfer squared $-t$ in (BeV/c)^2 up to $-t$ about 0.5. The data are from Ref. 4. The 18.9-BeV data are not plotted because they do not have points below $-t=0.1$.

good fit would be that α' and t_0 obtained from different sets are essentially the same.

In Figs. 1 to 6 are shown the fits to the elastic-scattering data. Our fits to K^+p, K^-p, and pp are quite good. The fits to $\pi^\pm p$ become better as the energy is increased. In the $\bar{p}p$ case it should be noted that, unlike the pp and $\pi^\pm p$ cases, data are available for energies less than 12 BeV. For these energies it is very possible that the three-pole approximation is inadequate. It should be further noted that, unlike the pp case where the pole

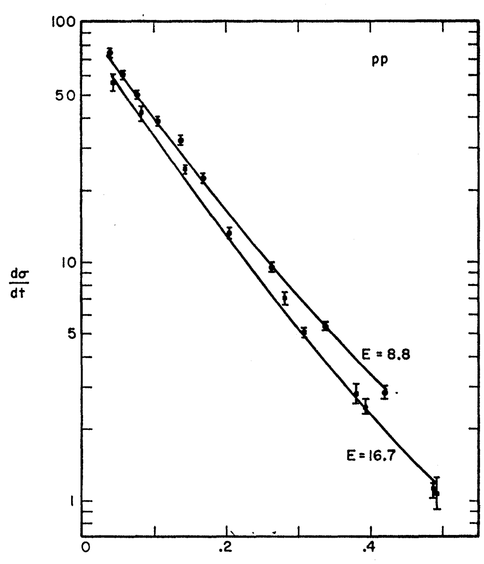

FIG. 5. Fits to the pp data at lab energies of 8.8 and 16.7 BeV. Differential cross section $d\sigma/dt$ in mb/(BeV/c)^2 is plotted versus momentum transfer squared $-t$ in (BeV/c)^2 up to $-t$ about 0.5. The data are from Ref. 4. The 19.6-BeV data are not plotted because they do not have points below $-t=0.1$.

TABLE I. The Regge parameters which give the best fit to the $K^\pm p$, $\pi^\pm p$ and pp-$\bar{p}p$ data of Ref. 4. The $\pi\pi$ parameters are obtained from the factorization theorem. α' and the b's are in units of (BeV/c)$^{-2}$, while t_0 is in units of (BeV/c)^2.

Exp	α'	b_P	$b_{P'}$	b_ω	t_0
$K^\pm p$	0.41	1.10	3.76	2.71	0.061
$\pi^\pm p$	0.41	1.39	2.84	···	···
pp-$\bar{p}p$	0.41	2.27	3.71	3.24	0.074
$\pi\pi$	0.41	0.51	1.97	···	···

contributions to the imaginary part of the amplitude can be of opposite sign, in the $\bar{p}p$ case they all occur with positive sign, and since $\bar{\gamma}_P(0)$ is quite large, an addition of another pole can give rise to a strong interference term with P and change the result significantly. Since the imaginary part of the amplitude is supposed to dominate for small $-t$, a similar effect will occur in $d\sigma/dt$.

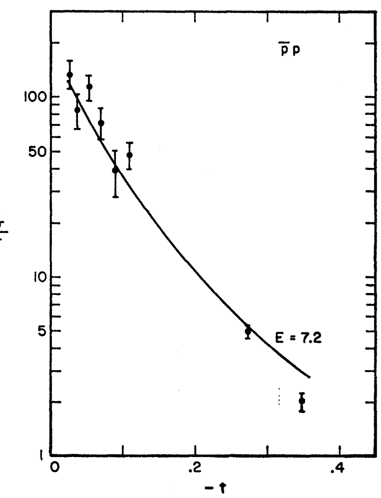

FIG. 6. Fits to the $\bar{p}p$ data at lab energy of 7.2 BeV. Differential cross section $d\sigma/dt$ in mb/(BeV/c)^2 is plotted versus momentum transfer squared $-t$ in (BeV/c)^2 up to $-t$ about 0.5. At higher energies the fits become poor. The data, from Ref. 4, are available only up to 12 BeV.

In Table I are given, for each set, the values of the Regge parameters which give rise to the least χ^2. Interestingly enough the best value we obtain for α' is the *same* in each set, namely 0.41 (BeV/c)$^{-2}$. The values of t_0 for $K^\pm p$ and pp-$\bar{p}p$ are 0.061 and 0.074 (BeV/c)^2 and are not too different. The behavior of the ω and P' terms relative to the P term is roughly the same in each set. For fixed s, both ω and P' fall off faster than P as $-t$ is increased. The logarithmic derivative of $\bar{\gamma}_P(t)$ at $t=0$ increases with the value of the total cross section. In Table I we have given the values of the $\pi\pi$ parameters obtained from the factorization theorem.[25] We have also fitted the data with α' and t_0 fixed *a priori*. The values of the Regge parameters are given in Table II.

The value 0.41 we have obtained for α' is comparable to some of the recent theoretical estimates.[15,26–28] It is

[26] H. Bransden, P. G. Burke, J. W. Mofat, R. G. Moorhouse, and D. Morgan, Nuovo Cimento **20**, 206 (1963). The value of α' was found to be 0.30.

[27] H. Cheng and D. Sharp, Phys. Rev. **132**, 1854 (1963) obtained $\alpha' \sim 0.1$. Here the effect of inelastic scattering was not included.

[28] K. Igi, Phys. Rev. **136**, B773 (1964). The value of α' was between 0.91 and 0.67 depending on the choice of the strip width.

TABLE II. The values of the Regge parameters obtained when α' and t_0 are fixed at values different from those in Table I. These parameters give rise to larger χ^2. The fits become poorer as α' is decreased and as t_0 is increased. α' and the b's are in units of $(\text{BeV}/c)^{-2}$ and t_0 is in units of $(\text{BeV}/c)^2$.

Exp	α'	b_P	$b_{P'}$	b_ω	t_0
$K^\pm p$	0.41	1.13	2.69	1.10	0.15
	0.20	1.40	2.65	3.14	0.061
	0.20	1.40	2.26	1.44	0.15
	0	1.75	1.86	3.15	0.061
	0	1.71	1.96	1.88	0.15
$\pi^\pm p$	0.20	1.72	2.56	⋯	⋯
	0	2.09	2.38	⋯	⋯
pp-$\bar{p}p$	0.41	2.20	3.27	2.22	0.15
	0.20	2.69	3.14	3.76	0.075
	0.20	2.59	2.94	2.84	0.15
	0	3.12	2.87	4.52	0.075
	0	3.00	2.78	3.85	0.15

a factor of two larger than the upper limit of Foley *et al.* based on $\pi^\pm p$ scattering with P alone,[3] while it is about a factor of two smaller than the earlier estimate of Chew and Frautschi.[6] It roughly corresponds to having the square of the radius of interaction equal to $m_\rho^2/4$, where m_ρ is the mass of the ρ meson. For the P trajectory the behavior of $\alpha_P(t)$ may be determined following the prescription of Ref. 15 but with a subtraction at $t=0$ $[\alpha_P(0)=1]$,

$$\alpha_p(t)=1+\frac{t}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dt'}{t'(t'-t)}\,\text{Im}\alpha_p(t')\,.$$

Instead of $\alpha(\infty)$, the value of α' we have obtained can be used to determine one of the parameters in the expression for Imα and thus avoid the possible critical dependence on $\alpha(\infty)$.

As mentioned earlier we have assumed the slopes of all the trajectories P, P', and ω to be the same. This was done primarily to avoid having too many parameters. Experience with potential scattering also shows that as long as the threshold value of α is $\geqslant\frac{1}{2}$, the slopes are not too different. In order to determine the slopes more accurately one must await the experiments at higher energies on both the total and differential cross sections.

Finally, let us compare our results with those of Refs. 12 and 13. Recall that

$$d\ln\bar{\gamma}_i(t)/dt\,|_{t=0}=c_ib_i\,,\quad i=P,\,P',\,\omega\,,$$

where $c_i=2.5$ for $i=P,P'$, and $=3.5$ for $i=\omega$. If we denote by $\bar{\Gamma}_i$ the total $(\text{width})^{-1}$ contributed by pole i in (1), then

$$\bar{\Gamma}_i=2\left[\frac{d\ln\bar{\gamma}_i(t)}{dt}\bigg|_{t=0}+\alpha'\ln\left(\frac{s}{2M^2}\right)\right].$$

The values of b_P and $b_{P'}$ for $\pi^\pm p$ obtained in Ref. 12 are 0.81 and 1.43 $(\text{BeV}/c)^{-2}$, respectively. In Ref. 13 for pp-$\bar{p}p$ the values of b_P, $b_{P'}$, and b_ω are found to be 1.46, 3.36, and 3.37 $(\text{BeV}/c)^{-2}$, respectively. The above values are generally smaller than the results given in Table I, presumably because the fits in Refs. 12 and 13 were confined to $-t\gtrsim 0.2$ $(\text{BeV}/c)^2$ where $d\sigma/dt$ does not fall off as sharply as it does for $-t<0.2$.

III. AN ESTIMATE OF THE CUT CONTRIBUTION

In the following we give a crude estimate of the cut contribution.[19,29,30] A typical cut term obtained from n iterations of the P pole is[31]

$$\frac{\gamma_c^{(n)}(t)(s/s_0)^{[\alpha_c^n(t)-1]}}{[\ln(s/s_0)]^n}\,,$$

where $\alpha_c^{(n)}(0)=\alpha_P(0)=1$, $0<\alpha_c'^{(n)}(0)<\alpha_P'(0)$. Presumably $\gamma_c^{(n)}(t)$ are bounded so that for all n and for all negative t, $|\gamma_c^{(n)}(t)|<\gamma_c$, a fixed number. As an extreme case we take $\alpha_c^{(n)\prime}(0)=0$ and $\gamma_c^{(n)}(t)=\gamma_c$, for all n. Then the sum of the above terms from $n=1$ to ∞ is

$$\frac{\gamma_c}{\ln(s/s_0)-1}\,. \tag{10}$$

At some negative t, (10) will dominate the P term,

$$\bar{\gamma}_P(t)(s/s_0)^{(\alpha_P(t)-1)}\,. \tag{11}$$

Now consider πp scattering and take $s_0=2M^2$ as before, so that $s/s_0=E$, the lab momentum in BeV/c. In place of P' consider the contribution (10). For sufficiently high energy, the P term will dominate as far as the total cross section is concerned $(t=0)$. The experimental data on the πp total cross section indicates that at $E=10$ BeV, the term (10) is 20% of (11) at $t=0$ and, therefore, $\gamma_c/\bar{\gamma}_P(0)$ is about $\frac{1}{4}$. The value of $-t$ at which (10) and (11) become comparable is given by

$$-t_c=(1/a+\alpha'\ln E)\ln[\bar{\gamma}_P(0)(\ln E-1)/\gamma_c]\,, \tag{12}$$

where a is the logarithmic derivative of $\bar{\gamma}_P(t)$ at $t=0$. In the previous section the values of a and α' were found to be about 5 and 0.4 $(\text{BeV}/c)^{-2}$, respectively. In the region of E about 10 to 20 BeV the value of $-t_c$ then turns out to be about 0.3, larger than the width of the diffraction pattern. The cut contribution (10), of course, is highly overestimated. It would not be surprising, therefore, if a more correct estimate showed that the cut dominates well beyond the region, $0\leqslant -t\leqslant 0.5$, in which we are interested. Note also that as $E\to\infty$,

$$-t_c\approx(1/\alpha'\ln E)\ln(\ln E)\,, \tag{13}$$

and therefore $-t_c$ approaches zero very slowly.

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[29] Here we take the specific form of the cut contribution suggested by D. Amati, S. Fubini, and A. Staghelini, Phys. Letters 1, 29 (1962). Even though Mandelstam has shown that such cuts do not exist, the position of the branch points of the new cuts (see Ref. 19) is unchanged.

[30] We are grateful to K. Igi for discussions on this subject.

[31] We consider only the contribution of the imaginary part for the purpose of the present crude calculation.

Recently Gribov *et al.*[32] have claimed that when proper account is taken of the existence of cuts and multiparticle unitarity, then the amplitude has the form

$$[i\phi_0(\xi\Upsilon)-\tfrac{1}{2}\pi\Upsilon\phi_0'(\xi\Upsilon)]+\Upsilon\phi_1(\xi\Upsilon),$$

where $\xi=\ln E$, $\Upsilon=-t\alpha'$, and ϕ_0 is a universal function of $\xi\Upsilon$ but ϕ_1, in general, is not. Accordingly when $\xi\gg1$, $\Upsilon<1$ and $\xi\Upsilon\sim1$, the amplitudes of all the reactions have the same behavior. If one considers the region above $E=10$ BeV to be essentially asymptotic, then experiments have now been done which also satisfy the criterion $\Upsilon<1$ and $\xi\Upsilon\sim1$.[3,4] It turns out that the experimental results do not agree with the prediction of Gribov *et al.* The behavior of the different amplitudes is not universal in character.

As pointed out in the beginning of Sec. I, three facts, namely, (i) a strong difference in the shrinkage properties (e.g., between πp and pp), (ii) a strong difference between the diffraction widths (e.g., between K^+p and $\bar{p}p$), and (iii) a sharp exponential falloff shared by all, are hard to understand within the framework of a universal function. It is quite conceivable that the dominance of the cut contribution near $-t=0$, as well as a universal character of all the amplitudes, can be achieved if E is large enough so that [see expression (12)]

$$\alpha'\ln E>a.$$

On the basis of our estimate of a ($\sim$5.0) and α' ($\sim$0.4), this would be above $E=10^6$ BeV, well beyond the highest energies of present-day accelerators.

One way to determine whether the cuts dominate or not would be to proceed as we have done, namely, to assume only the poles and see whether consistent fits are obtained. Our results, as given in Sec. II, show that there is no reason to abandon the pole approximation at least up to $E=20$ BeV, and $-t<0.5$ (BeV/c)^2.

IV. CONCLUSIONS

Our results indicate that if the pole hypothesis is adequate then the Regge parameters must satisfy the following properties:

(i) The slope α' must be small, about 0.41 (BeV/c)$^{-2}$.

(ii) More than one pole should play an important role, at least in the energy region up to 20 BeV. The shrinkage or absence of it can be understood if one considers P' and ω in addition to P.

(iii) The P' and ω terms must fall off faster than the P term as $-t$ increases for fixed s.

(iv) The sharp exponential falloff of $d\sigma/dt$ should come mainly from the reduced residue $\bar{\gamma}_P(t)$. The logarithmic derivative of $\bar{\gamma}_P(t)$ at $t=0$ must be larger the larger the value of $\gamma_P(0)$, the total cross section.

(v) The quantities $(d\sigma_{\bar{p}p}/dt-d\sigma_{pp}/dt)$ and $(d\sigma_{K^-p}/dt-d\sigma_{K^+p}/dt)$ change sign at $-t$ about 0.07 (BeV/c)^2. Since $\bar{\gamma}_\omega(t)$ occurs with opposite signs, positive for antiparticle and negative for particle cross sections, it should have a zero at the same point.

The property (i) is quite reasonable. Our value is in good agreement with some of the recent theoretical estimates. The fact that more than one pole must play an important role is not surprising. It is known that in fitting the total cross section [$\sim(d\sigma/dt)^{1/2}$ at $t=0$, essentially], P' and ω make an important contribution. Therefore, for $t\lesssim0$ they should continue to play a useful role. Property (iii) does not seem in contradiction with the results of potential theory. In the following paper properties (iv) and (v) are discussed.[17]

Note added in proof. A number of calculations have recently been done to obtain partial wave amplitudes for a given channel, e.g., the s channel, from the knowledge of the Born terms corresponding to the resonances in the crossed-channel, e.g., the t channel. These Born terms, in general, do not have zeros as a function of t. Now for the case of the ω pole we have seen that one is forced to have a zero in the residue for negative t. It is, therefore, of interest to know the effect of these zeroes in the above calculations. For negative t, a single pole contribution is given by $\beta/(t_r-t)$ which near a zero at $t=t_0(<0)$ can be written as

$$\frac{\beta(t)}{t_r-t}=\frac{(t_0-t)C(t_0)}{t_r-t}=C(t_0)+\frac{d(t_0)}{t_r-t},$$

where β is the residue and t_r is the position of a resonance. Thus we see the effect would be only to change the S wave amplitude in the s channel. In the calculations mentioned above, the S waves are normally always substracted out and, therefore, the presence of zeroes will not alter their results.

ACKNOWLEDGMENTS

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[32] V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martiorosyan (to be published).