Selection Rules for Parafields and the Absence of Para Particles in Nature

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(Received 7 March 1963; revised manuscript received 2 September 1964)

Green's parafield quantization is reviewed. It is shown, both for a single field and for sets of fields, that all Fock-like representations of Green's trilinear commutation rules are realized by Green's ansatz with anticommuting (commuting) Bose (Fermi) component fields for para-Bose (para-Fermi) fields. Restrictions on the form of the interaction Hamiltonian density $H_I(x)$ are derived from the requirement that $H_I(x)$ be a paralocal operator. From these restrictions on H_I, selection rules on the S matrix are proved to all orders of perturbation theory. The most important such rule prohibits all reactions in which the total number of para particles of order $p>1$ in the initial and final states is one. This last selection rule, together with experimental information, leads to the conclusion that no presently known particle can be para.

1. INTRODUCTION AND RESULTS

Résumé of Earlier Paper

In an earlier paper,[1] we studied the symmetrization postulate (SP) that states of more than one identical particle must be either symmetric or antisymmetric under permutations, from both the theoretical and experimental points of view, within the framework of particle quantum mechanics. We did not find any *a priori* argument against the existence of particles other than bosons and fermions in nature. However, we did find that there are absolute selection rules concerning the reactions of such particles. Our argument, based on the precise formulation of the requirement of the indistinguishability of identical particles in quantum mechanics together with certain other assumptions[2] led to the conclusion that if particles other than bosons and fermions could be produced, they would not be produced in any experiment which at present can be carried out. From this point of view, present tests of SP appear really as tests of quantum mechanics, and experiments of a new kind, involving two or more of the unstable particles whose statistics is being questioned in the initial state, would lead to new information. We also made a direct phenomenological analysis of particle-physics experiments, carried out with as few theoretical assumptions as possible, to see if experiments rule out the possibility of statistics other than Bose or Fermi for each particle separately. The Pauli principle for electrons and the rotational spectra of homonuclear diatomic molecules (among other arguments) establish that the electron and nucleon are fermions. Blackbody radiation and the quantitative success of quantum electrodynamics show that photons are bosons. The suppression of the decays $K_2{}^0 \rightarrow 2\pi$ and $K^+ \rightarrow 2\pi$ relative to $K_1{}^0 \rightarrow 2\pi$ give evidence that pions are bosons. We did not find direct evidence for the statistics of K, Λ, Σ, Ξ, or μ. We proposed feasible tests for the statistics of K and of those hyperons which have an asymmetric decay.

Conclusions of Present Paper

In this paper, we study parafields,[3] which are certain kinds of second quantized fields for which the particles are neither bosons nor fermions, together with Bose and Fermi fields. Our main interests are to find the selection rules for para and ordinary particles which follow from the requirement that the interaction Hamiltonian density be a paralocal operator, i.e., commute with itself at space-like separation in the Hilbert space, called $\mathfrak{B}$ in Sec. 2, of the Green component fields, and to find the implications of these rules, together with with experimental evidence, for the existence of para particles in nature. The selection rules which we find are absolute selection rules. These rules are: (1) the total number of Fermi and para-Fermi particles (counting both particles and antiparticles as positive) on both sides of a reaction must be even, i.e., the total number of such particles is conserved, modulo 2; (2) for para particles of each even order p, the total number on both sides of a reaction must be even; and, (3) for each odd order p, the total number on both sides of a reaction can be any even number or any odd number $\geq p$. From (2) and (3) the important conclusion follows that for each order p at least two para particles must enter into every reaction. Finally, from this result, together with

* Supported in part by the U. S. Air Force Office of Scientific Research under Contract AFOSR 500-64 and by the National Science Foundation under Contract GP 3221.

[1] A. M. L. Messiah and O. W. Greenberg, Phys. Rev. **136**, B248 (1964).

[2] The other assumptions are time-reversal or TCP invariance of the laws of motion, and full coherence of each superselecting sector $\mathfrak{F}_{QBL}{}^x$, where $\mathfrak{F}^x$ is the subspace of states in Fock space which have at most one particle whose permutation character might be questioned, and the subscripts Q, B, and L specify a sector with definite fixed values of the electric, baryonic, and leptonic charges.

[3] H. S. Green, Phys. Rev. **90**, 270 (1953).

experimental information, we conclude that no presently known particles are para.

Content of Present Paper

Section 2 starts with a review of Green's work[3] on parafields, and then proves that all representations of Green's trilinear commutation rules which are analogous to the Fock representation of the Bose and Fermi commutation rules are realized by Green's ansatz. This proof is given both for a single field and for sets of fields with ordinary or para relative commutation rules, and with any assortment of "normal" or "anomalous" rules between each pair of fields. The importance of this demonstration is that it allows the Green component fields to be used in the analysis of interactions containing parafields. Section 3 derives the selection rules following from the paralocality of $H_I(x)$. As we point out in Sec. 3, we do not know whether or not locality of $H_I(x)$ implies paralocality. If not, then our selection rules might be too restrictive; however, it is possible that our selection rules could hold even if paralocality of $H_I(x)$ is not implied by locality. The analysis is valid both for the "normal" case and for the most general "anomalous" case. First the restrictions on H_I itself are found, and then selection rules for the S matrix are proved to all orders of perturbation theory. Illustrative examples are given for all allowed interactions. We also give explicit counterexamples to show that the modulo-two selection rule stated earlier under the name "conservation of statistics" by Kamefuchi and Strathdee[4] is incorrect. In Sec. 4, we draw the conclusion that no presently known particle can be para.

2. PARAFIELD QUANTIZATION

In this section, we study Green's theory of parafield quantization.[3–5] This theory generalizes the usual method of second quantization, and allows particles not obeying the symmetrization postulate. In general, trilinear commutation relations replace the usual bilinear ones for the parafields; however, Bose and Fermi quantization still occur as special cases. We first review Green's work, and then prove that Green found all the solutions of his trilinear commutation relations which correspond to the Fock representations of the Bose and Fermi commutation relations. This discussion considers both the para commutation relations of a field with itself and the relative para commutation relations of fields with other fields. We conclude this section by indicating how to carry out para quantization in a manner consistent with internal symmetries.

(A) Review of Green's Work

Green[3] studied a quantum theory of free particles not obeying the symmetrization postulate, and chose as his basic requirement that the free Hamiltonian, i.e., the generator of time translations for freely moving particles, be bilinear and properly symmetrized in annihilation and creation operators. Green's requirement is

$$[H_0,a_l^\dagger]_- = \omega_l a_l^\dagger, \tag{1}$$

where

$$H_0 = \tfrac{1}{2}\textstyle\sum_k \omega_k [a_k^\dagger, a_k]_\pm \tag{2}$$

is the free Hamiltonian, a_k and $a_k^\dagger$ are the annihilation and creation operators, ω_k is the energy of a free particle in quantum state k, and H_0 is symmetrized or antisymmetrized for parabosons (upper sign) or parafermions (lower sign), respectively. Green replaced Eqs. (1) and (2) by the stronger conditions[6]

$$[[a_k^\dagger,a_l]_\pm, a_m]_- = -2\delta_{km}a_l, \tag{3}$$

$$[[a_k,a_l]_\pm, a_m]_- = 0, \tag{4}$$

from which the relation

$$[[a_k,a_l]_\pm, a_m^\dagger]_- = 2\delta_{lm}a_k \pm 2\delta_{km}a_l$$

follows by Jacobi's identity

$$[[A,B]_\pm, C]_- + [[C,A]_\pm, B]_- + [[B,C]_\pm, A]_- = 0,$$

from which still other relations follow by taking the adjoint of both sides of the above equations. From Eqs. (1) and (2) it follows that there is a commutative set of operators n_k, defined up to a constant by

$$n_k = \tfrac{1}{2}[a_k^\dagger, a_k]_\pm + \text{const},$$

where, as usual, the upper sign is for parabosons and the lower one for parafermions, which have the property of number operators

$$[n_k, a_l^\dagger]_- = \delta_{kl}a_l^\dagger.$$

Green found an infinite set of solutions, labeled by the integers $p \geq 1$, for each case. His solutions for $p=1$ are the usual Bose and Fermi operators. His solutions for $p>1$, which for reasons described below we call "para-Bose" or "para-Fermi" operators, can be exhib-

[4] S. Kamefuchi and J. Strathdee, Nucl. Phys. **42**, 166 (1963).

[5] E. P. Wigner, Phys. Rev. **77**, 711 (1950); D. V. Volkov, Zh. Eksperim. i Teor. Fiz. **36**, 15–60 (1959); **38**, 518 (1960) [English transls.: Soviet Phys.—JETP **9**, 1107 (1959); **11**, 375 (1960)]; G. F. Dell' Antonio, O. W. Greenberg, and E. C. G. Sudarshan, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by Feza Gürsey (Gordon and Breach Publishers, Inc., New York, 1964); H. Scharfstein, New York University, thesis, 1962 (unpublished); L. O'Raifeartaigh and C. Ryan, Proc. Royal Irish Acad. **62A**, 93 (1963); C. Ryan and E. C. G. Sudarshan, Nucl. Phys. **47**, 207 (1963); T. F. Jordan, N. Mukunda, and S. V. Pepper, J. Math. Phys. **4**, 1089 (1963); D. G. Boulware and S. Deser, Nuovo Cimento **30**, 230 (1963); A. Galindo and F. J. Yndurain, *ibid.* **30**, 1040 (1963); I. Bialynicki-Birula, Nucl. Phys. **49**, 605 (1963); O. W. Greenberg and A. M. L. Messiah, J. Math. Phys. **6**, 500 (1965).

[6] Bialynicki-Birula (Ref. 5), showed that Eqs. (3) and (4) follow from the requirement that Eqs. (1) and (2) be invariant under unitary transformations of the a_k's.

ited in terms of a canonical ansatz.[7] The para-Bose operator of order p can be represented by

$$a_k=\sum_{\alpha=1}^{p} b_k^{(\alpha)},$$

where for a given value of α the $b_k^{(\alpha)}$ and $b_k^{(\alpha)\dagger}$ operators obey the Bose commutation relations

$$[b_k^{(\alpha)},b_l^{(\alpha)\dagger}]_-=\delta_{kl},\quad [b_k^{(\alpha)},b_l^{(\alpha)}]_-=0,$$

and for $\alpha\neq\beta$ all the operators anticommute

$$[b_k^{(\alpha)},b_l^{(\beta)\dagger}]_+=[b_k^{(\alpha)},b_l^{(\beta)}]_+=0,\quad (\alpha\neq\beta).$$

The para-Fermi operator of order p can be represented by the same formulas, but the roles of commutation and anticommutation relations for the operators are reversed. At this point, the reader should verify that these two ansatzes satisfy Green's conditions.

Since the b and $b^\dagger$ operators have many inequivalent irreducible representations in Hilbert space, we specify a unique representation (up to unitary equivalence) by requiring the existence of a unique no-particle vector Φ_0,

$$b_k^{(\alpha)}\Phi_0=0,\quad \text{for all } k,\alpha.$$

The Hilbert space $\mathcal{B}$ on which the b and $b^\dagger$ act is the closure of vectors of the form $\mathcal{P}(b^\dagger)\Phi_0$, where $\mathcal{P}$ is an arbitrary polynomial. The representation of the a and $a^\dagger$ on $\mathcal{B}$ which is defined by Green's ansatz is reducible, even for a finite number of degrees of freedom. We will show below that there is a representation of the a and $a^\dagger$ on the Hilbert space $\mathcal{A}$ which is the closure of vectors of the form $\mathcal{P}(a^\dagger)\Phi_0$, and that this representation is irreducible. For this representation, Φ_0 is the unique no-particle vector for the a's,

$$a_k\Phi_0=0,\quad \text{for all } k,\tag{5}$$

and

$$a_ka_l^\dagger\Phi_0=p\delta_{kl}\Phi_0,\quad \text{for all } k,\ l.\tag{6}$$

Using Eqs. (5) and (6) we can complete the definition of the number operator,

$$n_k=\tfrac{1}{2}[a_k^\dagger,a_k]_\pm\mp\tfrac{1}{2}p.\tag{7}$$

The representation specified by Eqs. (5) and (6) plays the same role for the theory of parabosons and parafermions that the usual Fock representation which has a no-particle state plays for bosons or fermions.[8]

[7] Some readers may suspect that the equation for the a_k in terms of the $b_k^{(\alpha)}$ may introduce (a) some composite structure or (b) degeneracy into the description of the *individual* particles annihilated by the a_k (or created by the $a_k^\dagger$). That (a) does not occur should be clear since Green's ansatz is linear, while in contrast a compound structure would require a multiplicative relation. Neither does (b) occur since the state $a_k^\dagger\Phi_0$ is nondegenerate, in contrast to a particle with a hidden degree of freedom. We emphasize that Green's ansatz is only a mathematical device, and that the $b_k^{(\alpha)}$ and $b_k^{(\alpha)\dagger}$ by themselves have no physical significance. Later in this section we discuss parafields without using Green's ansatz.

[8] L. Garding and A. S. Wightman, Proc. Nat. Acad. Sci. U. S. **40**, 617, 622 (1954).

The N-particle states ($N>1$) have properties which justify calling the field quanta parabosons and parafermions. All the N-particle states corresponding to a given set of single quantum states $(k_1,k_2,\cdots k_n)$ can be expressed as linear combinations of the $N!$ vectors obtained by permuting the order of the creation operators in $a_1^\dagger a_2^\dagger\cdots a_N^\dagger\Phi_0$ (here $a_i^\dagger$ denotes $a_{k_i}^\dagger$ for short). Although the number of such states which are linearly independent is generally greater than one, a certain number of these linear combinations may vanish. Consider in particular the "symmetric" and "antisymmetric" states[9]

$$\Psi_N^{(s)}=\sum_Q a_{\mu_1}^\dagger a_{\mu_2}^\dagger\cdots a_{\mu_N}^\dagger\Phi_0,\tag{8}$$

$$\Psi_N^{(a)}=\sum_Q \delta_Q a_{\mu_1}^\dagger a_{\mu_2}^\dagger\cdots a_{\mu_N}^\dagger\Phi_0,\tag{9}$$

where Q is the permutation taking $(1,2,\cdots N)$ into $(\mu_1,\mu_2,\cdots\mu_N)$, δ_Q is the signature of Q, and the sum runs over all $N!$ permutations. For order p parafermions, $\Psi_N^{(s)}=0$ if $N>p$; similarly for order p parabosons, $\Psi_N^{(a)}=0$ if $N>p$. In other words, p is the maximum number of identical parafermions (parabosons), of order p, which can occur in a "symmetric" ("antisymmetric") state. In particular, there can be any number of parabosons, but at most p parafermions of order p in the same quantum state. All this is easily proved by using Green's ansatz. For example, from the fact that each b is a Fermi operator in the case of parafermions, one readily deduces

$$(a_k^\dagger)^p\Phi_0=p!b_k^{(1)\dagger}b_k^{(2)\dagger}\cdots b_k^{(p)\dagger}\Phi_0;$$

hence, $(a_k^\dagger)^{p+1}\Phi_0=0$, which proves the very last property stated above about parafermions.

(B) Proof That All Fock Representations of Eqs. (3) and (5) Are Given by Green's Ansatz

For the Bose and Fermi commutation relations there are continuum-many unitarily-inequivalent irreducible representations in a Hilbert space. This is a large number indeed; however, among these the one used in quantum physics, the Fock representation, is singled out up to unitary equivalence by the no-particle condition[8] that there exist a *unique* vector Φ_0 which is annihilated by all the a_k [Eq. (5)]. We expect that

[9] Whereas, the relation between the Bose and Fermi field-theory Hilbert spaces and the corresponding quantum-mechanical Hilbert spaces (Fock space) with symmetric and antisymmetric many-particle states is well known, the relation, if any, between the parafield-theory Hilbert spaces and quantum-mechanical Hilbert spaces is open. In particular, although permutations of the $a^\dagger$'s correspond to permutations of particles in quantum-mechanical states for the Bose and Fermi theories, permutations of the $a^\dagger$'s do not have this significance for the parafield theories. The difficulties which Galindo and Yndurain found in parafield theories are due to a misinterpretation of the significance of permutations of the $a^\dagger$'s (cf., footnote 4, p. 1041, in the article of Galindo and Yndurain cited in Ref. 5).

there are also very many representations of Green's two kinds of trilinear commutation relations, Eqs. (3) and (4), which we call the para-Bose and para-Fermi commutation relations, respectively. In this subsection, we investigate the representations in a Hilbert space of these relations which are analogous to the Fock representation of the Bose and Fermi rules. We find:

Theorem: All irreducible representations in a Hilbert space of the para-Bose and para-Fermi commutation relations, Eqs. (3) and (4), which have a unique no-particle state obeying Eq. (5), also satisfy Eq. (6), in which p is a positive integer, and are characterized up to unitary equivalence by Eqs. (5) and (6). These representations are included in the reducible representations given by Green's ansatz.

Proof: Equation (3) acting on Φ_0 gives

$$a_m(a_l a_k^\dagger)\Phi_0=0 \quad \text{for all } k, l, m. \tag{10}$$

Since Φ_0 is unique, this implies

$$a_l a_k^\dagger\Phi_0=c_{lk}\Phi_0 \quad (c_{lk} \text{ a number}). \tag{11}$$

From Eq. (3) again, we find[10]

$$[[a_k^\dagger,a_l]_\pm, a_m a_n^\dagger]_-=2(\delta_{ln}a_m a_k^\dagger-\delta_{km}a_l a_n^\dagger). \tag{12}$$

Letting both sides of Eq. (12) act on Φ_0 and using Eq. (11), we find

$$0=2(\delta_{ln}c_{mk}-\delta_{km}c_{ln})\Phi_0;$$

hence,

$$\delta_{ln}c_{mk}=\delta_{km}c_{ln} \quad \text{for all } k, l, m, n, \tag{13}$$

which obviously implies $c_{mk}=0$ if $m\neq k$, and $c_{mm}=c_{ll}$. Thus,

$$c_{kl}=p\delta_{kl},$$

with

$$p=\|a_k^\dagger\Phi_0\|^2>0 \quad \text{independent of } k.$$

Finally, p has to be integral to avoid vectors having negative norm. For the para-Fermi case, the calculation of a relevant norm is particularly simple; for

$$\chi_N=(a_k^\dagger)^N\Phi_0,$$

the norm squared is $\prod_{r=1}^N r(p+1-r)$, which assumes negative values for $N>p+1$ unless p is integral. We give this calculation in the Appendix. For the para-Bose case, the square of the norm of the vector $\Psi_N^{(a)}$, Eq. (9), is $\prod_{r=1}^N r^2(p+1-r)$ provided that all N momenta are different. Again the square of the norm becomes negative for $N>p+1$ unless p is integral. This calculation is also in the Appendix. [The square of the norm of the vector $\Psi_N^{(s)}$, Eq. (8), can be calculated for the para-Fermi case in a similar way, when all the momenta are different, and yields exactly the same result.] This completes the proof that Eq. (6) holds with p a positive integer.

With p a positive integer, the (reducible) Green ansatz actually gives an irreducible representation of a and $a^\dagger$ on the space $\mathcal{A}$ of the para-Bose (or para-Fermi) commutation relations obeying the no-particle conditions, Eqs. (5) and (6). For each p, the representation on $\mathcal{A}$ is therefore equivalent to the representation (with the same p) just characterized in our theorem. That $\mathcal{A}$ is invariant under the action of the a's follows straightforwardly from Eqs. (3)–(6). The space $\mathcal{A}$ is a proper subspace of the representation space $\mathcal{B}$ (defined above) of the Fock representation of the b and $b^\dagger$ operators occurring in the Green ansatz. Finally the space $\mathcal{A}$ ($\mathcal{B}$) is irreducible under the a and $a^\dagger$ (b and $b^\dagger$) operators. The irreducibility follows from a lemma of Haag and Schroer[11] that an algebra of operators is irreducible if the representation contains a cyclic vector and the algebra contains a projection Λ onto the cyclic vector. For both $\mathcal{A}$ and $\mathcal{B}$, the cyclic vector is Φ_0. For $\mathcal{A}$ the projection is[12]

$$\Lambda=\prod_k \frac{\sin\pi n_k}{\pi n_k}$$

and for $\mathcal{B}$

$$\Lambda=\prod_{k,\alpha} \frac{\sin\pi b_k^{(\alpha)\dagger}b_k^{(\alpha)}}{\pi b_k^{(\alpha)\dagger}b_k^{(\alpha)}}.$$

The no-particle conditions, Eqs. (5) and (6), uniquely characterize the irreducible representations of the para commutation rules, Eqs. (3) and (4), up to unitary equivalence, because Eqs. (3)–(6) suffice to compute the expectation value of any polynomial in the a_k and $a_k^\dagger$ in the no-particle state.[13]

The usual Bose and Fermi operators satisfy Eqs. (3)–(6) with $p=1$. Since these equations uniquely characterize the irreducible representation, they necessarily lead to fields obeying the usual Bose and Fermi rules. Thus, ordinary fields are special cases of parafields.

Another special case is $p=2$, where simpler trilinear commutation relations hold:

$$a_k a_l^\dagger a_m \mp a_m a_l^\dagger a_k=2\delta_{kl}a_m \mp 2\delta_{ml}a_k,$$
$$a_k^\dagger a_l a_m \mp a_m a_l a_k^\dagger=\mp 2\delta_{kl}a_m,$$
$$a_k a_l a_m \mp a_m a_l a_k=0.$$

(C) Case of Several Different Fields

We want to generalize the preceding discussion to the case of several different fields. To this effect, we have to define the relative commutation rules between different fields. The situation here is somewhat more complicated than in the case of ordinary fields, but not essentially different.

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[10] This argument that p is independent of k is due to D. W. Robinson. We thank Dr. Robinson for allowing us to quote his argument.

[11] R. Haag and B. Schroer, J. Math. Phys. **3**, 248, 252 (1962).

[12] A. S. Wightman and S. S. Schweber, Phys. Rev. **98**, 812 (1955), Table I.

[13] A. S. Wightman, Phys. Rev. **101**, 860 (1956); W. Schmidt and K. Baumann, Nuovo Cimento 4, 860 (1956); R. Haag and B. Schroer, Ref. 10; and M. A. Naimark, *Normed Rings*, translated by L. F. Boron (P. Noordhoff, Ltd., Groningen, 1960), especially Chap. IV.

With ordinary fields, one has to specify whether a pair commutes or anticommutes. The usual terminology labels as "normal" the case where any two Bose fields commute, any two Fermi fields anticommute and each Bose field commutes with each Fermi field. Beside this "normal" case, there is a whole set of "anomalous" cases, where some commutators are replaced by anti-commutators and vice versa. There is no *a priori* reason to exclude the anomalous cases. The main difference with the normal case is that, in general, additional restrictions have to be set on the interaction Hamiltonian in order to preserve locality, the practical consequence of which is the occurrence of some additional conservation rules, modulo two.[14]

When dealing with several parafields, it is also possible to assume bilinear commutation or anticommutation relations. However, in keeping with the spirit of parafield theory, there is no good *a priori* reason to exclude the possibility of trilinear relations between some of the parafields.

Let us investigate this point more closely.

There is considerable leeway in the choice of trilinear relations between different fields, and we clearly have to adopt some requirements in order to limit the choice. These requirements ought to be such that the trilinear commutation relations introduced by Green for a single parafield appear as a special case of the trilinear relations between different fields. We demand the following:

(i) The left-hand side must have the form

$$[[A,B]_\epsilon, C]_\eta$$

with ϵ, $\eta=\pm1$, and the right-hand side must be linear;

(ii) when the internal pair $[A,B]_\epsilon$ refers to the same field, its ϵ must have the form related to the number operator ($\epsilon=+1$ for para-Bose, $\epsilon=-1$ for para-Fermi) and it must commute with $C(\eta=-1)$ if C refers to another field;

(iii) these relations must be satisfied by ordinary Bose or Fermi fields.

Let us apply these conditions to two parafields ϕ_a, ϕ_b. We use ϵ^a, $\epsilon^b=\pm1$ to label their respective types ($\epsilon=+1$ for para-Bose, $\epsilon=-1$ for para-Fermi). We want to show that conditions (i), (ii), and (iii) allow only two sets of trilinear relations.

We recall the trilinear relations involving one of the fields alone, ϕ_a, say. We have

$$[[a_k^\dagger,a_l]_{\epsilon^a}, a_m]_- = -2\delta_{km}a_l\,, \tag{14}$$

$$[[a_k,a_l]_{\epsilon^a}, a_m]_- = 0\,, \tag{15}$$

from which all other relations can be deduced by applying Jacobi's identity or Hermitian conjugation. We next consider the trilinear relations involving ϕ_a twice and ϕ_b once. Conditions (i) and (ii) imply

$$[[a_k^\dagger,a_l]_{\epsilon^a}, b_m]_- = 0\,, \tag{16}$$

$$[[a_k,a_l]_{\epsilon^a}, b_m]_- = 0\,, \tag{17}$$

$$[[a_k^\dagger,a_l^\dagger]_{\epsilon^a}, b_m]_- = 0\,. \tag{18}$$

Using the generalized version of Jacobi's identity

$$[[A,B]_\epsilon, C]_- + [[C,A]_\eta, B]_{-\eta\epsilon} + \eta\epsilon[[B,C]_\eta, A]_{-\eta\epsilon} = 0\,,$$

together with conditions (i) and (iii), we find only two possible sets of relations "permuted" from Eq. (16)

$$[[b_m,a_k^\dagger]_\eta, a_l]_{-\eta\epsilon^a} = -2\epsilon^a\delta_{kl}b_m\,, \tag{16a}$$

$$[[a_l,b_m]_\eta, a_k^\dagger]_{-\eta\epsilon^a} = 2\eta\delta_{kl}b_m\,. \tag{16b}$$

For ordinary fields, $\eta=+1$ corresponds to commuting fields and $\eta=-1$ to anticommuting fields. Hermitian conjugation further gives

$$[[a_k,a_l^\dagger]_{\epsilon^a}, b_m^\dagger]_- = 0\,, \tag{16c}$$

$$[[b_m^\dagger,a_k]_\eta, a_l^\dagger]_{-\eta\epsilon^a} = 2\delta_{kl}b_m^\dagger\,, \tag{16d}$$

$$[[a_l^\dagger,b_m^\dagger]_\eta, a_k]_{-\eta\epsilon^a} = -2\eta\epsilon^a\delta_{kl}b_m^\dagger\,. \tag{16e}$$

The same arguments applied to Eq. (17) and to Eq. (18), give similar sets of five relations, the sole difference being that all the right-hand sides vanish. We find

$$[[b_m,a_k]_\eta, a_l]_{-\eta\epsilon^a} = 0\,, \tag{17a}$$

$$[[b_m,a_k^\dagger]_\eta, a_l^\dagger]_{-\eta\epsilon^a} = 0\,. \tag{18a}$$

The 18 trilinear relations involving ϕ_b twice and ϕ_a once are obtained in exactly the same way; they are readily deducible from the 18 relations above by exchanging the letters a and b everywhere.

The important point in all this is that, owing to condition (iii), the same value of η must be taken everywhere. Thus, we find in all two possible sets of trilinear relations. The set corresponding to $\eta=+1$ will be called *relative para-Bose*; that corresponding to $\eta=-1$ will be called *relative para-Fermi.*

Next we look for Fock representations of fields obeying these rules, i.e., representations possessing a *unique* no-particle state Φ_0

$$a_k\Phi_0 = b_m\Phi_0 = 0\,. \tag{19}$$

Solutions to this problem are provided by the following straightforward generalization of Green's ansatz. Given a positive integer p, we expand each field operator into "Green components"

$$a_k = \sum_{\alpha=1}^{p} a_k^{(\alpha)}, \quad b_m = \sum_{\beta=1}^{p} b_m^{(\beta)}.$$

For each pair of components belonging to the same field, we assume the commutation rules of Green's ansatz, i.e., the para-Bose rule if $\epsilon=+1$ and the para-Fermi rule if $\epsilon=-1$. For each pair $a_k^{(\alpha)}$, $b_m^{(\beta)}$, we assume the para-Bose rule if $\eta=+1$ and the para-Fermi rule if

[14] H. Araki, J. Math. Phys. 2, 267 (1961), and references cited therein.

$\eta=-1$, that is

$$[a_k^{(\alpha)},b_m^{(\alpha)}]_{-\eta}=0\,,\quad [a_k^{(\alpha)},b_m^{(\alpha)\dagger}]_{-\eta}=0\,,$$
$$[a_k^{(\alpha)},b_m^{(\beta)}]_{\eta}=0\,,\quad [a_k^{(\alpha)},b_m^{(\beta)\dagger}]_{\eta}=0\,,\quad \alpha\neq\beta.$$

As can be seen easily, fields obeying these rules verify the set of trilinear relations stated above.

Using this generalized version of Green's ansatz, we build, for each possible value of p, a Fock representation for the pair of parafields ϕ_a, ϕ_b, exactly in the same way as for a single parafield. The no-particle condition (5) is generalized to

$$a_k a_l^\dagger\Phi_0=p\delta_{kl}\Phi_0\,, \tag{19a}$$
$$b_m b_n^\dagger\Phi_0=p\delta_{mn}\Phi_0\,, \tag{19b}$$
$$a_k b_m^\dagger\Phi_0=0\,, \tag{19c}$$
$$b_m a_k^\dagger\Phi_0=0\,. \tag{19d}$$

Conversely, as a generalization of the theorem stated above, it can be proved that *all* Fock representations are given by this ansatz. The proof exactly follows the line of argument used in subsection 2(B) for a single parafield, with the sole difference that condition (5) is replaced by the set of conditions (19).

The latter can be *derived* from the parafield commutation relations and from the uniqueness of the no-particle state Φ_0 as follows: The results for single fields applied to ϕ_a and ϕ_b separately lead to Eqs. (19a) and (19b), with p a positive integer, except that the values of p might be different,

$$a_k a_l^\dagger\Phi_0=p^{(a)}\delta_{kl}\Phi_0\,, \tag{20a}$$
$$b_m b_n^\dagger\Phi_0=p^{(b)}\delta_{mn}\Phi_0\,. \tag{20b}$$

From Eq. (16a), with a and b interchanged, acting on the vacuum,

$$b_l(a_k b_m^\dagger\Phi_0)=0\,,\quad \text{for all } k,\, l,\, m$$

and from this the uniqueness of Φ_0 implies

$$a_k b_m^\dagger\Phi_0=c_{km}\Phi_0\,,\quad c_{km}\text{ a number}. \tag{20c}$$

Now using Eqs. (14) and (15), we get

$$[[a_k,a_l^\dagger]_{\epsilon^a},\, a_l b_m^\dagger]_-=-2\epsilon^a a_k b_m^\dagger\,,$$

and, using Eqs. (20a) and (20c) the two members applied to Φ_0 give

$$0=-2\epsilon^a a_k b_m^\dagger\Phi_0\,,$$

which proves Eq. (19c). Equation (19d) is proved the same way. Then, from Eqs. (16a) and (16d) with a and b interchanged, we obtain

$$[b_m,a_k^\dagger]_\eta a_k b_m^\dagger=2\epsilon^a(a_k a_k^\dagger-b_m b_m^\dagger)+\epsilon^a\epsilon^b a_k b_m^\dagger[b_m,a_k^\dagger]_\eta$$

and the two members applied to Φ_0 give, if we take into account Eqs. (19c) and (19d),

$$0=(a_k a_k^\dagger-b_m b_m^\dagger)\Phi_0=(p^{(a)}-p^{(b)})\Phi_0\,,$$

hence

$$p^{(a)}=p^{(b)}\equiv p\,.$$

The considerations just given for two fields can be extended, without difficulty, to any number of fields. In particular, *all* Fock representations of any systems of parafields are given by Green's ansatz. With this last remark, we are now ready to describe the whole variety of theories involving several parafields.

For each pair of parafields, we have in all four possible sets of relative commutation relations: straight commutation (B), straight anticommutation (F), relative para-Bose (pB) and relative para-Fermi (pF). The last two are allowed only for parafields of equal order; they are *not* equivalent to the first two, respectively, except for ordinary fields ($p=1$). All field operators obey the no-particle conditions, Eqs. (19), and are conveniently described by Green's ansatz with suitable bilinear commutation relations between Green components.

The set of parafields entering the theory can be divided into subsets or "families," such that (i) any two fields belonging to the same family have equal order, and (ii) any two fields belonging to different families have ordinary relative commutation rules, B or F. Then each field $\phi_{i\lambda}$ is denoted with two subscripts, i and λ, with i labeling the family and λ the particular field within the ith family. We choose i so that $p_i\leqslant p_j$ for $i<j$. The case $i=0$ labels the family of ordinary Bose and Fermi fields ($p_0=1$). For $p>1$, we allow the possibility that more than one para family can occur with the same order.

In order to specify the relative commutation relations between two parafields, $\phi_{i\lambda}$ and $\phi_{j\mu}$, we use two dichotomic variables $\theta_{i\lambda,j\mu}$ and $\tau_{i\lambda,j\mu}$, each of which may be equal to 0 or 1. The four possible sets of commutation rules are associated to the four possible sets of values of these variables according to the following tableau:

$\theta_{i\lambda,j\mu}$ \ $\tau_{i\lambda,j\mu}$	0	1
0	B	F
1	pB	pF

Thus, θ indicates whether we have to deal with para or ordinary relations, τ indicates whether the type is Bose or Fermi. Note that these numbers are symmetric in the exchange of subscripts $i\lambda\leftrightarrow j\mu$, and that:

$$\theta_{i\lambda,j\mu}=0\quad\text{if}\quad i\neq j\,, \tag{21}$$
$$\theta_{i\lambda,i\lambda}=1\,. \tag{22}$$

In the following we shall repeatedly use the Green expansion of these parafields

$$\phi_{i\lambda}(x)=\sum_{\alpha=1}^{p_i}\phi_{i\lambda}^{(\alpha)}(x) \tag{23}$$

and the commutation relations between the Green components $\phi_{i\lambda}^{(\alpha)}(x)$, $\phi_{j\mu}^{(\beta)}(y)$ for $x-y$ space-like. Using the variables θ and τ permits to write these relations in the following completely general form

$$\phi_{i\lambda}^{(\alpha)}\phi_{i\mu}^{(\alpha)}=(-)^{\tau_{i\lambda,i\mu}}\phi_{i\mu}^{(\alpha)}\phi_{i\lambda}^{(\alpha)}\,,\quad i=j\,,\quad \alpha=\beta\,, \tag{24a}$$

$$\phi_{i\lambda}{}^{(\alpha)}\phi_{i\mu}{}^{(\beta)}=(-)^{\tau_{i\lambda,i\mu}+\theta_{i\lambda,i\mu}}\phi_{i\mu}{}^{(\beta)}\phi_{i\lambda}{}^{(\alpha)}, \quad i=j,\quad \alpha\neq\beta, \tag{24b}$$

$$\phi_{i\lambda}{}^{(\alpha)}\phi_{j\mu}{}^{(\beta)}=(-)^{\tau_{i\lambda,j\mu}}\phi_{j\mu}{}^{(\beta)}\phi_{i\lambda}{}^{(\alpha)},\quad i\neq j. \tag{24c}$$

As a natural generalization of ordinary field theory, we single out as the "normal" case the one in which (i) there is only one family for each p, (ii) inside a family the relative commutation rules are all para, and (iii) the relative rules are of Fermi type if and only if both fields are Fermi or para-Fermi. In equations, these conditions are

$$\theta_{p\lambda,p'\mu}=\delta_{pp'}, \tag{25a}$$

$$\tau_{p\lambda,p'\mu}=\tau_{p\lambda,p\lambda}\tau_{p'\mu,p'\mu}, \tag{25b}$$

where we have replaced the family index i by the order p because of (i). The motivation for this terminology is that it gives a unique normal case, and, as we will see below, one in which there are as few restrictions as possible on the interaction. We call all other cases "anomalous."

As a final remark, we mention that for a set of parafields of the same type which transform as a representation of some internal symmetry group the normal relative rules are preserved by the internal transformations. Thus, the normal case is appropriate for fields with an *internal variable*.

3. DERIVATION OF SELECTION RULES FOR PARAFIELDS FROM PARALOCALITY OF $H_I(x)$

Our basic requirements are that the Hamiltonian density $H_I(x)$, expressed in free fields in the interaction picture, be a polynomial in the fields all taken at the same point x, and that $H_I(x)$ be paralocal,[15]

$$[H_I(x),H_I(y)]_-=0,\quad x\sim y, \tag{26}$$

in the space $\mathcal{B}$ associated with the Green component fields acting on the vacuum ($x\sim y$ means $x-y$ spacelike).

This imposes on $H_I(x)$ severe limitations which are most conveniently explored by using Green's ansatz, and expressing Eq. (26) as a set of conditions on the terms occurring in the expression of $H_I(x)$ in Green components. We carry out this program in subsection (A). The conditions which we obtain are discussed and illustrated by examples in subsection (B). Then [subsection (C)] we deduce a set of selection rules for the S matrix. The results are illustrated on some simple examples in subsection (D). The notations of subsection 2(C) are used throughout.

(A) Conditions on $H_I(x)$

Expansion of H_I in Green components. Applying Green's ansatz, Eq. (23), to each field in $H_I(x)$ we express H_I as a finite sum of monomials in the Green component fields:

$$H_I(x)=\sum_{(m)}\mathcal{H}^{(m)}(x), \tag{27a}$$

$$\mathcal{H}^{(m)}(x)\equiv H^{(m_0)}(x)H^{(m_1)}(x)\cdots H^{(m_q)}(x). \tag{27b}$$

Here $(m)=(m_0,m_1,\cdots,m_q)$ is a short notation for the set of indices defining each term in the sum, and $H^{(m_i)}(x)$ contains all the field components of the ith family.

For definiteness, we adopt a standard way of ordering the Green components $\phi_{i\lambda}{}^{(\alpha)}$ within each term $\mathcal{H}^{(m)}(x)$. As exhibited by Eq. (27b), they are arranged in increasing order of family indices; furthermore, within each $H^{(m_i)}(x)$, i.e., for fixed i, they are arranged in such a way that λ increases from left to right; for fixed i and λ, the index α increases from left to right; and finally for fixed i, λ and α, the fields are normal ordered.

For later use, we define the following quantities relating to a given $H^{(m_i)}$:

$\nu_{i\lambda}{}^{(\alpha)}$, degree of the Green component $\phi_{i\lambda}{}^{(\alpha)}$;

$\nu_i{}^{(\alpha)}\equiv\sum_\lambda \nu_{i\lambda}{}^{(\alpha)}$, total degree of the αth Green components belonging to the ith family;

$\sigma_{i\lambda}\equiv\sum_{\alpha=1}^{p_i}\nu_{i\lambda}{}^{(\alpha)}$, total degree of the field $\phi_{i\lambda}$;

$\sigma_i\equiv\sum_\lambda\sigma_{i\lambda}$, total degree of the fields of the ith family.

We shall also use the total degree N_f of fermion and parafermion fields in $\mathcal{H}^{(m)}$:

$$N_f\equiv\sum_{i\lambda}\tau_{i\lambda,i\lambda}\sigma_{i\lambda}. \tag{28}$$

We point out two important properties of the expansion Eq. (27).

First, because we have chosen a well-defined standard order for each of the terms, *the expansion is unique*, and no cancellations can occur between different terms.

Secondly, the expansion is invariant under all permutations of the Green indices α which preserve the algebraic relations satisfied by the Green component fields. These algebraic relations are Green's ansatz, Eq. (23), and the commutation rules, Eq. (24). The permutations which preserve these equations are the simultaneous permutations of the Green indices for all fields in a given family. Therefore, if a term $\mathcal{H}^{(m)}$ occurs in the expansion, then another term $\mathcal{H}^{(P_im)}$ also occurs

[15] Strictly speaking, we should only assume that $H_I(x)$ is local in the space $\mathcal{A}$, since $\mathcal{A}$ is the space of physical states, rather than the stronger condition that $H_I(x)$ is paralocal, i.e., local in the space $\mathcal{B}\supset\mathcal{A}$, which we have made. It is an open question whether or not locality implies paralocality. Thus the selection rules which we derive might be more restrictive than those which follow from locality, although it is also possible that our selection rules may be valid even if locality does not imply paralocality. The assumption of paralocality of the fields has been used by Dell' Antonio *et al.*, cited in Ref. 5, to derive the connection of spin and type of paralocality and the *TCP* theorem in the framework of general parafield theory. We are grateful to Professor C. N. Yang for pointing out that our derivation of selection rules requires the assumption of paralocality.

which differs from $\mathcal{H}^{(m)}$ only by the replacement of the factor $H^{(m_i)}$ by a factor $H^{(P_i m_i)}$ obtained from it by simultaneous permutation of all the Green indices. The important fact, which we use later, is that the sets $\{\nu_{i\lambda}{}^{(\alpha)}\}$ for $H^{(m_i)}$ and $H^{(P_i m_i)}$ are related by the same permutation of the α indices that relates the two H's.

The paralocality conditions in terms of Green's ansatz. We want to show that the paralocality condition, Eq. (26), can be expressed as a set of conditions on the various degrees, ν and σ, relating to each $H^{(m_i)}$ according to the definitions given above.

Let us show first that paralocality of H_I is equivalent to relative locality between any two terms of its Green expansion. The expansion Eq. (27a) gives

$$H_I(x)H_I(y)=\sum_{(m');(m'')}\mathcal{H}^{(m')}(x)\mathcal{H}^{(m'')}(y) \qquad (29)$$

and

$$[H_I(x),H_I(y)]_-=\sum_{(m');(m'')}[\mathcal{H}^{(m')}(x),\mathcal{H}^{(m'')}(y)]_-. \qquad (30)$$

Since the terms in Eq. (29) are products of monomials which either commute or anticommute, each term in Eq. (30) either vanishes or equals twice the correspond-corresponding term in Eq. (29). Because of the standard ordering of the monomials $\mathcal{H}^{(m)}(x)$ and the independence of the fields at x and at y, no cancellation is possible among the terms in Eq. (29). The nonvanishing terms in Eq. (30) are multiples of terms in Eq. (29); thus the terms in Eq. (30) also cannot cancel, and the vanishing of the sum implies the vanishing of each term taken separately. Therefore, the paralocality condition, Eq. (26), on H_I implies

$$[\mathcal{H}^{(m')}(x),\mathcal{H}^{(m'')}(y)]_-=0\,,\quad x\sim y\,,\quad \text{for all } m',\, m''. \qquad (31)$$

Since the converse obviously holds, we have proved the equivalence stated above.

If $x\sim y$, each Green component field in $\mathcal{H}^{(m')}(x)$ either commutes or anticommutes with each Green component field in $\mathcal{H}^{(m'')}(y)$. Hence we have

$$\mathcal{H}^{(m')}(x)\mathcal{H}^{(m'')}(y)=(-)^{\eta(m',m'')}\mathcal{H}^{(m'')}(y)\mathcal{H}^{(m')}(x), \qquad (32)$$

where $\boldsymbol{\eta}(m',m'')$ is the number, modulo 2, of minus signs introduced by having each field component in $\mathcal{H}^{(m')}(x)$ travel from the left to the right across $\mathcal{H}^{(m'')}(y)$. The paralocality condition Eq. (31) means that this number must be even for all pairs (m',m'').

From now on, we use the symbol $\cong$ to indicate equality modulo 2. With this notation, the paralocality condition thus reads

$$\boldsymbol{\eta}(m',m'')\cong 0 \quad \text{for all } m',\, m''. \qquad (33)$$

It remains to compute $\boldsymbol{\eta}$. We first compute the number of minus signs $\eta_{ij}\equiv\eta(m_i',m_j'')$ defined by

$$H^{(m_i')}(x)H^{(m_j'')}(y)=(-)^{\eta_{ij}}H^{(m_j'')}(y)H^{(m_i')}(x)\,,\quad x\sim y.$$

Using Eq. (24), we find that

$$\eta_{ij}\cong\sum_{\lambda\mu}\Big[\tau_{i\lambda,j\mu}\sigma_{i\lambda}'\sigma_{j\mu}'' +\delta_{ij}\theta_{i\lambda,j\mu}\Big(\sigma_{i\lambda}'\sigma_{j\mu}''+\sum_{\alpha=1}^{p_i}\nu_{i\lambda}'^{(\alpha)}\nu_{j\mu}''^{(\alpha)}\Big)\Big].$$

Since, from Eq. (27b),

$$\boldsymbol{\eta}(m',m'')\cong\sum_{ij}\eta_{ij}\,,$$

we have

$$\boldsymbol{\eta}(m',m'')\cong\sum_{ij;\lambda\mu}\tau_{i\lambda,j\mu}\sigma_{i\lambda}'\sigma_{j\mu}'' +\sum_{i;\lambda\mu}\theta_{i\lambda,i\mu}\Big(\sigma_{i\lambda}'\sigma_{i\mu}''+\sum_{\alpha=1}^{p_i}\nu_{i\lambda}'^{(\alpha)}\nu_{i\mu}''^{(\alpha)}\Big). \qquad (34)$$

Thus, an alternative way of expressing the paralocality conditions for H_I is that the expressions given by Eq. (34) must satisfy Eq. (33).

Interesting properties can be derived from this.

Consider first Eq. (34) for $(m'')=(m')$. Because of the symmetry of the τ's and the θ's, all cross terms in the sums occur twice and do not contribute. Using, in addition, the property $n^2\cong n$, we find

$$\boldsymbol{\eta}(m',m')\cong\sum_{i\lambda}\tau_{i\lambda,i\lambda}\sigma_{i\lambda}'+\sum_{i\lambda}\theta_{i\lambda,i\lambda}\Big[\sigma_{i\lambda}'+\sum_{\alpha=1}^{p_i}\nu_{i\lambda}'^{(\alpha)}\Big].$$

The last sum cancels (mod 2) as a simple consequence of the definition of $\sigma_{i\lambda}'$, and the first sum precisely equals the number of Fermi-like particles N_f' [Eq. (28)]. Therefore,

$$\boldsymbol{\eta}(m',m')\cong N_f'\,,\quad \text{for all } m'\,,$$

and the paralocality conditions Eq. (33) lead to the rule

$$N_f\cong 0\,,\quad \text{for all } m. \qquad (35)$$

Otherwise stated, the total number of Fermi-like fields must be even for all terms, a conclusion which could also be reached using rotation invariance and the connection of spin and statistics.

Another consequence of Eq. (33) is that $\boldsymbol{\eta}(m',m'')$ must not change parity if a specific factor $H^{(m_i')}$ in $\mathcal{H}^{(m')}$ is replaced by the factor $H^{(P_i m_i')}$ related to the original factor $\mathcal{H}^{(m')}$ by an over-all permutation P_i of the α indices in the ith family. This is true because, as discussed above, terms $\mathcal{H}^{(m')}$ and $\mathcal{H}^{(P_i m')}$ differing *only* by the replacement of $H^{(m_i')}$ by $H^{(P_i m_i')}$ both occur in Eq. (27a). Thus, paralocality implies

$$\boldsymbol{\eta}(m',m')\cong\boldsymbol{\eta}(P_i m',m')\,.$$

When we replace the two sides by their expressions as given by Eq. (34), the equality simplifies enormously because the permutation P_i permutes only the indices in the ith family and, moreover, leaves the σ's invariant.

One finds

$$\sum_{\lambda\mu} \theta_{i\lambda,i\mu} \sum_{\alpha} (\nu_{i\lambda}'^{(\alpha)}+\nu_{i\lambda}'^{(P_i\alpha)})\nu_{i\mu}'^{(\alpha)} \cong 0.$$

Suppose now that the permutation P_i is a transposition, for example the one exchanging Green indices 1 and 2. The above mod 2 equality then reads

$$\sum_{\lambda\mu} \theta_{i\lambda,i\mu}(\nu_{i\lambda}'^{(1)}+\nu_{i\lambda}'^{(2)})(\nu_{i\mu}'^{(1)}+\nu_{i\mu}'^{(2)}) \cong 0.$$

Here again, because of the symmetry of $\theta_{i\lambda,i\mu}$, all cross terms (i.e., $\lambda\neq\mu$) occur twice and can be dropped. Then, using Eq. (22) and the property $n^2\cong n$, we are left with

$$\sum_{\lambda} (\nu_{i\lambda}'^{(1)}+\nu_{i\lambda}'^{(2)}) \cong 0,$$

that is,

$$\nu_i'^{(1)}+\nu_i'^{(2)} \cong 0.$$

Repeating the argument with all possible transpositions, we obtain the following important property:

$$\nu_i^{(1)} \cong \nu_i^{(2)} \cong \cdots \cong \nu_i^{(p_i)}, \quad \text{for all } m_i. \tag{36}$$

Otherwise stated, paralocality implies that in each term of the Green expansion of H_I, the total degree $\nu_i^{(\alpha)}$ of the αth components of the parafields of the ith family have a parity independent of α. However, the parity of $\nu_i^{(\alpha)}$ can be different in different terms in the expansion.

The normal case. In the normal case, as defined by Eqs. (25), condition (36) simply reads

$$\nu_p^{(1)} \cong \nu_p^{(2)} \cong \cdots \cong \nu_p^{(p)}, \quad \text{for all } m,\, p \tag{37}$$

and Eq. (34) simplifies to

$$\begin{aligned}\eta(m',m'') &\cong \sum_{pp';\lambda\mu} \tau_{p\lambda,p\lambda}\tau_{p'\mu,p'\mu}\sigma_{p\lambda}'\sigma_{p'\mu}'' \\ &\quad + \sum_{p'} \sum_{\lambda\mu} (\sigma_{p\lambda}'\sigma_{p\mu}'' + \sum_{\alpha=1}^{p} \nu_{p\lambda}'^{(\alpha)}\nu_{p\mu}''^{(\alpha)}) \\ &\cong N_f'N_f'' + \sum_{p} (\sigma_p'\sigma_p'' + \sum_{\alpha=1}^{p} \nu_p'^{(\alpha)}\nu_p''^{(\alpha)}).\end{aligned}$$

The first term on the right-hand side vanishes (mod 2) as a consequence of Eq. (35). As for the second term, since $\sigma_p=\sum_{\alpha=1}^{p}\nu_p^{(\alpha)}$, it can be written $\sum_p(p^2\nu_p'^{(1)}\nu_p''^{(1)} + p\nu_p'^{(1)}\nu_p''^{(1)})$ as a consequence of Eq. (37); it is therefore a sum of even numbers and also vanishes (mod 2). We see that the set of conditions Eqs. (35) and (37) implies the paralocality condition Eq. (33).

Thus, Eqs. (35) and (37), which have been shown to follow in all cases from the requirement that $H_I(x)$ be paralocal, express the full content of this requirement in the normal case. This confirms the indication given at the end of Sec. 2, that the normal case is the one which implies the least amount of restrictions on the interaction.

Anomalous cases. We do not give here a systematic discussion of the anomalous cases, but merely point out that in general additional restrictions can arise.

For example, a particularly simple anomalous case occurs if there is more than one family for a given p, and if all fields inside a given family have normal para rules, and fields in different families have normal relative bilinear rules. This case can be treated exactly as the normal case; the additional restriction is that Eq. (36) holds separately for each family with a given p, rather than for all fields with the same p, as is the situation in the normal case when there is only one family for each p [condition (37)].

In general, the paralocality condition, Eq. (33), leads to further restrictions on H_I which are similar to those which occur for ordinary fields.[14]

(B) Properties of Paralocal $H_I(x)$: Discussion and Examples

Since we are interested in selection rules which *necessarily* follow from the paralocality condition for parafields, we discuss only the version of the theory which leads to the minimum restrictions, that is, the theory with normal commutation rules. Then, the paralocality condition is equivalent to conditions (35) and (37).

Condition (35) means that $H_I(x)$ does not contain any term with an odd number of Fermi-like fields. This result is a straight generalization of the result relating to fermions for ordinary fields.

Condition (37), on the contrary, has no analog in the ordinary field theory and leads to new results.

In particular, it leads to

$$\sigma_p \equiv \sum_{\alpha=1}^{p} \nu_p^{(\alpha)} \cong p\nu_p^{(1)} \quad \text{for all } m,p. \tag{38}$$

The interesting feature of σ_p is that, although it has been defined, like $\nu_p^{(\alpha)}$, as a property of the term $\mathcal{H}^{(m)}$ in the Green expansion of H_I, it can also be defined, contrary to $\nu_p^{(\alpha)}$, as a property of the terms of H_I, without any reference to the Green expansion. Recall that H_I is a polynomial in the parafields. Then, assuming normal ordering, σ_p is the number of times parafields of order p appear in the terms of this polynomial.

Equation (38) summarizes the restrictions on σ_p. They are given in a more detailed and transparent way in Table I. That all the allowed values of σ_p indicated in this Table can actually occur is best demonstrated by producing specific examples. Let us give examples of local fields in each of the four cases entering Table I.

(1) $\nu_p^{(\alpha)}$ *even*, p *even* (σ_p *even*). Let ϕ_1 and ϕ_2 be

TABLE I. Restrictions on σ_p from paralocality of H_I.

	p even	p odd
$\nu_p^{(\alpha)}$ even	σ_p even	σ_p even
$\nu_p^{(\alpha)}$ odd	σ_p even $\geqslant p$	σ_p odd $\geqslant p$

para-Bose fields of order p with relative para-Bose rules, and let A be a Bose field with relative Bose rules with respect to ϕ_1 and ϕ_2. Then

$$H_I(x)=gA(x)[\phi_1(x),\phi_2(x)]_+=2gA(x)\sum_{\alpha=1}^{p}\phi_1^{(\alpha)}(x)\phi_2^{(\alpha)}(x)$$

is local. In this example, $\sigma_p=2$. We can obtain any even σ_p by raising $[\phi_1(x),\phi_2(x)]_+$ to some power. Here, as below, similar examples can be given with para-Fermi fields. Here $[\phi_i,H_I]_-$ is local.

(2) $\nu_p^{(\alpha)}$ *even*, p *odd* (σ_p *even*). Same as in (1).

(3) $\nu_p^{(\alpha)}$ *odd*, p *even* (σ_p *even* $\geqslant p$). Let $\phi_1, \phi_2, \cdots, \phi_p$ be para-Bose fields of order p with relative para-Bose rules. Then

$$H_I(x)=gA(x)[\cdots[[\phi_1(x),\phi_2(x)]_-,\phi_3(x)]_+,\cdots\phi_p(x)]_-$$
$$=2^{p-1}gA(x)\sum_{\substack{i_1,\cdots i_p\\ \text{all different}}}\phi_1^{(i_1)}(x)\phi_2^{(i_2)}(x)\cdots\phi_p^{(i_p)}(x)$$

is local and has $\sigma_p=p$. Examples with σ_p even $>p$ can be constructed by multiplying this example by a factor of the type given in (1). Here $[\phi_i,H_I]_+$ is local. (Note anticommutator for $[\phi_i,H_I]_+$.)

(4) $\nu_p^{(\alpha)}$ *odd*, p *odd* (σ_p *odd* $\geqslant p$). Same as (3), except that the outer bracket is an anticommutator:

$$H_I(x)=gA(x)[\cdots[[\phi_1(x),\phi_2(x)]_-,\phi_3(x)]_+,\cdots\phi_p(x)]_+.$$

Here $[\phi_i,H_I]_-$ is local.

The existence of local interactions of this type escaped the attention of previous workers.[4,16] Note however that it is impossible to produce a paralocal H_I with terms in which parafields of order $p>1$ occur singly.

A more practical example of type (4), which will be used below for further illustration, is the Yukawa interaction:

$$H_I(x)=\tfrac{1}{4}g[[\bar\psi(x),\psi(x)]_+-\langle[\bar\psi(x),\psi(x)]_+\rangle_0,\phi(x)]_+$$
$$=g\sum_{\substack{i_1,i_2,i_3\\ \text{all different}}}\bar\psi^{(i_1)}(x)\psi^{(i_2)}(x)\phi^{(i_3)}(x),\qquad(39)$$

where ψ and ϕ are para-Fermi and para-Bose fields of order 3, respectively, and have relative para-Bose commutation rules.

(C) Selection Rules for The S Matrix

Conditions (35) and (37), which follow from the assumption that $H_I(x)$ is paralocal, and which in fact express the full content of the paralocality condition in the case of parafield theories with normal commutation rules, are symmetry properties and lead to selection rules.

From condition (35), according to which $H_I(x)$ does not contain any term with an odd number of Fermi-like particles, it obviously follows that *the total number of Fermi-like particles*, i.e., fermions and parafermions, *is absolutely conserved modulo 2 in all reactions*.

The selection rules associated with condition (37) are somewhat more involved. Here, we derive them by looking for symmetry properties common to all Feynman diagrams. It should be emphasized that the selection rules thereby obtained are absolute selection rules (in all orders of perturbation theory), and that phenomenological nonlocal interactions which are mediated by repeated local interactions must also obey these selection rules.

Standard perturbation theory applies to the theory expressed in terms of the Green component fields $\phi_{p\lambda}^{(\alpha)}$. Let us consider a particular Feynman diagram in the perturbation expansion. Let $N_p^{(\alpha)\mathrm{ext}}(N_p^{(\alpha)\mathrm{int}})$ be the number of external (internal) lines associated with the αth Green components of the fields of the pth family, and let $N_p^{(\alpha)}$ be the total number of these components present in the entire diagram. We have

$$N_p^{(\alpha)\mathrm{ext}}=N_p^{(\alpha)}-2N_p^{(\alpha)\mathrm{int}}\cong N_p^{(\alpha)}\qquad(40)$$

and

$$N_p^{(\alpha)}=\sum_v \nu_p^{(\alpha,v)},\qquad(41)$$

where the sum has to be taken over all vertices of the diagram and $\nu_p^{(\alpha,v)}$ is the number of Green component fields of order p with Green index α at vertex v. Applying condition (37), Eqs. (40) and (41) lead to

$$N_p^{(1)\mathrm{ext}}\cong N_p^{(2)\mathrm{ext}}\cong\cdots\cong N_p^{(p)\mathrm{ext}}.\qquad(42)$$

Since Eq. (42) hold for all Feynman diagrams, it is an absolute selection rule for the elements of the S matrix. Conversely, as is easily seen, this selection rule necessarily implies that $H_I(x)$ obey condition (37). Therefore it expresses the full content of condition (37).

Among the properties, which follow from (42), we are primarily interested in those which can be expressed without explicit reference to the Green expansion. Here, we shall focus on the selection rules for the total number N_p^{ext} of external lines relating to parafields of a given order p in a Feynman diagram. Equation (42) leads to

$$N_p^{\mathrm{ext}}\equiv\sum_{\alpha=1}^{p}N_p^{(\alpha)\mathrm{ext}}\cong pN_p^{(1)\mathrm{ext}}.\qquad(43)$$

Equation (43) summarizes the restrictions on N_p^{ext}. We display these restrictions in Table II. Inspection of this table leads to the following selection rules for N_p^{ext} as a consequence of paralocality:

(1) *For each even p, the total number of para particles of order p on both sides of a reaction must be even.*

[16] S. Kamefuchi and J. Strathdee, Ref. 4, stated that parafields can only occur in pairs in H_I. Their condition on H_I (which is more restrictive than the condition we have used) was that the Euler-Lagrange variational equations for the fields, deduced using commuting or anticommuting variations, should agree with the equations found from the commutation relations, $-i\partial_\mu\chi(x)=[\chi(x),P_\mu]_-$, where P_μ is the total energy-momentum operator. This condition is satisfied if $[\chi(x),H_I(y)]_-=0$ when $x\sim y$, which is the case for the examples given in (4). The parafields in these examples do not occur in pairs in (4), which shows that Kamefuchi and Strathdee's analysis was incorrect.

(2) *For each odd p, the total number of para particles of order p on both sides of a reaction can take any value except for odd numbers smaller than p.*

It follows from this set of selection rules that $N_p^{\text{ext}}=1$ is absolutely forbidden when $p>1$. Therefore, we reach the important conclusion that

Reactions with any number of ordinary particles and only one para particle are absolutely forbidden.

Hence, the decay of para particles into ordinary particles, and the production of a single para particle by ordinary particles are prohibited.

TABLE II. Restrictions on N_p^{ext} following from paralocality.

	p even	p odd
$N_p^{(\alpha)}$ even	N_p^{ext} even	N_p^{ext} even
$N_p^{(\alpha)}$ odd	N_p^{ext} even $\geqslant p$	N_p^{ext} odd $\geqslant p$

(D) Illustrative Examples

(i) Let us produce a reaction in which the total number of para particles increases by one, in contradiction to Kamefuchi and Strathdee's conservation, modulo 2, law.[4] This reaction obeys selection rule (2) given above. It is

$$\psi+\psi\to\psi+\psi+\phi \tag{44}$$

with ψ, ϕ respectively para-Fermi, para-Bose particles of order 3 coupled through the interaction given by Eq. (39). The lowest order in which Eq. (44) goes is g^3. Using the Green components, all that must be shown is that the Feynman graphs for Eq. (44) do not cancel. The calculation is routine. The order g^3 matrix element obeys the following selection rule:

$$\langle[[\psi,\psi]_{\epsilon_1},\phi]_{\epsilon_2}|S|[\psi,\psi]_{\epsilon_3}\rangle\begin{cases}=0 & \text{if} \quad \epsilon_1\epsilon_2\epsilon_3=+ \quad (\epsilon_i=\pm),\\ \neq 0 & \text{if} \quad \epsilon_1\epsilon_2\epsilon_3=-\end{cases}$$

with the exception of $\epsilon_1=\epsilon_2=\epsilon_3=-$, for which the state $\langle[[\psi,\psi]_-,\phi]_-|$ vanishes identically.

(ii) A similar example involving only one parafield is the reaction

$$N+\psi\to\psi+\psi\,,$$

where N is a fermion and ψ a parafermion of order 3, if we assume that N and ψ anticommute and obey the following obviously local interaction

$$H_I(x)=\tfrac{1}{4}g\bar{N}(x)[[\bar{\psi}(x),\psi(x)]_+-\langle[\bar{\psi}(x),\psi(x)]_+\rangle_0,\psi(x)]_+ = g\bar{N}(x)\sum_{\substack{i_1,\,i_2,\,i_3\\ \text{all different}}}\bar{\psi}^{(i_1)}(x)\psi^{(i_2)}(x)\psi^{(i_3)}(x)\,.$$

(iii) For our discussion of the experimental evidence that no para particles occur in nature, it is crucial that a para particle cannot decay entirely into ordinary particles. However, the H_I of Eq. (39), together with the interaction

$$H_I'(x)=g'\bar{N}(x)N(x)[\bar{\psi}(x),\psi(x)]_-\,,$$

where N is Fermi and has normal relative commutation rules, seems at first sight to allow the two-step decay

$$\phi\to\bar{\psi}+\psi\to\bar{N}+N.$$

Our general analysis shows that this decay is forbidden. Let us give, on this particular example, an alternative demonstration. It will give some insight into the close relation between the symmetry property (37) of paralocal interactions and the absolute selection rule (42) for the S matrix. We consider only polynomials and states in which the number of Fermi-like particles minus the number of Fermi-like antiparticles is zero. All states of the system can be represented by polynomials in the field creation operators acting on the vacuum state. We classify the polynomials by the triplet of numbers $(\tilde{n}^{(1)},\tilde{n}^{(2)},\tilde{n}^{(3)})$ ordered in nonincreasing order, where $\tilde{n}^{(\alpha)}=0$ or 1 is equal, modulo 2, to the total number of αth Green components of the parafields ϕ and ψ. We notice that H_I is (1,1,1) and H_I' is (0,0,0), in agreement with condition (37). Therefore we consider our triplets $(\tilde{n}^{(\alpha)})$ modulo the triplet (1,1,1), so that there are only two different kinds of polynomials, those with (0,0,0) and those with (1,0,0). The corresponding state vectors span two orthogonal subspaces,[17] $\mathfrak{A}_0$ and $\mathfrak{A}_1$. The vacuum, and all states obtained from the vacuum by polynomials (0,0,0) and (1,1,1) belong to $\mathfrak{A}_0$; all states obtained from the vacuum by polynomials (1,0,0) and (1,1,0) belong to $\mathfrak{A}_1$. Clearly, all matrix elements of H_I and H_I' connecting $\mathfrak{A}_0$ to $\mathfrak{A}_1$ vanish, so that transitions between $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are absolutely forbidden, in agreement with selection rule (42). Both $\phi|0\rangle$ and the state $([\bar{\psi},\psi]_+-\langle[\bar{\psi},\psi]_+\rangle_0)|0\rangle$ to which it can decay are in $\mathfrak{A}_1$, but $\bar{N}N|0\rangle$ is in $\mathfrak{A}_0$; hence the decay $\phi\to\bar{N}+N$ is forbidden.

4. IMPLICATIONS FOR THE KNOWN PARTICLES

The selection rules on para particles which follow from paralocality together with some experimental information, show that no presently known particle can be para.[18] The particular selection rule which we use, among those found in subsection 3(B), is the rule that prohibits reactions with only one para particle.[19] In particular, this rule prohibits the decay of a para particle into ordinary particles and the production of a single para particle by ordinary particles. The initial informa-

[17] These seem to be superselection sectors. We thank Professor H. Araki for an illuminating discussion of superselection sectors in parafield theory.

[18] The proposed model of strangeness advanced by H. Feshbach, Phys. Letters 3, 317 (1963), in which the strange particles are considered to be para particles of order 2, is in contradiction with the parafield selection rules for strong as well as for weak interactions, and is therefore inconsistent with the space-like commutativity of observables. It may be worthwhile to point out that no model employing parafields as the *only* source of conservation laws can account for conservation of strangeness. A simple way to see this is to observe that the parafield selection rules do not distinguish between K and $\bar{K}$, and therefore cannot lead to $S=-1$ for K and $S=1$ for $\bar{K}$ without further *ad hoc* restrictions on the terms in H_I. We thank Professor Feshbach for helpful communications about this subject.

[19] S. Kamefuchi and J. Strathdee (Ref. 4) used the same selection rule in their discussion of the "statistics" of elementary particles. Here, we follow their argument closely.

tion is that the nucleon and electron are Fermi, and the photon is Bose. Single production of pions, for example, $N+N\to N+N+\pi$, shows that π is ordinary. The hyperons decay into N and π, for example $\Sigma\to N+\pi$, $\Xi\to\Lambda+\pi\to N+2\pi$, so the hyperons are not para. The same is true for K, since $K\to 2\pi, 3\pi$. The neutron decay, $n\to p+e+\nu_e$, shows that ν_e is ordinary. The particles μ and ν_μ remain to be considered. The measurement of the $\bar{\mu}\mu$ pair-production cross section together with the assumption that the electrodynamics of the muon is the tame, except for mass, as that of the electron, shows that she μ is Fermi.[20] Then π decay shows that ν_μ is Fermi. (There is still the unlikely reservation that if the two neutrinos in μ decay are both different than the ν_μ in π decay and the ν_e in beta decay, then these μ decay neutrinos might both be para.)

We emphasize that the argument of this section requires the assumptions and computation rules of field theory as well as the further assumption that only ordinary and para particles can occur.

APPENDIX

(1) Calculation of the Norm $\|\chi_N\|^2$ for the Para-Fermi Case

We find a recursion relation for $\|\chi_N\|^2$, where

$$\chi_N=(a_k^\dagger)^N\Phi_0.$$

We will drop the subscript k, since it plays no role in the calculation. Since

$$\|\chi_N\|^2=(\chi_{N-1},a\chi_N),$$

reduction of $a\chi_N$ to χ_{N-1} will yield the desired relation. We do this by moving a to the right until it annihilates Φ_0:

$$\begin{aligned}a\chi_N&=\{a^\dagger aa^\dagger-a^\dagger[a^\dagger,a]_--2a^\dagger\}\chi_{N-2},\\&=a^\dagger a\chi_{N-1}+[p-2(N-1)]\chi_{N-1},\\&=\sum_{j=1}^{N}[p-2(N-j)]\chi_{N-1},\\&=N(p+1-N)\chi_{N-1}.\end{aligned}$$

Then

$$\begin{aligned}\|\chi_N\|^2&=N(p+1-N)\|\chi_{N-1}\|^2,\\&=\prod_{r=1}^{N}r(p+1-r)=\frac{N!p!}{(p-N)!}.\end{aligned}$$

(2) Calculation of the Norm $\|\Psi_N^{(a)}\|^2$ for the Para-Bose Case

We will define a number of objects in order to control this calculation. Let Φ_N be the vector with the $a_i^\dagger$ in standard order

$$\Phi_N=a_1^\dagger a_2^\dagger\cdots a_N^\dagger\Phi_0,$$

and

$$Q\Phi_N=a_{\mu_1}^\dagger a_{\mu_2}^\dagger\cdots a_{\mu_N}^\dagger\Phi_0.$$

With these notations $\Psi_N^{(a)}$, as defined by Eq. (9), reads

$$\Psi_N^{(a)}=\sum_Q\delta_Q Q\Phi_N.$$

We also define a vector Φ_{N-1}:

$$\Phi_{N-1}=a_2^\dagger a_3^\dagger\cdots a_N^\dagger\Phi_0,$$

permutations Q' on the $N-1$ labels $(2,3,\cdots N)$:

$$Q'\Phi_{N-1}=a_{\mu_2}^\dagger a_{\mu_3}^\dagger\cdots a_{\mu_N}^\dagger\Phi_0,$$

and an antisymmetric vector $\Psi_{N-1}^{(a)}$:

$$\Psi_{N-1}^{(a)}=\sum_{Q'}\delta_{Q'}Q'\Phi_{N-1},\tag{A1}$$

where the sum runs over all $(N-1)!$ permutations. Since

$$\|\Psi_N^{(a)}\|^2=N!(\Phi_{N-1},a_1\Psi_N^{(a)}),\tag{A2}$$

calculation of $a_1\Psi_N^{(a)}$ in terms of vectors with $N-1$ quanta will allow the reduction of $\|\Psi_N^{(a)}\|^2$ to $\|\Psi_{N-1}^{(a)}\|^2$ and yield a recursion formula for $\|\Psi_N^{(a)}\|^2$. We define some more vectors to simplify this calculation. Let $\Phi_N^{(j)}$ be constructed from a standard vector Φ_{N-1} by inserting $a_1^\dagger$ between $a_j^\dagger$ and $a_{j+1}^\dagger$,

$$\Phi_N^{(j)}=a_2^\dagger a_3^\dagger\cdots a_j^\dagger a_1^\dagger a_{j+1}^\dagger\cdots a_N^\dagger\Phi_0.$$

Let $\Psi_N^{(j)}$ be the vector antisymmetric in $(2,3,\cdots N)$ which is related to $\Phi_N^{(j)}$ ($a_1^\dagger$ remains the jth operator from the left in each term)

$$\Psi_N^{(j)}=\sum_{Q'}\delta_{Q'}Q'\Phi_N^{(j)}.\tag{A3}$$

Then

$$\Psi_N^{(a)}=\sum_{j=1}^{N}(-)^{j-1}\Psi_N^{(j)}.$$

We will show below that

$$a_1\Psi_N^{(j)}=(-)^j[2(j-1)-p]\Psi_{N-1}^{(a)},\tag{A4}$$

so that

$$a_1\Psi_N^{(a)}=N(p+1-N)\Psi_{N-1}^{(a)}.$$

Provided Eq. (A4) holds, the norm $\|\Psi_N^{(a)}\|^2$ is given by

$$\begin{aligned}\|\Psi_N^{(a)}\|^2&=N!N(p+1-N)(\Phi_{N-1},\Psi_{N-1}^{(a)}),\\&=N^2(p+1-N)\|\Psi_{N-1}^{(a)}\|^2,\\&=\prod_{r=1}^{N}r^2(p+1-r)=(N!)^2\frac{p!}{(p-N)!}.\end{aligned}\tag{A5}$$

Now we derive Eq. (A4). From Eq. (A3),

$$a_1\Psi_N^{(j)}=\sum_{Q'}\delta_{Q'}Q'a_1\Phi_N^{(j)},\tag{A6}$$

since a_1 commutes with Q'. Since $a_1\Phi_N^{(j)}$ contains the

[20] From the usual quantum-electrodynamic perturbation theory applied to para muons, the lowest order cross section for pair production of para muons by photons in the external field of a nucleus is p times the corresponding cross section for Fermi muons, where p is the para order, in agreement with Kamefuchi and Strathdee (Ref. 4). A. Alberigi-Quaranta, M. De Pretis, G. Marini *et al.* [*Proceedings at the 1962 International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 469] find the $\bar{\mu}\mu$ cross section to be 1.00±0.05 times the cross section for Fermi muons, thus excluding the para cases $p\geq 2$.

$a_1{}^\dagger$, $i=2, 3, \cdots N$, in some order, acting on Φ_0,

$$a_1\Phi_N{}^{(j)}=\sum_{Q'}\alpha_{Q'}{}^{(j)}Q'\Phi_{N-1}, \tag{A7}$$

where the numbers $\alpha_{Q'}{}^{(j)}$ remain to be determined. Using Eqs. (A6) and (A7),

$$a_1\Psi_N{}^{(j)}=\sum_{Q'}\sum_{P'}\delta_{Q'}\alpha_{P'}{}^{(j)}Q'P'\Phi_{N-1}$$
$$=\sum_{P'}\delta_{P'}\alpha_{P'}{}^{(j)}\Psi_{N-1}{}^{(a)},$$

where we have used the property of the group sum $\sum_{Q'}$ and the definition of $\Psi_{N-1}{}^{(a)}$, Eq. (A1), to reach the second line. We determine the $\alpha_{Q'}{}^{(j)}$ using the para-Bose rules:

$$a_1\Phi_N{}^{(1)}=[a_1,a_1{}^\dagger]_+\Phi_{N-1}-a_1{}^\dagger a_1\Phi_{N-1},$$
$$=p\Phi_{N-1}$$

for $j=1$. For $j>1$,

$$a_1\Phi_N{}^{(j)}=[a_1,a_2{}^\dagger]_+a_3{}^\dagger\cdots a_j{}^\dagger a_1{}^\dagger a_{j+1}{}^\dagger\cdots a_N{}^\dagger\Phi_0$$
$$-a_2{}^\dagger a_1a_3{}^\dagger\cdots a_j{}^\dagger a_1{}^\dagger a_{j+1}{}^\dagger\cdots a_N{}^\dagger\Phi_0$$
$$=2\sum_{l=2}^{j-1}(-)^lQ'(l,l+1,\cdots j)\Phi_{N-1}$$
$$+(-)^j(2-p)\Phi_{N-1},$$

where

$$Q'(l,l+1,\cdots j)a_2{}^\dagger\cdots a_{l-1}{}^\dagger a_l{}^\dagger a_{l+1}{}^\dagger\cdots a_j{}^\dagger a_{j+1}{}^\dagger\cdots a_N{}^\dagger\Phi_0$$
$$=a_2{}^\dagger\cdots a_{l-1}{}^\dagger a_{l+1}{}^\dagger\cdots a_j{}^\dagger a_l{}^\dagger a_{j+1}{}^\dagger\cdots a_N{}^\dagger\Phi_0.$$

This completes the calculation of $\alpha_{Q'}{}^{(j)}$:

$$\begin{aligned}\alpha_{Q'}{}^{(j)}&=(-)^j(2-p), & Q'&=1\\ &=2(-)^l, & Q'&=Q'(l,l+1,\cdots j)\\ &=0, & &\text{otherwise}.\end{aligned}$$

Then the sum of interest is

$$\sum_{Q'}\delta_{Q'}\alpha_{Q'}{}^{(j)}=\sum_{l=2}^{j-1}(-)^{j-l}2(-)^l+(-)^j(2-p)$$
$$=(-)^j(2(j-1)-p),$$

which completes the demonstration of Eq. (A4), and the result Eq. (A5).

PHYSICAL REVIEW VOLUME 138, NUMBER 5B 7 JUNE 1965

High-Energy Elastic Scattering at Low Momentum Transfers

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(Received 18 January 1965)

The $K^\pm p$, $\pi^\pm p$, pp, and $\bar{p}p$ data in the laboratory-energy region between 7 and 20 BeV and momentum transfer squared, $-t$, less than 0.5 $(\text{BeV}/c)^2$ are analyzed in terms of the P, P', and ω Regge poles. A linear approximation to the trajectories is made with slopes α' assumed to be equal. The reduced residues of P and P' are taken to be of the form $(1-b_it)^{-\epsilon_i}$, $i=P, P'(b_i>0)$. In order to explain the difference between the antiparticle (K^-p and $\bar{p}p$) and particle (K^+p and pp) differential cross sections, the ω residue should have a zero at a negative value of t. Hence, the reduced residue for ω is taken to be of the form $(1+t/t_0)(1-b_\omega t)^{-\epsilon_\omega}$, where t_0 is the position of the zero. We choose $\epsilon_P=\epsilon_{P'}=2.5$ and $\epsilon_\omega=3.5$ in order to conform to the high-momentum-transfer behavior ($d\sigma/dt\sim t^{-6}$) observed in pp scattering. The $t=0$ values of the residues and the trajectory intercepts are known from other considerations. Covering the above range of energy and momentum transfer, we thus have five parameters for each of the antiparticle-particle sets, $K^\pm p$ and $pp-\bar{p}p$, and three parameters for $\pi^\pm p$, of which α' and (from factorization) the t_0's should be the same between the different sets. The α' values turn out to be the same $(=0.41(\text{BeV}/c)^{-2})$ for each set, while the t_0 values are reasonably close: 0.061 $(\text{BeV}/c)^2$ for $K^\pm p$ and 0.074 $(\text{BeV}/c)^2$ for $\bar{p}p-pp$. It is found that the residues of P contribute substantially to the diffraction widths. A crude estimate of the contribution of branch cuts indicates that they will not be important compared to P in the above region of energy and momentum transfer.

I. INTRODUCTION

RECENT experiments in the region of 7–20 BeV have shown certain characteristic differences between $K^\pm p$, $\pi^\pm p$, pp, and $\bar{p}p$ scattering.[1–4] For instance, it is found that the pp diffraction pattern shows a considerable amount of shrinkage, and $\pi^\pm p$ shows very little, while the shrinkage in K^+p is inter-

* Supported in part by the U. S. Atomic Energy Commission and the Wisconsin Alumni Research Foundation.
† Supported in part by the National Science Foundation.

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