# Low-Energy $\pi$-$\Lambda$ Interaction, and the $\Sigma\Lambda\pi$ Coupling Constant

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A semiphenomenological study of the dynamics of the first hyperon resonance $Y_1^*$ (1385) has been made in a one-channel approximation by the techniques of partial-wave dispersion relations. Contributions from $\Sigma$ and $Y_1^*$ exchange, and the exchange of a low-energy $s$-wave $\pi$-$\pi$ pair, to dispersion relations for the $P_{3/2}$ $\pi$-$\Lambda$ scattering amplitude were evaluated on the physical and crossed physical cuts. The coupling constant $g_{\Sigma\Lambda\pi}$ and a parameter related to the $P_{3/2}$ $\pi$-$\Lambda$ scattering length were varied so that the amplitude calculated from the dispersion relation was consistent with the input resonant amplitude on both cuts simultaneously. This could be achieved for $g_{\Sigma\Lambda\pi}^2=10.9\pm0.3$. The dynamics of $Y_1^*$ are found to depend almost as much on the exchange of the $s$-wave $\pi$-$\pi$ pair as on $\Sigma$ exchange.

## 1. INTRODUCTION

THE existence of a wealth of accurate experimental pion-nucleon scattering data, together with the fact that $\pi$-$N$ scattering may be treated as an elastic problem to a good approximation, at least to medium energies, has doubtless contributed to the fact that the majority of successful applications of dispersion relations in the past have been in the field of pion physics. In strange-particle physics, however, there is a paucity of experimental data, and few, if any, of the strange-particle coupling constants are known from experiment with confidence. We present here a calculation of one of these coupling constants $g_{\Sigma\Lambda\pi}$ by the phenomenological application of partial-wave dispersion relations to a system which, we have reason to hope, may be treated adequately in a single-channel approximation, namely, the production of the first hyperon resonance $Y_1^*(1385)$ [1] by $p_{3/2}$ pion-lambda scattering.

The quantum numbers of $Y_1^*(1385)$, i.e.,[2] $s=-1$, $B=+1$, $J=\frac{3}{2}$, allow it to be coupled to $p$-wave boson-baryon systems of strangeness minus one. The three lowest-lying such states are $\pi$-$\Lambda$, $\pi$-$\Sigma$, and $\bar{K}$-$N$. Conservation of energy allows decay only to the two former channels. The branching ratio for the $\pi$-$\Lambda$ channel is greater than 95%.[2] The branching ratio for the $\pi$-$\Sigma$ channel, although correspondingly small, is still the subject of some doubt.[3] We interpret the branching ratios as experimental evidence that $Y_1^*$ is weakly coupled to the $\pi$-$\Sigma$ system, and that the forces giving rise to the resonance come from the $\pi$-$\Lambda$ channel. Possible theoretical reasons why this should be so are briefly mentioned in Sec. 9. The $\bar{K}$-$N$ channel, although closed, could influence $Y_1^*$ if there were appreciable $p$-wave $\bar{K}$-$N$ scattering. However, $K^-p$ scattering data are well-fitted by the assumption of a pure $s$-wave interaction to kaon momenta of 200 MeV/$c$.[4] In view of these facts we have analyzed the dynamics of $Y_1^*$ assuming it to be a pure $p_{3/2}$ resonance in the $\pi$-$\Lambda$ channel.

The results of the calculation are (i) that a self-consistent resonance can be achieved for $g_{\Sigma\Lambda\pi}^2=10.9\pm0.3$, and (ii) that the dynamics of the resonance depend on the exchange of a low-energy $T=0$ $s$-wave $\pi$-$\pi$ pair as much as on the exchange of a $\Sigma$ hyperon.

### Summary of Method

Partial-wave dispersion relations for the amplitudes $F_{l\pm}(s)\equiv f_{l\pm}(s)/q^{2l}(s)$ were evaluated on both the physical and crossed physical cuts. A three-parameter form of the resonant $p_{3/2}$ amplitude was used in the rescattering integral. Two of the parameters were fixed by assuming the position and width of $Y_1^*$, and the third parameter $a$, which was related to the $p_{3/2}$ scattering length, was free. In terms of $a$ and $g_{\Sigma\Lambda\pi}$, contributions to Re$p_{3/2}$ were calculated from the exchange of $\Sigma$ and $Y_1^*$ in the $u$ channel, and from the exchange of a low-energy $T=0$, $J=0$ $\pi$-$\pi$ pair in the $t$ channel. In the latter case the form of the low-energy $s$-wave phase shifts for $\pi+\pi\rightarrow\pi+\pi$ was assumed. The two parameters $a$ and $g_{\Sigma\Lambda\pi}^2$ were varied so that the calculated real parts were consistent with those given by the resonance formula on both the physical and the crossed physical cuts simultaneously. These parameters were then used to calculate $p_{1/2}$ contributions to the $p_{3/2}$ amplitude, arising from the crossed channels, and the two parameters again varied to achieve consistency. Good self-consistent solutions could be found for $g_{\Sigma\Lambda\pi}^2=10.9\pm0.3$ and $a=0.33\pm0.02$.

### Contents

In Sec. 2 we present the notation and kinematics to be used throughout the rest of the paper, and in Sec. 3 we discuss the partial-wave amplitudes $F_{l\pm}(s)$ and the dispersion relation. In Sec. 4 we describe the treatment of the physical region, and in Sec. 5 we discuss the cal-

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[1] M. Alston and M. Ferro-Luzzi, Rev. Mod. Phys. 3, 416 (1961); see also Ref. 2 for other references.

[2] A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Roos, Rev. Mod. Phys. 36, 977 (1964).

[3] Measurements of the branching ratio into the $\pi$-$\Sigma$ channel have been made by several groups (see Ref. 2 for the complete list) and up to the end of 1964 the value quoted was from 1 to 4% ($\pm$3%). However, in a recent measurement, D. O. Huwe, University of California Radiation Laboratory Report 11291, 1964 (unpublished), has reported a value of 9$\pm$4%.

[4] R. H. Dalitz, Rev. Mod. Phys. 33, 471 (1961).

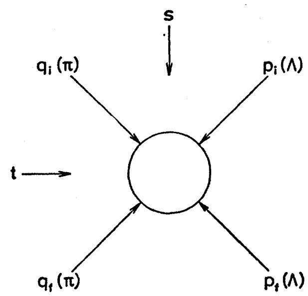


FIG. 1. $\pi$-$\Lambda$ scattering and crossed processes.

culation of the unphysical-cut terms as functions of the parameters $a$ and $g_{\Sigma\Lambda\pi}$. Details of the calculational procedure are presented in Sec. 6, followed in Sec. 7 by a discussion of the short-range forces in the $p_{3/2}$ and $p_{1/2}$ channels. In this latter section we also estimate the effect of neglecting $s$-waves. Finally, in Sec. 8 we compare the value of $g_{\Sigma\Lambda\pi}^2$ obtained here with other published values, and in Sec. 9 we summarize the results obtained and briefly comment on the implications for other strange-particle processes.

## 2. NOTATION AND KINEMATICS

The notation and kinematics of $\pi$-$\Lambda$ scattering are identical with those of the $(+)$ charge combination in $\pi$-$N$ scattering.[5] The scattering is in a pure $T=1$ isotopic spin state. We present here the relevant formulas that we shall need in later sections.

We take $p_i(q_i)$ to be the initial four-momentum of the lambda (pion) and $p_f(q_f)$ to be the corresponding final four-momentum. These are shown in Fig. 1 which represents the three processes

$$\pi+\Lambda\rightarrow\pi+\Lambda\,, \qquad \text{(I)}$$

$$\pi+\Lambda\rightarrow\pi+\Lambda\,, \qquad \text{(II)}$$

$$\pi+\pi\rightarrow\Lambda+\bar{\Lambda}\,. \qquad \text{(III)}$$

The four-momenta are formally written as ingoing. Thus, for any of the three scattering processes two of the momenta are negative. We define the usual Lorentz invariants,

$$\begin{aligned} s&=-(q_i+p_i)^2=-(q_f+p_f)^2\,,\\ t&=-(q_i+q_f)^2=-(p_i+p_f)^2\,,\\ u&=-(q_i+p_f)^2=-(q_f+p_i)^2\,, \end{aligned} \qquad (1)$$

where, by virtue of momentum conservation,

$$s+t+u=2(\Lambda^2+\mu^2)\,. \qquad (2)$$

In channel I

$$s\equiv w^2=[(\Lambda^2+q^2)^{1/2}+(\mu^2+q^2)^{1/2}]^2\,, \qquad (3)$$

and

$$t=-2q^2(s)[1-\cos\vartheta_s]\,, \qquad (4)$$

where $q(s)$ is the magnitude of the c.m. momentum and $\vartheta_s$ is the scattering angle. $q^2(s)$ may be written

$$4sq^2(s)=[s-(\Lambda+\mu)^2][s-(\Lambda-\mu)^2]\,. \qquad (5)$$

The $S$ matrix for channel I may be written[5]

$$S_{fi}=\delta_{fi}-(2\pi)^4 i\delta^{(4)}(q_i+p_i+q_f+p_f)\times\left[\frac{\Lambda^2}{4\omega_i\omega_fE_iE_f}\right]^{1/2}\tau_{fi}\,, \qquad (6)$$

where

$$\tau_{fi}=\bar{u}_f(p_f)T_{fi}u_i(p_i)\,.$$

$\omega(E)$ is the c.m. energy of the pion ($\Lambda$). The problem of spin is treated by writing the $T$-matrix element $T_{fi}$ in terms of two scalar invariant amplitudes $A_{fi}$, $B_{fi}$.

$$T_{fi}=[-A_{fi}+\tfrac{1}{2}i\gamma\cdot(q_i+q_f)B_{fi}]\,. \qquad (7)$$

The amplitudes $A$ and $B$ satisfy the Mandelstam representation.[6] In the c.m. system the differential cross section may be written

$$\left(\frac{d\sigma}{d\Omega}\right)_{fi}=\sum\left|\left\langle f\left|f_1+\frac{(\boldsymbol{\sigma}\cdot\mathbf{q}_i)(\boldsymbol{\sigma}\cdot\mathbf{q}_f)}{q_iq_f}f_2\right|i\right\rangle\right|^2\,, \qquad (8)$$

where the matrix element is taken between two-component spinors and the expression is summed over final spin states and averaged over initial spin states. The helicity amplitudes[7] $f_1$ and $f_2$ are related to the amplitudes $A$ and $B$ by

$$f_1=[((w+\Lambda)^2-\mu^2)/16\pi s][A+(w-\Lambda)B]\,, \qquad (9a)$$

and

$$f_2=[((w-\Lambda)^2-\mu^2)/16\pi s][-A+(w+\Lambda)B]\,. \qquad (9b)$$

The inverse equations are

$$A=8\pi w\left[\frac{w+\Lambda}{(w+\Lambda)^2-\mu^2}f_1-\frac{w-\Lambda}{(w-\Lambda)^2-\mu^2}f_2\right]\,, \qquad (10a)$$

and

$$B=8\pi w\left[\frac{1}{(w+\Lambda)^2-\mu^2}f_1+\frac{1}{(w-\Lambda)^2-\mu^2}f_2\right]\,. \qquad (10b)$$

The amplitudes $f_1$ and $f_2$ are related to the phase shifts by

$$f_1=\sum_{l=0}^{\infty}f_{l+}P_{l+1}'(x)-\sum_{l=2}^{\infty}f_{l-}P_{l-1}'(x)\,, \qquad (11a)$$

and

$$f_2=\sum_{l=1}^{\infty}[f_{l-}-f_{l+}]P_l'(x)\,, \qquad (11b)$$

where $x\equiv\cos\vartheta_s$ and

$$f_{l\pm}(s)=\exp[i\delta_{l\pm}(s)]\sin\delta_{l\pm}(s)/q(s) \qquad (12)$$

[5] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[6] S. Mandelstam, Phys. Rev. **112**, 1344 (1958): **115**, 1741, 1752 (1959).

[7] W. Jacob and G. C. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).

is the partial-wave amplitude for scattering in a state of total angular momentum $J=l\pm\frac{1}{2}$. The inverse of Eqs. (11) is

$$f_{l\pm}(s)=\frac{1}{2}\int_{-1}^{+1}dx[f_1P_l(x)+f_2P_{l\pm1}(x)]. \qquad (13)$$

Finally, using Eqs. (9) and (13) we have

$$f_{l\pm}(s)=\frac{(w+\Lambda)^2-\mu^2}{16\pi s}[A_l+(w-\Lambda)B_l]$$
$$+\frac{(w-\Lambda)^2-\mu^2}{16\pi s}[-A_{l\pm1}+(w+\Lambda)B_{l\pm1}], \qquad (14)$$

where

$$[A_l(s);B_l(s)]=\frac{1}{2}\int_{-1}^{+1}dx[A(s,t);B(s,t)]P_l(x). \qquad (15)$$

### 3. PARTIAL-WAVE DISPERSION RELATIONS

The invariant amplitudes $[A(s,t);B(s,t)]$ satisfy the Mandelstam representation[6]

$$[A(s,t);B(s,t)]=\frac{[R_s;R_s']}{s-\Sigma^2}+\frac{[R_u;R_u']}{u-\Sigma^2}$$
$$+\frac{1}{\pi^2}\int_{(\Lambda+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\frac{[A_{13}(s',t');B_{13}(s',t')]}{(s'-s)(t'-t)} \qquad (16)$$

plus similar integrals in $(s,u)$ and $(u,t)$. The functions $[A_{jk};B_{jk}]$, $jk=13$, 23, 12, are real weight functions, and

$$R_s=R_u=g_{\Sigma\Lambda\pi}^2(\Sigma-\Lambda), \qquad (17a)$$

$$R_s'=-R_u'=-g_{\Sigma\Lambda\pi}^2. \qquad (17b)$$

$g_{\Sigma\Lambda\pi}$ is the renormalized pseudoscalar coupling constant (e.g., $g_{NN\pi}^2/4\pi\simeq15$). The analytic properties of $f_{l\pm}(s)$ are easily obtained[8,9] from Eqs. (14), (15), and (16) (see Fig. 2). Here as throughout this paper we use units such that $\hbar=\mu=c=1$. The singularities consist of the physical cut $s_0=80.8\leqslant s<\infty$, and a number of unphysical cuts. In particular the short cut labeled "Born" is

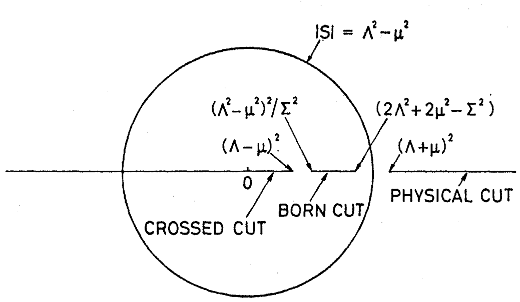


FIG. 2. The singularities of $f_{l\pm}(s)$ in the $s$ plane.

[8] S. W. McDowell, Phys. Rev. **116**, 774 (1960).

[9] J. Hamilton and T. D. Spearman, Ann. Phys. (N.Y.) **12**, 172 (1961).

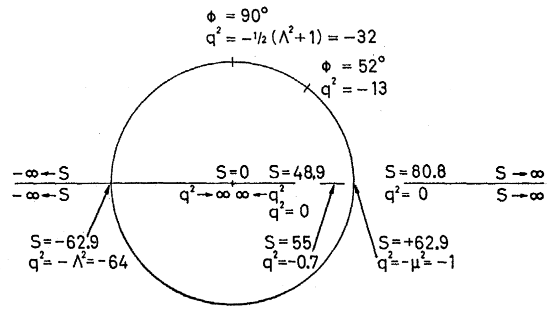


FIG. 3. Values of $q^2(s)$ on the cuts of $f_{l\pm}(s)$.

due to $\Sigma$ exchange in the $u$ channel, and the cut $-\infty<s\leqslant\bar{s}_0=48.9$ is due to the region $u\geqslant(\Lambda+\mu)^2$. The line $0<s\leqslant\bar{s}_0$ we call the crossed physical cut.[9] The circular cut $|s|=(\Lambda^2-\mu^2)$ is due to the region $t\geqslant4\mu^2$. In addition there is a pole at $s=\Sigma^2$ (not shown in Fig. 2) which occurs in the $p_{1/2}$ amplitude only.

In practice we shall work with the amplitude $F_{l\pm}(s)=f_{l\pm}(s)/q^{2l}(s)$. This function is bounded at the physical threshold and has a square-root branch point there. Thus we may write the dispersion relation

$$F_{l\pm}(s)=\frac{1}{\pi}\int_{s_0}^{\infty}ds'\frac{\mathrm{Im}F_{l\pm}(s')}{s'-s}+\frac{1}{\pi}\int_{0}^{\bar{s}_0}ds'\frac{\mathrm{Im}F_{l\pm}(s')}{s'-s}$$
$$+\frac{1}{2\pi i}\int_{\text{other unphysical cuts}}ds'\frac{\Delta F_{l\pm}(s')}{s'-s}. \qquad (18)$$

Equation (18) may be evaluated on the physical cut, where the first integral is principal-valued, and on the crossed cut, where the second integral is principal-valued. The advantages of using the amplitude $F_{l\pm}(s)$ have been stressed by Hamilton and co-workers.[10] On all the cuts $q^2(s)$ is real, and increases as $s$ moves further from the physical region (see Fig. 3). Thus the factor $q^{-2l}(s)$ suppresses contributions from those parts of the unphysical cuts where the interaction is unknown, i.e., the back of the circle, and the line $-\infty<s\leqslant0$. Furthermore, at high physical energies the factor $q^{-2l}(s)$ is small, and so the high-energy portion of the rescattering integral is also suppressed. This is particularly important in the situation treated here where only the low-energy resonance is known.

### 4. THE PHYSICAL REGION

In the absence of detailed experimental information about the $p_{3/2}$ $\pi$-$\Lambda$ scattering amplitude we are forced to use a resonance formula in the rescattering integrals and for the real parts of the amplitude. The most commonly used of such forms is the relativistic Breit-Wigner formula,

$$F_{l\pm}(s)=\gamma/[\omega_R-\omega-i\gamma q^{2l+1}(s)], \qquad (19)$$

[10] See, e.g., A. Donnachie, J. Hamilton, and A. T. Lea, Phys. Rev. **135**, B515 (1964).

where $\omega=(q^2+\mu^2)^{1/2}$, and the real parameters $\omega_R$ and $\gamma$ are related to the position and width of the resonance, respectively. However, the choice of physical amplitude should be such as to approximate the singularities shown in Fig. 2 and exhibited by the dispersion relation, Eq. (18). Investigation of the singularities of Eq. (19) to the left of $s_0$ shown that apart from the cut $-\infty<s\leqslant\bar{s}_0$, the function $F_{1+}(s)$ has a pair of complex-conjugate poles close to the real axes in the region $0\leqslant\mathrm{Re}s\leqslant\bar{s}_0$, and another pair of complex-conjugate poles on the circle at about $\phi\sim50°$ (for a $Y_1^*$ width of 50 MeV). This region of the circle corresponds to momentum transfers of $t\sim50\mu^2$, and since we believe low momentum transfers to be more important, because of the factor $q^{-2l}(s)$, such an amplitude was not used. A three-parameter formula suggested by Layson,[11] as an empirical fit to the experimental $N^*(1238)$ phase shifts turns out to be very satisfactory. For a $p$-wave resonance it is

$$F_p(s)=\gamma_1/[\omega_R-\omega-i\gamma_1q^3(s)],\tag{20a}$$

where

$$\gamma_1=\frac{2\Lambda\gamma_L}{\omega+\omega_R}\cdot\frac{a^3}{1+(qa)^2}.\tag{20b}$$

The parameters $\omega_R$ and $\gamma_L$ are related to the position and width of the resonance, and the parameter $a$ determines the over-all shape of the amplitude. The $p_{3/2}$ $\pi$-$\Lambda$ scattering length $a_3$ is given in terms of $a$ by

$$a_3\equiv\mathrm{Re}F_{1+}(s_0)=2\Lambda a^3\gamma_L/q_R^2.\tag{21}$$

The singularities of Eq. (20) are similar to those of the Breit-Wigner form, but now the poles on the circle are at $\phi\sim10°$ which corresponds to low momentum transfers $t\sim8\mu^2$. The exact positions and residues of these poles depend on the three parameters $\omega_R$, $\gamma_L$, and $a$. We may look at the Layson formula in another, more physical[12] way. In the Breit-Wigner formula, the centrifugal-barrier-penetration effect is described by the factor $q^3(s)$ in the denominator. In Layson's form, this is replaced by the factor

$$\frac{(qa)^3}{1+(qa)^2}\cdot\frac{1}{\omega+\omega_R},$$

which even in a relativistic process should be a considerable improvement, since a high-energy particle does not have to penetrate such an extensive potential barrier as a low-energy particle.

The parameter $a$ is of the order of the range of the interaction. Layson found a good fit to the experimental $N^*$ data for $a=0.714$. A Layson form with $a=0.718$ has also been found to be a good solution of the $p$-wave $\pi$-$N$ dispersion relation.[12] However, this value of $a$ cannot simply be taken over to the $Y_1^*$ calculation, as the following simple model shows. Consider the nucleon as a hard core with a surrounding pion cloud. Then, the radius of the cloud, i.e., the range of the pions, will be limited by the uncertainty principle $\Delta E\Delta R\sim\hbar c$, with $\Delta E=1\,\mu$ in the $\pi$-$N$ case we have $\Delta R\sim1$ ($\hbar=\mu=c=1$). For $\pi$-$\Lambda$ scattering, $\Delta E=\Sigma-\Lambda+\mu=1.5\,\mu$ and $\Delta R\sim\frac{2}{3}$. Thus from the value of $a$ found in $\pi$-$N$ scattering, i.e., $a=0.7$, we would expect for $\pi$-$\Lambda$ scattering $a\sim0.4$. In the actual calculation the parameter $a$ was left free and determined by the requirements of self-consistency.

## 5. CALCULATION OF THE UNPHYSICAL-CUT TERMS

### (i) The Born Terms

From the Mandelstam representation Eq. (16) we have

$$A_{\mathrm{Born}}(s,t)=g_{\Sigma\Lambda\pi}{}^2(\Sigma-\Lambda)/(u-\Sigma^2),\tag{22a}$$

$$B_{\mathrm{Born}}(s,t)=g_{\Sigma\Lambda\pi}{}^2/(u-\Sigma^2).\tag{22b}$$

Using Eqs. (22) in Eqs. (10), (14), and (15) of Sec. 2 we may easily calculate the contribution of the Born terms to the $\mathrm{Re}F_{l\pm}(s)$, on both the physical and crossed physical cuts,[13] in terms of the single parameter $g_{\Sigma\Lambda\pi}$. On the physical cut the Born terms for the $p_{3/2}$ amplitude are positive, and on the crossed cut they are negative.

### (ii) The Crossed Physical Integral

$F_{l\pm}(s)$ on the line $0<s\leqslant\bar{s}_0$ may be related to physical $\pi$-$\Lambda$ scattering by the use of the crossing relations

$$A(s,u,t)=A(u,s,t),\tag{23a}$$

$$B(s,u,t)=-B(u,s,t).\tag{23b}$$

The crossing relations (23) enable $f_1$ and $f_2$ for $0<s\leqslant\bar{s}_0$ to be calculated from the equations of Sec. 2. The crossed region contains contributions from all waves in the physical region $s\geqslant s_0$. Equation (13) then enables $\mathrm{Im}F_{l\pm}(s)$ to be calculated on the crossed cut, and hence the crossed-cut contribution to $\mathrm{Re}F_{l\pm}(s)$ on both cuts can be calculated. Provided only $p$ waves are crossed from the physical region, there is no trouble at $\bar{s}_0$ and $\mathrm{Im}F_{l\pm}(\bar{s}_0)$ is finite there.[14] The real parts on the crossed cut may be found by using the crossing relations applied to the real parts in the physical region. At the crossed threshold,

$$\mathrm{Re}F_{1+}(\bar{s}_0)\simeq\tfrac{1}{3}a_3+\tfrac{2}{3}a_1-a_0/6\Lambda^2,\tag{24a}$$

$$\mathrm{Re}F_{1-}(\bar{s}_0)\simeq\tfrac{4}{3}a_3-\tfrac{1}{3}a_1-2a_0/3\Lambda^2,\tag{24b}$$

where $a_{2J}$ is the scattering length for scattering in a state of total angular momentum $J$. Equations (24) show that the $s$-wave contributions to the real parts of the $p$-wave

[11] W. M. Layson, Nuovo Cimento **20**, 1207 (1961), and CERN Report, 1961 (unpublished).
[12] A. Donnachie and J. Hamilton, Phys. Rev. **133**, B1053 (1964).
[13] See Ref. 9 for the details.
[14] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962). This paper will be referred to as HMOV.

amplitudes at the crossed threshold are suppressed with respect to the $p$-wave contributions by factors of $\Lambda^2$. Our estimate of the $s$-wave scattering length, to be made in Sec. 7, shows that the $s$-wave contributions are negligible.

### (iii) $T=0$ $\pi$-$\pi$ Term

The amplitudes $A$ and $B$ have discontinuities on the circle $|s|=(\Lambda^2-\mu^2)$ which are given by the absorptive parts of the helicity amplitudes for $\pi+\pi\rightarrow\Lambda+\bar{\Lambda}$, as defined by Jacob and Wick,[7] which we denote by $h_\pm{}^J(t)$ in analogy with the amplitudes $f_\pm{}^J(t)$ introduced by Frazer and Fulco[15] for the $\pi$-$N$ problem. The absorptive parts are given by[15]

$$\begin{aligned}\mathrm{Im}A_{\pi\pi}(s,t)\\ =\frac{8\pi}{p_-{}^2}\sum_{J=0}^{\infty}(J+\tfrac{1}{2})(ip_-q_3)^J\bigg[P_J(\cos\vartheta_t)\,\mathrm{Im}h_\pm{}^J(t)\\ -\frac{\Lambda}{[J(J+1)]^{1/2}}\cos\vartheta_t P_J'(\cos\vartheta_t)\,\mathrm{Im}h_\pm{}^J(t)\bigg],\end{aligned}\tag{25a}$$

$$\begin{aligned}\mathrm{Im}B_{\pi\pi}(s,t)=8\pi\sum_{J=1}^{\infty}\frac{(J+\frac{1}{2})}{[J(J+1)]^{1/2}}\\ \times(ip_-q_3)^{J-1}P_J'(\cos\vartheta_t)\,\mathrm{Im}h_\pm{}^J(t),\end{aligned}\tag{25b}$$

where $q_3$ ($ip_-$) is the pion ($\Lambda$) c.m. momentum in the channel $\pi+\pi\rightarrow\Lambda+\bar{\Lambda}$. They are given by

$$\begin{aligned}q_3{}^2&=(\tfrac{1}{4}t-\mu^2),\\ p_-{}^2&=(\Lambda^2-\tfrac{1}{4}t).\end{aligned}\tag{26}$$

Also,

$$\cos\vartheta_t=(s-p_-{}^2+q_3{}^2)/(2ip_-q_3)\tag{27}$$

is the channel-III scattering angle.

Bose statistics applied to the initial-state pions shows that isospin $T=0$ occurs only with even $J$ values. The Legendre expansions (25) converge only on that arc of the circle $|\phi|\leqslant 52°$, which corresponds to $4\mu^2\leqslant t\leqslant 52\mu^2$. Thus we require $\mathrm{Im}h_+{}^J(t)$ for even $J$, and $t$ in the range $4\mu^2\leqslant t\leqslant 52\mu^2$. Oades[16] has shown that $T=0$, $J=2$ $\pi$-$\pi$ scattering is not strong at these energies. Therefore we consider only $s$-wave $\pi$-$\pi$ scattering and take over the results of HMOV,[14] who determined the low-energy $T=0$ $s$-wave $\pi$-$\pi$ phase shifts from an analysis of low-energy $s$-wave $\pi$-$N$ scattering data. We shall discuss the use of their phase shifts in Sec. 5(iv).

If we neglect $J\geqslant 2$, Eqs. (25) become

$$\mathrm{Im}A_{\pi\pi}(s,t)=[4\pi/(\Lambda^2-\tfrac{1}{4}t)]\,\mathrm{Im}h_+{}^0(t)\tag{28a}$$

and

$$\mathrm{Im}B_{\pi\pi}(s,t)=0.\tag{28b}$$

[15] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1609 (1960).
[16] G. C. Oades, Phys. Rev. **132**, 1277 (1963).

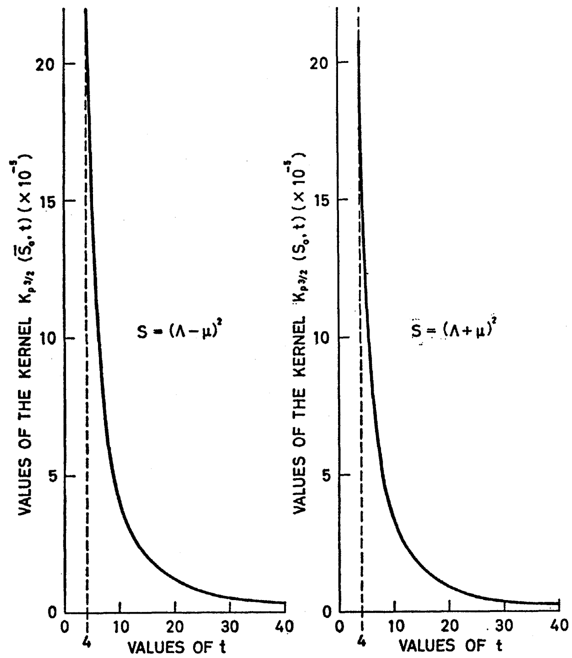


FIG. 4. The kernels $K_{1+}(s)$ at the physical and crossed physical threshold.

Equations (28) may now be used in Eqs. (14) and (15), for $s$ on the circle,

$$\mathrm{Im}F_{l\pm}(s)=\frac{1}{8sq^{2l}(s)}\int_{-1}^{+1}dx\,\mathrm{Im}h_+{}^0(t)H_{l\pm}(s,t),\tag{29a}$$

where

$$\begin{aligned}H_{l\pm}(s,t)=[(w+\Lambda)^2-\mu^2]P_l(x)\\ -[(w-\Lambda)^2-\mu^2]P_{l\pm1}(x).\end{aligned}\tag{29b}$$

Now on the circle

$$t=-2q^2(s)[1-x]\tag{30a}$$

and

$$q^2=-[\Lambda^2\sin^2(\tfrac{1}{2}\phi)+\mu^2\cos^2(\tfrac{1}{2}\phi)].\tag{30b}$$

Thus, contributions to $q^2$ on the circle come from the helicity amplitudes for all values of $t$ in the range

$$4\mu^2\leqslant t\leqslant t_{\max}(\phi),\tag{31}$$

where

$$t_{\max}(\phi)=-4q^2=4[\Lambda^2\sin^2(\tfrac{1}{2}\phi)+\mu^2\cos^2(\tfrac{1}{2}\phi)].\tag{32}$$

Hence we may change the variable of integration in Eq. (29) and, after some manipulation, write the circle contribution to $\mathrm{Re}F_{l\pm}(s)$ as

$$\mathrm{Re}F_{l\pm}(s)=\int_{4\mu^2}^{t_{\max}(\phi_{\max})}dt\,K_{l\pm}(s,t)\,\mathrm{Im}h_+{}^0(t).\tag{33}$$

The kernels $K_{l\pm}(s,t)$ are complicated functions of various kinematical factors, and $t_{\max}(\phi_{\max})$ is taken as $52\mu^2$ because of the convergence restriction.

The form of $K_{l\pm}(s,t)$ is shown in Fig. 4 at both thresholds for the $p_{3/2}$ case. The $p_{1/2}$ kernels are of similar form. It should be noted that the kernels are strongly peaked in the forward direction, i.e., about $t=4\mu^2$, and that they are slightly larger on the crossed cut than on the physical cut. The strong peaking means that low $t$ values will be far more important than high $t$ values in Eq. (33). This, in turn, means that the error in $\mathrm{Im}h_+{}^0(t)$ for high $t$ values should be unimportant, and that the precise value of the cutoff $t_{\max}$ ($\phi_{\max}$) should not significantly affect the circle contribution. This latter statement is found to be true in practice.

### (iv) The Helicity Amplitudes $h_+{}^0(t)$

The calculation of $\mathrm{Im}h_+{}^0(t)$ for $t\geqslant 4\mu^2$ may be effected in terms of the low-energy $T=0$ $s$-wave $\pi$-$\pi$ phase shifts and physical $\pi$-$\Lambda$ scattering parameters by means of the Omnès method.[17]

By examining the analytic properties of $[A_J(t),B_J(t)]$ in channel III,[15] we find that the amplitudes $h_\pm{}^J(t)$ are analytic in the cut plane $4\mu^2\leqslant t<\infty$ and $-\infty<t\leqslant a'$, where

$$a'\equiv(2\Lambda^2+2\mu^2-\Sigma^2)-(\Lambda^2-\mu^2)^2/\Sigma^2=2.59\,.$$

The function

$$g(t)=h_+{}^0(t)\exp[-i\delta_0{}^0(t)] \tag{34}$$

has only the cut $-\infty<t\leqslant a'$, and we may write the dispersion relation

$$g(t)=\frac{1}{\pi}\int_{-\infty}^{a'}dt'\frac{\mathrm{Im}g(t')}{t'-t} \tag{35}$$

or a subtracted form of this equation if the integral does not converge. To bring in the $T=0$, $J=0$ $\pi$-$\pi$ phase shifts we write

$$A(\nu)=((\nu+1)/\nu)^{1/2}\exp[i\delta_0{}^0]\sin\delta_0{}^0, \tag{36}$$

the invariant $T=0$ $\pi+\pi\rightarrow\pi+\pi$ amplitude, in the $N/D$ form.[18] Here $\nu=t/4-1$, and $A(\nu)$ is analytic in the cut plane $-\infty<\nu\leqslant-1$ and $0\leqslant\nu<\infty$; thus

$$A(\nu)=N(\nu)/D(\nu), \tag{37}$$

where

$$\begin{aligned}\mathrm{Im}N(\nu)&=D(\nu)\,\mathrm{Im}A(\nu), && \nu\leqslant-1\\ &=0 && \nu>-1\end{aligned} \tag{38}$$

$$\begin{aligned}\mathrm{Im}D(\nu)&=0, && \nu<0\\ &=-RN(\nu)(\nu/(\nu+1))^{1/2}, && \nu\geqslant0\end{aligned} \tag{39}$$

and $R=(\sigma_{\mathrm{tot}}/\sigma_{\mathrm{el}})_{J=0,T=0}$. Taking $R=1$ and making one subtraction in $D(\nu)$ at $\nu=\nu_0$ to normalize, we have

$$D(\nu)=1-\frac{(\nu-\nu_0)}{\pi}\int_0^\infty d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{N(\nu')}{(\nu'-\nu_0)(\nu'-\nu)}\,. \tag{40}$$

If we replace the left-hand cut by a single pole at $\nu=\nu_1$, then

$$\mathrm{Im}A(\nu)=-\pi\Gamma\delta(\nu-\nu_1),\quad \Gamma=\mathrm{const},\quad \nu_1<0 \tag{41}$$

and

$$N(\nu)=\Gamma D(\nu_1)/(\nu-\nu_1)\,. \tag{42}$$

We use $\nu_0=\nu_1$, i.e., $D(\nu_1)=1$. The solutions of (40) and (42) are

$$\mathrm{Re}D(\nu)=1+\frac{\Gamma}{\pi}\left[\frac{F(\nu)-F(\nu_1)}{\nu-\nu_1}-F'(\nu_1)\right], \tag{43}$$

$$\begin{aligned}\mathrm{Im}D(\nu)&=(-\Gamma/(\nu-\nu_1))(\nu/(\nu+1))^{1/2}, && \nu\geqslant0\\ &=0, && \nu<0\end{aligned} \tag{44}$$

where

$$\begin{aligned}F(\nu)&=M\ln\{|(1+M)/(1-M)|\}, && \nu\geqslant0\nu\leqslant-1,\\ &=M'\arctan(1/M'), && 0>\nu>-1,\end{aligned} \tag{45}$$

and

$$\begin{aligned}M&=[\nu/(1+\nu)]^{1/2},\\ M'&=[-\nu/(1+\nu)]^{1/2}.\end{aligned} \tag{46}$$

$D(\nu)$ has the phase $-\delta_0{}^0$ on $4\mu^2\leqslant t<16\mu^2$, and if we assume this is true for higher values of $t$ we can replace Eq. (34) by

$$g(t)=D(\nu)h_+{}^0(t)\,, \tag{47}$$

and Eq. (35) becomes

$$h_+{}^0(t)=\frac{1}{\pi D(\nu)}\int_{-\infty}^{a'}dt'\frac{D(\nu')\,\mathrm{Im}h_+{}^0(t')}{t'-t}, \tag{48}$$

and is formally true. Inspection of Eq. (48), however, shows that, because of the contribution of $Y_1{}^*$ (for $t\leqslant0$) to $\mathrm{Im}h_+{}^0(t')$, the equation requires two subtractions to ensure convergence. These are made at $t=0$. Thus, we finally have

$$h_+{}^0(t)=\frac{1}{D(\nu)}\left\{D(\nu=-1)\,\mathrm{Re}h_+{}^0(0)+t\frac{\partial}{\partial t''}[D(\nu'')\,\mathrm{Re}h_+{}^0(t'')]\bigg|_{t''=0}+\frac{1}{\pi}t^2\frac{\partial}{\partial t''}\int_{-\infty}^{a'}dt'\frac{D(\nu')\,\mathrm{Im}h_+{}^0(t')}{(t'-t'')(t'-t)}\bigg|_{t''=0}\right\}. \tag{49}$$

Values of $\mathrm{Im}h_+{}^0(t)$ for $t\geqslant4\mu^2$ may now be found from Eq. (49) if we know $\mathrm{Im}h_+{}^0(t)$ for $-\infty<t\leqslant a'$, the two subtraction constants $\mathrm{Re}h_+{}^0(0)$ and $\mathrm{Re}h_+{}^{0\prime}(0)$, and the $\pi$-$\pi$ pole parameters $\Gamma$ and $\nu_1$. For the latter we use the best-fit solution of HMOV, i.e., $\Gamma=15$ and $\nu_1=-30$. The form of the $\pi$-$\pi$ phase shift $\delta_0{}^0$ is shown in Fig. 5. The $\pi$-$\pi$ scattering length is $a_0{}^0=1.3$.

The region of convergence of the Legendre expansions limits us to finding $\mathrm{Im}h_+{}^0(t)$ for $a''\leqslant t\leqslant a'$, where $a''=-26.5\mu^2$. $\mathrm{Im}h_+{}^0(t)$ in this range may be found in terms of the two parameters $a$ and $g_{\Sigma\Lambda\pi}$ by writing dis-

[17] R. Omnès, Nuovo Cimento 8, 316 (1958).
[18] G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

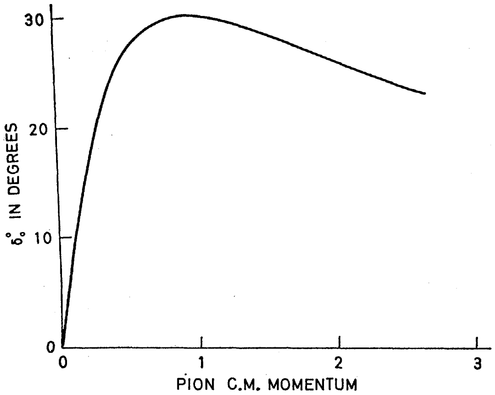


FIG. 5. The values of the $T=0$, $J=0$ $\pi$-$\pi$ phase shifts $\delta_0^0$.

persion relations for $A$ and $B$ at fixed momentum transfers $t\leqslant a'$ and using physical $\pi$-$\Lambda$ scattering parameters to evaluate them. The subtraction constants may be found in a similar manner in terms of forward $\pi$-$\Lambda$ scattering data by an extension of the method of Ball and Wong,[19] as used by Menotti,[20] in a similar calculation for $\pi$-$N$ scattering. In practice the dispersion relation for $A(s,t)$ requires a subtraction and we make this at $t=0$ in the amplitude

$$F(s,t)\equiv A(s,t)+(\Lambda/4p_-^2)B(s,t). \quad (50)$$

This has the practical advantage of removing the contributions of the $p$-wave scattering lengths to $\mathrm{Re}h_+^0(0)$, and the $d$-wave scattering lengths to $\mathrm{Re}h_+^{0\prime}(0)$. The scattering-length contributions to the two subtraction constants are then

$$\mathrm{Re}h_+^0(0)_{\mathrm{SL}}=\Lambda(\Lambda+1)a_0, \quad (51a)$$

$$\mathrm{Re}h_+^{0\prime}(0)_{\mathrm{SL}}=\Lambda(\Lambda+1)a_3+\tfrac{1}{2}\Lambda(\Lambda+1)a_1 -((\Lambda-1)/8\Lambda)a_0. \quad (51b)$$

Equations (51) show that repulsive $s$-wave $\pi$-$\Lambda$ scattering would increase $\mathrm{Re}h_+^{0\prime}(0)$ and attractive scattering would decrease it, but in both cases the effect is heavily suppressed compared with the contributions from the $p$-wave scattering lengths. Our estimate of $a_0$ in Sec. 7 shows that at most the $s$-wave contribution is 1% of $\mathrm{Re}h_+^{0\prime}(0)$. In the case of $\mathrm{Re}h_+^0(0)$, the $s$-wave contribution is not small and our estimate of $a_0$ shows that this term could be comparable to the Born and rescattering contributions to $\mathrm{Re}h_+^0(0)$. However, the contribution of $\mathrm{Re}h_+^0(0)$ to Eq. (49) is very small, and altering the value of $\mathrm{Re}h_+^0(0)$ by a factor as large as two does not appreciably affect the helicity amplitudes $\mathrm{Im}h_+^0(t)$.

### 6. CALCULATION PROCEDURE

The mass of $Y_1^*$ was taken[2] to be 1385 MeV. This fixes the value of the parameter $\omega_R=3.218$. The width of the resonance is a function of both $\gamma_L$ and $a$. For fixed values of the parameter $a$, values of $\gamma_L$ were calculated for a $Y_1^*$ width[2] of 50 MeV. With fixed values of $a$ and $g_{\Sigma\Lambda\pi}$, the Born term, the crossed integral, and the rescattering integral were evaluated on both the physical and crossed physical cuts by the methods of Secs. 4, 5(i), and 5(ii). The Layson formula was used in the rescattering integrals. In the crossed integrals only the $Y_1^*$ was used ($p_{1/2}$ contributions are discussed later). The difference between the real parts of the $p_{3/2}$ amplitude given by the Layson formula and the sum of the contributions from the Born term, rescattering, and crossed integrals was then formed. This quantity we call the discrepancy.[9] By varying $g_{\Sigma\Lambda\pi}$ and $a$, and ignoring the $\pi$-$\pi$ effect, we find that there exists no pair of parameters such that the discrepancy is a monotonic function of $s$ across both cuts simultaneously, i.e., could be represented by a short-range term. This is due to the sign of the Born terms being different on the two cuts, and shows that a $\pi$-$\pi$ effect must be invoked in $\pi$-$\Lambda$ scattering. We fit the discrepancy by the contribution from the $T=0$ $\pi$-$\pi$ term and a linear term which we use to represent the unknown, neglected, short-range part of the interaction. We compare our estimate of the short-range term with that calculated from a unitary sum rule in Sec. 7.

Thus, using the same fixed values of $a$ and $g_{\Sigma\Lambda\pi}$ values of $\mathrm{Im}h_+^0(t)$ for $a''\leqslant t\leqslant a'$ were calculated by the method of Sec. 5(iv). Again only the $Y_1^*$ was used in the rescattering integral, but here $s_{1/2}$ and $p_{1/2}$ contributions are negligible because of the restriction implied by Eq. (31). The double subtraction in Eq. (49) means that values of $\mathrm{Im}h_+^0(t)$ for larger negative values of $t$ are well suppressed. Furthermore, it turns out that the second subtraction constant is more important than the integral term in Eq. (49). The two subtraction constants $\mathrm{Re}h_+^0(0)$ and $\mathrm{Re}h_+^{0\prime}(0)$ were also calculated in terms of $g_{\Sigma\Lambda\pi}$ and $a$, using the $p_{3/2}$ scattering length $a_3$ from the Layson formula, Eq. (21). Equation (49) was then used to calculate the values of $\mathrm{Im}h_+^0(t)$ for $t\geqslant 4\mu^2$ in terms of the $s$-wave $T=0$ $\pi$-$\pi$ phase shifts. Finally, Eq. (33) gives the $T=0$ $\pi$-$\pi$ contribution from the front of the circle to $\mathrm{Re}F_{1+}(s)$ on both the physical and crossed physical cuts.

The parameters $a$ and $g_{\Sigma\Lambda\pi}$ were then varied to make the difference between the discrepancies and the $T=0$ $\pi$-$\pi$ contribution a monotonic function of $s$ across both cuts simultaneously. In practice this condition was difficult to achieve near $s=s_0$ because, as Eq. (24a) shows, the $p_{1/2}$ contribution there is not negligible. Furthermore, the second subtraction constant $\mathrm{Re}h_+^{0\prime}(0)$ gives an important contribution to Eq. (49), and, as Eq. (51b) shows, the $p_{1/2}$ scattering length cannot be neglected in its calculation. Consequently, the best parameters found from the above were used to calculate $\mathrm{Re}F_{1-}(s)$ on the low-energy portion of the physical cut from the dispersion relation (18) in exactly the same

[19] J. S. Ball and D. Y. Wong, Phys. Rev. Letters 6, 29 (1961).
[20] P. Menotti, Nuovo Cimento 23, 931 (1962).

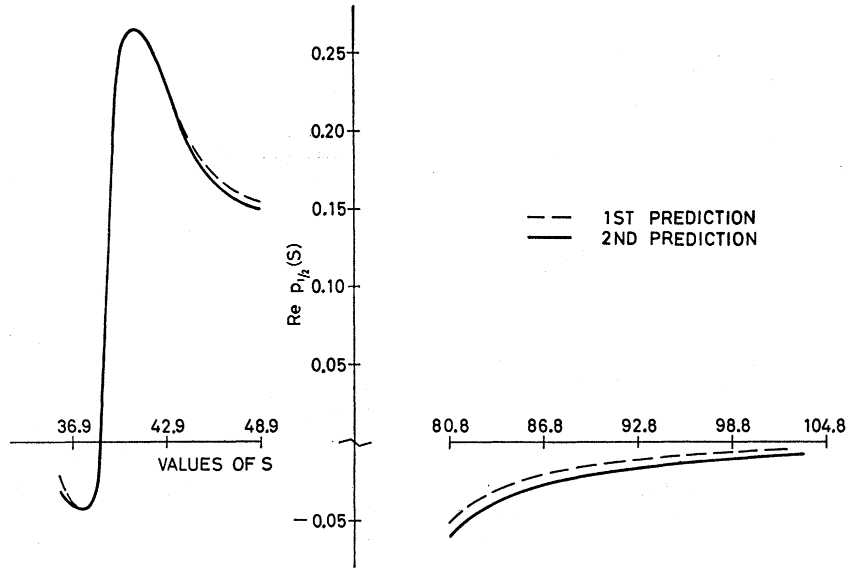


FIG. 6. The predicted values of $\mathrm{Re}p_{1/2}$ on the physical and crossed-physical cuts, without short-range effects.

way as the $p_{3/2}$ amplitude was calculated. Unitarity then gives $\mathrm{Im}F_{1-}(s)$. The predicted values of $\mathrm{Re}p_{1/2}$ on both cuts are shown in Fig. 6. The $p_{3/2}$ amplitude was recalculated incorporating the $p_{1/2}$ contributions to $F_{1+}(s)$ on the crossed cut, and to the circle contribution on both cuts. The procedure of fitting the discrepancies was then repeated by varying $a$ and $g_{\Sigma\Lambda\pi}$. The best-fit parameters found were $a=0.33\pm0.02$ and $g_{\Sigma\Lambda\pi}{}^2=10.9\pm0.3$. The errors are determined only by the goodness of fit, and do not represent any assumptions about the validity of the approximations made. Finally, the difference between the discrepancies and the circle contribution was fitted by a linear term of the form $A+Bs$, where $A$ and $B$ were chosen to give a best fit on both cuts simultaneously. The results are shown in Fig. 7. Table I shows the contributions to the Omnès-equation subtraction constants, and the corresponding values of $\mathrm{Im}h_+{}^0(t)$ for $a''\leqslant t\leqslant a'$ and $t\geqslant 4\mu^2$ are shown in Figs. 8 and 9, respectively. The values of $\mathrm{Re}F_{1+}(s)$ calculated from the dispersion relation (18) and the values given by the Layson formula, with short-range terms, are shown in Fig. 10, and the numerical values of the contributions at both thresholds are shown in Table II. The column headed "Short-Range" shows the

TABLE I. Contributions to the Omnès-equation subtraction constants.

| | Total | Born | Rescattering | From scattering lengths $p_{3/2}$ | $p_{1/2}$ |
|---|---|---|---|---|---|
| First subtraction constant $\mathrm{Re}h_+{}^0(0)$ | −9.02 | −4.74 | −4.28 | | |
| Second subtraction constant $\mathrm{Re}h_+{}^{0\prime}(0)$ | 6.21 | 2.84 | −1.85 | 7.06 | −1.84 |

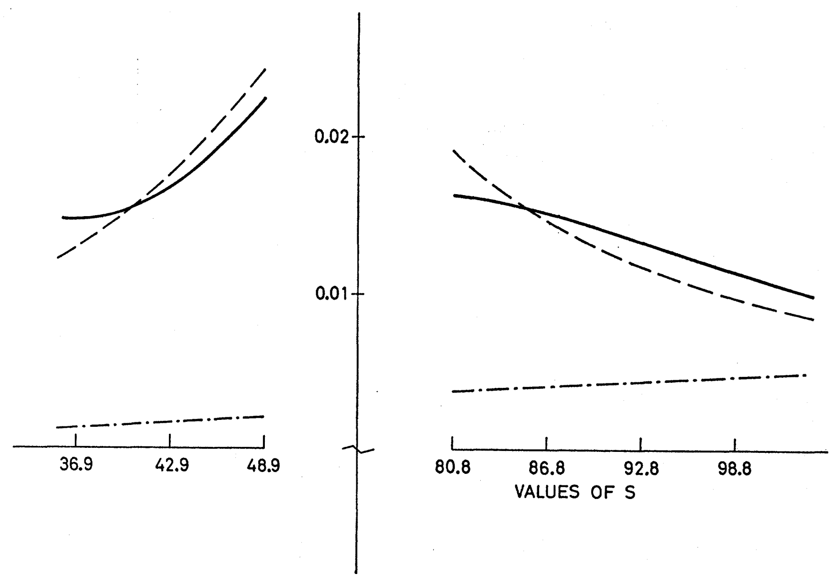


FIG. 7. The $p_{3/2}$ discrepancies with short-range term (solid lines), and the $T=0$ $\pi$-$\pi$ contributions (broken lines) for the best-fit parameters. The linear term is shown as a dotted line.

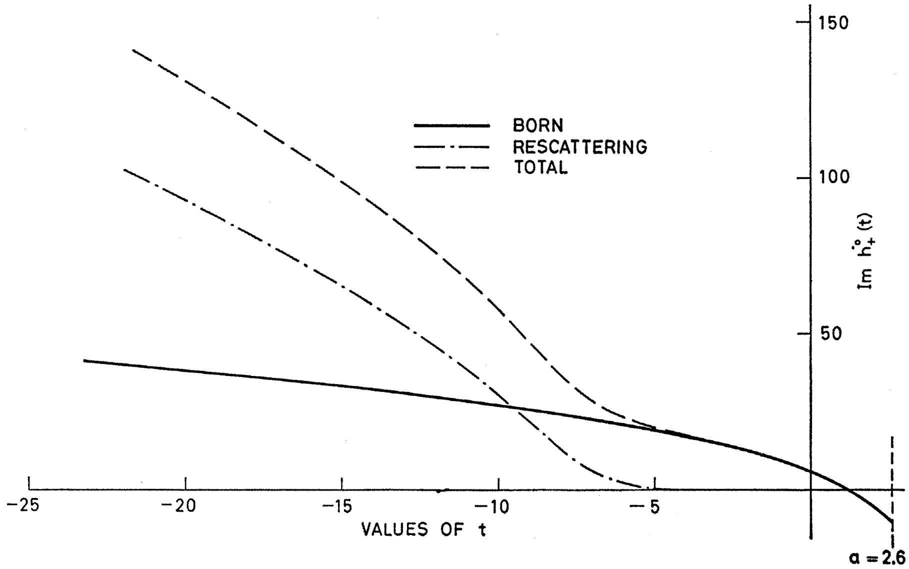


FIG. 8. Values of $\mathrm{Im}h_+^0(t)$ for $t \leqslant a'$.

value of the short-range forces necessary to match the value of $\mathrm{Re}F_{1+}(s)$ calculated from Eq. (18) with that given by the Layson formula. Finally, the $p_{1/2}$ amplitude was recalculated including $p_{1/2}$ contributions to the crossed region taken from the first prediction. The results are shown in Fig. 6.

TABLE II. Contributions to $\mathrm{Re}F_{1+}(s)$ at the physical and crossed-physical thresholds.

| $S$ | Born | Crossed integral | Rescattering | $T=0$ $\pi$-$\pi$ | $\mathrm{Re}F_{1+}(s)$ | $\mathrm{Re}F_{1+}(s)$ (Layson) | Short range |
|---|---|---|---|---|---|---|---|
| $s_0$ | 0.016 | 0.004 | 0.058 | 0.019 | 0.097 | 0.098 | 0.001 |
| $\bar{s}_0$ | −0.063 | 0.018 | 0.020 | 0.024 | −0.001 | 0.000 | 0.001 |

## 7. SHORT-RANGE FORCES AND $S$ WAVES

### (i) Short-Range Forces

The application of unitarity to the asymptotic form of the dispersion relation (18) leads to conclusions about the size of the short-range forces in the $p$-wave channels. Without proof, we state the resulting unitary sum rule,[21] for $l \geqslant 1$,

$$\int_{s_0}^{\infty} s'^{(l-1)}\,\mathrm{Im}F_{l\pm}(s')ds' + \int_{-\infty}^{s_1} s'^{(l-1)}\,\mathrm{Im}F_{l\pm}(s')ds' = 0, \quad (52)$$

where the second integral represent the integrals over all the left-hand unphysical cuts. Equation (52) enables information about the distant unphysical values of $\mathrm{Im}F_{l\pm}(s)$ to be deduced if both the "nearby" unphysical discontinuities of $F_{l\pm}(s)$ and the rescattering are known.

If we write $F_{l\pm}'(s)$ as the total left-hand unphysical contribution to $F_{l\pm}(s)$, i.e.,

$$F_{l\pm}'(s) = \frac{1}{\pi}\int_{-\infty}^{s_1} ds' \frac{\mathrm{Im}F_{l\pm}(s')}{s'-s}, \quad (53)$$

then the existence of scattering in the physical region $s_0 \leqslant s < \infty$ implies that

$$\int_{-\infty}^{s_1} ds'\,\mathrm{Im}F_{l\pm}(s') < 0, \quad (54)$$

and hence

$$F_{l\pm}'(s) \underset{s\to\infty}{\simeq} (A_{l\pm}/s), \quad (55)$$

where $A_{l\pm}$ is a positive constant.

In our treatment of $Y_1^*$ we have neglected contribu-

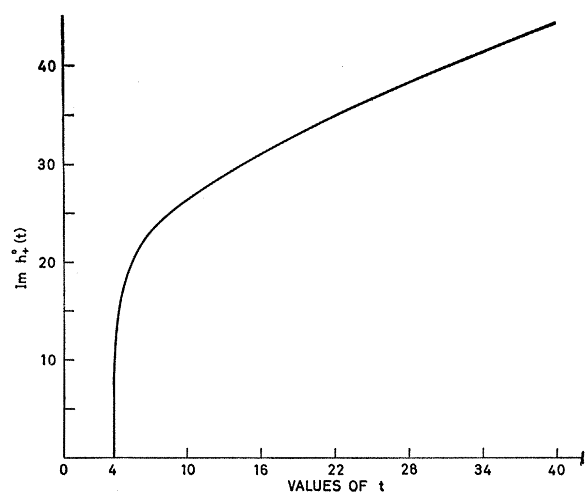


FIG. 9. Values of $\mathrm{Im}h_+^0(t)$ for $t \geqslant 4\mu^2$.

[21] A. Donnachie and J. Hamilton, University College, London, 1964 (unpublished).

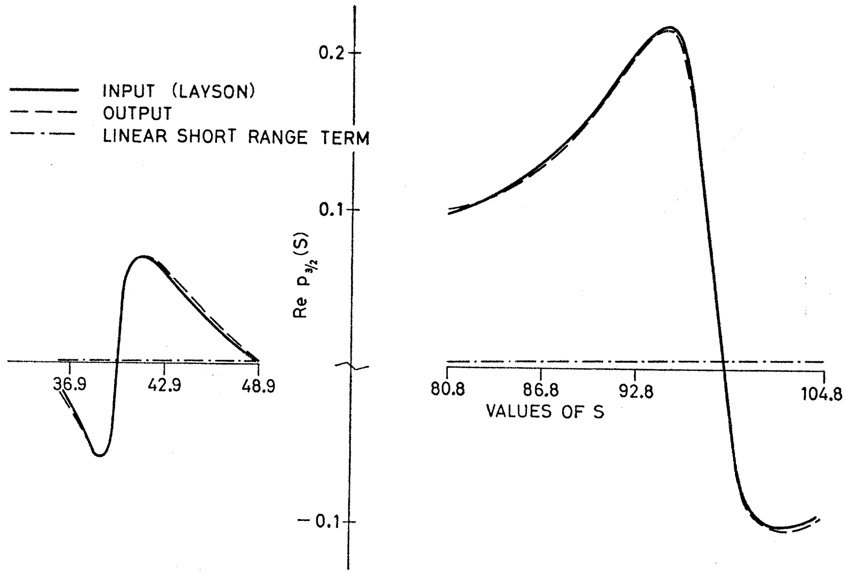


FIG. 10. Values of Re$p_{3/2}$ calculated from the dispersion relation (broken lines), and as given by the Layson formula (solid line) on both cuts, including short-range terms.

tions from the back of the circle and the line $-\infty < s \leqslant 0$. If we represent the sum of these contributions by a single pole on the line $-\infty < s \leqslant 0$, then the unitary sum rule (52) will enable us to determine its residue. The exact position of the pole is undetermined, but because of the factor $q^{-2l}(s)$ it would be unreasonable to place it too far to the left of the circle. At the same time, a pole on the line $0 \leqslant s \leqslant s_1$ would have most of the unknown cuts to its left. We will compromise by placing the pole at the rear of the circle, at $s = -(\Lambda^2 - 1)$, noting that this will give an upper estimate of the short-range forces, and that the value will decrease as we move the pole to the left.

If we define

$$J = \frac{1}{\pi} \int \mathrm{Im} F_{l\pm}(s') ds'$$

and

$$\Gamma = -\frac{1}{\pi} \int_{\mathrm{SR}} \mathrm{Im} F_{l\pm}(s') ds',$$

where

$$\frac{1}{\pi} \int_{\mathrm{SR}} \mathrm{Im} F_{l\pm}(s') ds' + \frac{1}{\pi} \int_{\mathrm{LR}} \mathrm{Im} F_{l\pm}(s') ds' = \frac{1}{\pi} \int_{-\infty}^{l_1} \mathrm{Im} F_{l\pm}(s') ds'$$

and

$$\frac{1}{\pi} \int_{\mathrm{LR}} \mathrm{Im} F_{l\pm}(s') ds'$$

is the unphysical contribution coming from the "nearby" singularities, then the residue of the short-range pole is $\Gamma$.

Using the results of Sec. 6 for the $p_{3/2}$ channel we find $J = 1.02$, $A_{1+} = 0.87$, and hence $\Gamma = -0.15$. For a pole at $s = -(\Lambda^2 - 1)$ this gives short-range contributions of 0.0011 at $s_0$ and 0.0013 at $\bar{s}_0$, which are consistent with those found in Sec. 6 (see Table II). In the $p_{1/2}$ channel we find $J < 0.005$ and $A_{1-} = -0.16$; hence, $\Gamma = -0.16$. We have calculated the short-range contribution to the $p_{1/2}$ amplitude given by the second predictions. The resulting $p_{1/2}$ phase shifts are shown in Fig. 11.

### (ii) *S* Waves

Throughout this work we have neglected the $s$-wave $\pi$-$\Lambda$ interaction and its contribution to $p$-wave scattering via the crossed channels. As in all $s$-wave problems, we are hindered in our calculation of the $s_{1/2}$ amplitude because the lack of a centrifugal barrier means that there undoubtedly exists an important short-range interaction. Furthermore, in this case we cannot deduce this short-range interaction because of our lack of knowledge of the $s$-wave rescattering. However, an estimate of the $s$-wave scattering length may be obtained by considering the weak decay

$$\Xi^- \rightarrow \Lambda^0 + \pi^-. \tag{56}$$

This decay may be specified by three parameters[22] $\alpha$, $\beta$, and $\gamma$ such that

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{57}$$

If $S$ and $P$ are the $s_{1/2}$ and $p_{1/2}$ $\pi$-$\Lambda$ scattering amplitudes in the final state, and we normalize

$$|S|^2 + |P|^2 = 1, \tag{58}$$

then[22]

$$\alpha = 2\,\mathrm{Re}(S^* P); \quad \beta = 2\,\mathrm{Im}(S^* P); \quad \gamma = |S|^2 - |P|^2. \tag{59}$$

The effect of the final-state interaction is to multiply the $s$- and $p$-wave amplitudes by a phase factor which

[22] T. D. Lee and C. N. Yang, Phys. Rev. **109**, 1755 (1958).

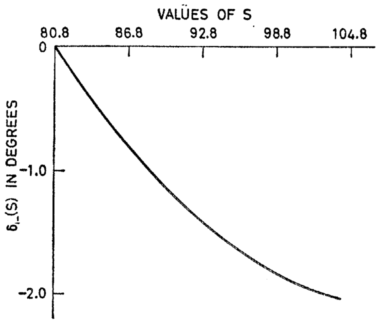


FIG. 11. Predicted values of the $p_{1/2}$ phase shifts.

depends on the $\pi$-$\Lambda$ phase shift. In Ref. 23 it is shown that, if time-reversal invariance holds for the decay, then

$$\tan(\delta_{p1/2}-\delta_{s1/2})=\beta/\alpha\,, \tag{60}$$

where $\delta$ are the phase shifts at the decay energy. Experiments agree[24] on the value of $\alpha$ but the value of $\beta$ is still subject to large errors.[25] We shall use the average results of Connolly *et al.*[25] and Alvarez *et al.*,[24] i.e.,

$$\alpha=-0.45\pm0.13;\quad \beta=-0.05\pm0.24\,. \tag{61}$$

Because of the large error on $\beta$ the precise value we use is not significant. Using these values of $\alpha$ and $\beta$ in Eq. (60) and taking the value of $\delta_{p1/2}$ at the decay energy $s=\bar{s}=89.5$ from Fig. 11, we have $\delta_{s1/2}(\bar{s})=-7°\pm13°$. Equation (12) then gives $\mathrm{Re}f_{0+}(\bar{s})=-0.12\pm0.20$.

The dispersion relation (18) may be written for $l=0$ and $s\geqslant s_0$,

$$\mathrm{Re}f_{0+}(s)=\Sigma(s)+\text{rescattering}+\text{short range}\,, \tag{62}$$

where $\Sigma(s)$ is the total contribution from the "nearly" singularities. The constituents of $\Sigma(s)$ at $s_0$ and $\bar{s}$ are shown in Table III. If we assume that both the rescattering and the short-range terms are slowly varying functions of $s$, then from Eq. (62)

$$\mathrm{Re}f_{0+}(s_0)\equiv a_0\simeq\mathrm{Re}f_{0+}(\bar{s})-\Sigma(\bar{s})+\Sigma(s)\,. \tag{63}$$

[23] T. D. Lee, T. Steinberger, G. Feinberg, P. K. Kabir, and C. N. Yang, Phys. Rev. **106**, 1764 (1959).

[24] See H. Ticho, Proceedings of the Brookhaven Conference on Weak Interactions, 1963 (unpublished).

[25] P. L. Connolly, E. L. Hart, G. Kalbfleisch, K. W. Lai, G. London, G. C. Moneti, R. R. Rau, N. P. Samios, I. O. Skillicorn, S. S. Yamamoto, M. Goldberg, M. Gundzik, J. Leitner, and S. Lichtman, Proceedings of Siena International Conference on Elementary Particles, 1963 (unpublished), find $\alpha=-0.49\pm0.16$ and $\beta=-0.02\pm0.22$ which are consistent with the results of L. W. Alvarez (Ref. 24) who finds $\alpha=-0.41\pm0.08$ and $\beta=-0.08\pm0.26$, and L. Jaureau, D. Morellet, U. Nguyen-Khac, A. Rousset, J. Six, H. H. Bingham, D. C. Cundy, W. Koch, M. Nikolić, B. Ronne, O. Skjeggestad, H. Sletten, A. K. Common, J. J. Esten, C. Henderson, C. M. Fisher, J. M. Scarr, R. H. Thomas, A. Haatuft, R. Møllerud, and K. Myklebost, Proceedings of the Siena Conference on Elementary Particles, 1963 (unpublished), who find $\alpha=-0.44\pm0.11$ and $\beta=-0.24\pm0.53$. However, D. Carmony, G. M. Pjerou, P. E. Schlein, W. G. Slater, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **12**, 482 (1964), find $\alpha=-0.64\pm0.13$, $\beta=+0.65\pm0.16$, and H. Schreider, Phys. Rev. Letters **4**, 360 (1963), finds $\alpha=-0.73\pm0.21$, $\beta=+0.44\pm0.36$, both of which appear to be incompatible with time reversal invariance which predicts $\beta\approx0$.

TABLE III. Contribution of nearby singularities to $\mathrm{Re}f_{0+}(S)$.

| $S$ | Born | $T=0$ $\pi$-$\pi$ | Crossed integral | $\Sigma(s)$ |
|---|---|---|---|---|
| $s_0$ | 0.000 | 0.212 | −0.035 | 0.177 |
| $\bar{s}$ | 0.000 | 0.170 | −0.030 | 0.140 |

Using the values given in Table III we have

$$a_0=-0.08\pm0.20\,. \tag{64}$$

Thus, although the sign is not determined, $|a_0|<0.3$ and hence the contribution of $a_0$ to $\mathrm{Re}h_+^{0\prime}(0)$, which is the most important term in Eq. (49), is less than 1%. Likewise, its contribution to $F_{1\pm}(s)$ on the crossed cut is negligible. Using the value of $a_0$ given by Eq. (64) and the value of $\Sigma(s_0)$ given in Table III, we see that the sum of the rescattering and short-range terms is $-0.26\pm0.20$. Since the rescattering will be positive at threshold, this means that there must exist a strong short-range repulsion in $s$-wave $\pi$-$\Lambda$ scattering, the value of which at threshold is less than $-0.26$. The situation is thus very similar to $s$-wave $\pi$-$N$ scattering, where there exists a strong short-range repulsion contributing about $-0.3$ to both the $s$-wave scattering lengths.[14]

Finally we note that accurate values of $\alpha$ and $\beta$ would enable us to predict $s$-wave $\pi$-$\Lambda$ phase shifts in the low-energy region. These would be valuable in the study of $\Lambda$-$N$ scattering.

## 8. OTHER EVIDENCE ON $g_{\Sigma\Lambda\pi}$

In this section we compare the value of $g_{\Sigma\Lambda\pi}{}^2$ obtained here with other determinations, with special reference to $SU(3)$ symmetry.[26] In terms of the $SU(3)$ mixing parameter $\alpha$ and the pion-nucleon coupling constant $g_{NN\pi}$, the coupling constant $g_{\Sigma\Lambda\pi}$ is given by[27]

$$g_{\Sigma\Lambda\pi}{}^2=\tfrac{4}{3}\alpha^2 g_{NN\pi}{}^2\,. \tag{65}$$

If we use $g_{NN\pi}{}^2=14.6$,[28] and $g_{\Sigma\Lambda\pi}{}^2=10.9$, we obtain $\alpha=0.75$.

Phenomenological determinations of $g_{\Sigma\Lambda\pi}$ have been made by many authors. Analyses of hyperon-nucleon scattering have been made by a variety of potential models and by effective-range theory,[29] but the results are inconclusive because of the lack of experimental data and the difficulty of fixing the core of the potential for $s$-wave scattering. Information on the pion-hyperon couplings may also be obtained from nonleptonic hyperon decays.[30] The most detailed calculations of

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[27] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963).

[28] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[29] C. Dullemond and J. J. De Swart, Ann. Phys. (N. Y.) **19**, 458 (1962); Nuovo Cimento **25**, 1075 (1962). J. J. De Swart and K. C. Iddings, Phys. Rev. **128**, 2910 (1962); **130**, 319 (1962). B. Ram and D. W. Downs, Phys. Rev. **133**, B420 (1964).

[30] G. Feldman, P. T. Matthews, and A. Salam, Phys. Rev. **121**, 302 (1961); S. K. Bose and R. Marshak, Nuovo Cimento **23**, 556 (1962); A. Fujii, Phys. Letters **1**, 75 (1962).

these processes have been carried out by McCliment and Nishijima.[31] These authors have used unsubtracted dispersion relations to calculate the rescattering contributions to the decay amplitudes. Their results are in rough agreement with experimental data on sigma decay for $g_{\Sigma\Lambda\pi}^2=7.1$ and $g_{\Sigma\Sigma\pi}^2=1.2$, which implies $0.60\leqslant\alpha\leqslant 0.86$.

Martin and Wali[32] have generalized Chew's "reciprocal bootstrap"[33] to a many-channel system based on $SU(3)$ symmetry. They have considered the scattering of the baryon octet $B$ by the pseudoscalar meson octet $P$, assuming the processes to be mediated by the exchange of $B$, the $B^*$ decuplet, and the vector-meson octet $V$. The coupling constants $g_{BBP}$ were taken from $SU(3)$ in terms of $\alpha$. The mixing parameter $\alpha$, together with two cutoff parameters, was then varied in a first iteration of the $N/D$ equations to yield a consistent closed $p_{3/2}$ $B^*$ decuplet and a $p_{1/2}$ $B$ octet. This could be achieved for $0.55\leqslant\alpha\leqslant 0.75$. This range of $\alpha$ includes $\alpha=0.66$ which was found by Cutkosky[34] using similar, but less detailed, self-consistent methods. However, these calculations do not include contributions from the exchange of scalar systems, which we find to be important for the dynamics of $Y_1^*$.

## 9. SUMMARY AND CONCLUSIONS

We have shown, in the framework of the approximation, that a self-consistent $Y_1^*(1385)$ resonance can be achieved for $g_{\Sigma\Lambda\pi}^2=10.9$, and that this value of the coupling constant is consistent with previous determinations and with unitary symmetry. Furthermore, self-consistency can be achieved only by the specific inclusion of the effects due to the exchange of a low-energy $s$-wave $\pi$-$\pi$ pair. The short-range force in the $P_{3/2}$ channel found for a unitary sum rule was shown to be consistent with the self-consistent determination. The short-range force in the $P_{1/2}$ channel was also calculated by a unitary sum rule and used, together with the contributions from nearby singularities, to predict $p_{1/2}$ $\pi$-$\Lambda$ phase shifts. The effect of $s_{1/2}$ $\pi$-$\Lambda$ scattering on the $p$ waves was shown to be small.

We have used the experimental branching ratios of $Y_1^*$ as part of the basis of the approximation used. However, some insight into the dynamical reasons for the weak coupling to the $\pi$-$\Sigma$ channel may be obtained by noting that for $\alpha=0.75$ the total Born term (due to both $\Sigma$ and $\Lambda$ exchange) in $p_{3/2}$ $T=1$ $\pi$-$\Sigma$ scattering is negative, i.e., repulsive. This means that in the Omnès equation [cf. Eq. (49)] for the helicity amplitudes for $\pi+\pi\to\Sigma+\bar{\Sigma}$ there would be cancellations between the integral term and the subtraction constants. Hence we would expect the $T=0$ $\pi$-$\pi$ effect to be small. Other possible attractive mechanisms are $\rho$ exchange and $T=2$ $\pi$-$\pi$ exchange. However, $\rho$ exchange occurs for $t=28\mu^2$ and would be well suppressed by the factor $q^{-2l}(s)$,[35] and $T=2$ $\pi$-$\pi$ scattering is known to be small.[36] Thus the total unphysical contributions to the $T=1$ $p_{3/2}$ elastic $\pi$-$\Sigma$ amplitude would be small. In the case of $\pi+\Lambda\to\pi+\Sigma$ the Born terms are positive in the $p_{3/2}$ state but are small because for $\alpha=0.75$, the quantity $g_{\Sigma\Sigma\pi}^2=3.7$, which is small. Furthermore, in the $t$ channel there is no $T=0$ $\pi$-$\pi$ exchange, and the arguments concerning $\rho$ exchange in $\pi$-$\Sigma$ scattering apply here too.

We conclude by remarking that in all previous self-consistent calculations of members of the $SU(3)$ $p_{3/2}$ decuplet[32,37] the scattering amplitude in the unphysical region has been assumed to be dominated by the baryon-exchange processes, with small contributions from vector-meson and isobar exchange, and although this is well known to be true for $\pi$-$N$ scattering[38] it has never explicitly been demonstrated for any of the other processes. Indeed, examinations of the strengths of the coupling constants involved and the relative distances of the nearest singularities would lead us to suspect that this assumption would not always be valid. In view of the importance of the $T=0$ $\pi$-$\pi$ exchange in this calculation, it would seem likely that the inclusion of the exchange of scalar systems would significantly affect the results of any such calculations.

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[33] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 395 (1961).

[34] R. Cutkosky, Ann. Phys. (N.Y.) **23**, 415 (1963).

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[38] Although even for $N^*(1238)$ the $T=0$ $\pi$-$\pi$ effect cannot be neglected. See, e.g., Ref. 14.