Coulombic Modified Effective-Range Theory for Long-Range Effective Potentials*

R. OPPENHEIM BERGER† AND LARRY SPRUCH
Physics Department, New York University, Washington Square, New York, New York
(Received 27 January 1965)

A modified effective-range theory (MERT) was introduced previously to describe the scattering of a charged particle by a neutral polarizable system. The long-range components of the resultant effective one-body (generalized optical-model) potential, which cause the phase shift to have a rapidly varying energy dependence, are taken account of exactly by solving a one-body problem numerically. The short-range potential components generate only a slowly varying energy-dependent effect on the phase shift, and this effect can be accounted for by a few terms in a power series in k^2. The procedure is here extended to the case for which the polarizable system is itself charged. An MERT expansion is derived for the difference $\delta(k)$ between the total phase shift $\eta(k)$ and the phase shift $\rho(k)$ due to the long-range tail alone; both $\eta(k)$ and $\rho(k)$ are defined relative to pure Coulomb scattering. With the strongly energy-dependent Coulombic and other long-range effects accounted for exactly by the numerical solution of a one-body scattering problem, low-energy scattering data can be matched by the proper choice of the coefficients of just the first few terms in the MERT expansion; these terms will then determine the scattering in the (experimentally inaccessible) energy domain extending down to zero energy. For a repulsive Coulomb field, the leading term in $\eta(k)$ is determined *exactly* for all L by the Born approximation; for $V(r)=(2\mu/\hbar^2)(-\beta^2/r^4)$, where $\alpha=\beta^2\hbar^2/|\mu e^2Z_1Z_2|=\beta^2a_0$ is the electric-dipole polarizability of the target, $\tan\eta(k)=\beta^2a_0{}^3k^5/15$ for $ka_0L\ll1$.

1. INTRODUCTION

THE problem of the elastic scattering of a particle by a compound system can be reduced, formally, to an equivalent one-body problem by introducing an effective (generalized optical-model) potential $\mathcal{V}$. The usual effective-range-theory (ERT) expansion of the phase shift is a very useful tool when $\mathcal{V}$ is a short-range potential. A version of ERT can also be used if $\mathcal{V}$ contains a Coulomb component plus only short-range components. Both versions of ERT fail if $\mathcal{V}$ contains additional long-range components.[1,2] We showed previously,[3] however, that if the long-range component is sufficiently well known, as it often will be, and if this long-range component does not contain a Coulomb contribution, then one can derive a modified ERT (MERT) expansion for the phase shift $\delta(k)$, *relative* to pure U-potential scattering, where U is known and contains all of the long-range component. [An expansion for the total phase shift $\eta(k)$ is possible,[1,2,4] but may not be useful, since the long-range interaction results in a strong energy dependence of the phase shift, so that many terms in the expansion, and hence many experimentally determined coefficients, are needed.] The basis of the technique is that since U is known, it is possible to solve (numerically, if necessary) the problem of pure U-potential scattering, which is trivial to do since it is a one-body problem, and thus to extract out exactly the strongly energy-dependent terms. Our present concern is to repeat the derivation for potentials which include a Coulomb potential as well as other long-range components.

The partial-wave solution to pure Coulomb scattering has long been known: it is the function $F_L(k,r)$ which is given[5,6] in terms of a confluent hypergeometric function and which has the asymptotic form

$$F_L(k,r)\to\sin(kr-n\ln 2kr-\tfrac{1}{2}L\pi+\sigma_L)\,,\quad r\to\infty \qquad (1.1)$$

where, in the scattering of a charged particle of charge Ze by a target of charge $Z'e$,

$$n=ZZ'e^2/(\hbar v)\,,\quad v=\text{the relative speed}\,,$$
$$\sigma_L=\arg\Gamma(L+1+in)\,.$$

When the potential consists of a short-range potential in addition to the Coulomb potential, the ERT expansion for the phase shift, η, relative to pure Coulomb scattering takes the form[7]

$$C_L{}^2(n)k^{2L+1}\cot\eta+h_L(n)=-1/A+\tfrac{1}{2}r_0k^2+\cdots\,, \qquad (1.2)$$

where $C_L(n)$ and $h_L(n)$ are given in Appendix A. The technique for deriving Eq. (1.2) was the model we used to derive the MERT expansion for the phase shift, δ, relative to pure U-potential scattering, where U, which is generally nonlocal and energy-dependent but which

* The research reported on in this article was sponsored by the Office of Ordnance Research under Contract No. DA-30-69-ORD-2581, Project No. 2360, the U. S. Office of Naval Research and the Advanced Research Projects Agency under Contract Nonr-285(49), NR 012-109, and by NASA under Contract No. NSG 699.
† Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at New York University.
[1] L. Spruch, T. F. O'Malley, and L. Rosenberg, Phys. Rev. Letters **5**, 375 (1960).
[2] T. F. O'Malley, L. Spruch, and L. Rosenberg, J. Math. Phys. **2**, 491 (1961).
[3] R. Oppenheim Berger, T. F. O'Malley, and L. Spruch, Phys. Rev. **137**, A1068 (1965).
[4] T. F. O'Malley, L. Rosenberg, and L. Spruch, Phys. Rev. **125**, 1300 (1962); B. R. Levy and J. B. Keller, J. Math. Phys. **4**, 54 (1963).
[5] See, for example, F. L. Yost, J. A. Wheeler, and G. Breit, Phys. Rev. **49**, 174 (1936).
[6] *Tables of Coulomb Wave Functions*, Natl. Bur. Std. Appl. Math. Ser. **17** (Washington, D. C., 1952), Vol. I. Our notation will largely follow that of this work.
[7] H. A. Bethe, Phys. Rev. **76**, 38 (1949); G. F. Chew and M. L. Goldberger, Phys. Rev. **75**, 1637 (1949); J. D. Jackson and J. M. Blatt, Rev. Mod. Phys. **22**, 77 (1950).

becomes local for large r, satisfies the condition

$$r^3U(k)\rightarrow 0\,, \quad r\rightarrow\infty\,. \tag{1.3}$$

In our present problem, the procedure is essentially the same as it was for the previous non-Coulombic MERT case: having expressed the potential as a sum of a Coulomb potential, a potential $U(k)$ which satisfies Eq. (1.3), and a short-range potential, one must first determine the phase shift, ρ, relative to Coulomb scattering [and two other functions of k which are the analogs of the $C_L(n)$ and $h_L(n)$ of Eq. (1.2)], due to $U(k)$ alone in the presence of the Coulomb potential, by solving a scattering problem which includes only $U(k)$ and the Coulomb potential, and then derive a MERT expansion for the *difference* between the total phase shift relative to Coulomb scattering and ρ.

While the technique we will use to derive the Coulombic MERT expansion is formally the same as the technique we used to derive the non-Coulombic MERT, the physical process under study is quite different, and we would like to discuss this here. To begin with, in the absence of a Coulomb field the contribution of the phase shift due to the long range tail of $U(k)$ is so strongly energy-dependent that it dominates over the short-range contribution. For example, for $L\geq 1$ the first term in an expansion of the phase shift ρ due to a potential which has a $1/r^4$ dependence for $r\geq R$ is proportional to k^2 rather than to k^{2L+1}, and the coefficient of k^2 is independent of R.[2] Since this first term is independent of R, it is due entirely to the long-range tail, and since the $1/r^4$ tail is asymptotically very weak, the Born approximation gives the first term *exactly*, that is, it gives both the energy dependence and the correct coefficient (in fact, all terms up to but not including the k^{2L+1} term are given exactly by the Born expansion). Consequently, the threshold energy dependence of the total phase shift, η, is known exactly, since any short-range potential contributes a higher order energy dependence. In Sec. 4 we derive an explicit expression for the Born approximation $\tan\eta_B$ for the potential

$$U(r)=-(\hbar^2/2\mu)(\beta^2/r^4)$$

in the presence of a Coulomb field, and discuss the results in some detail. Briefly, we prove in the case of the repulsive Coulomb field, if for $L\neq 0$ we have $Lka_0\ll 1$, or if for $L=0$ we have $ka_0\ll 1$, that to lowest order in k

$$\tan\eta_B=(\beta/a_0)^2(ka_0)^5/15\,,$$

and that this result is exact for η itself, to lowest order. In comparison, the phase shift for a short-range potential in the presence of a repulsive Coulomb field is proportional to $\exp[-2\pi/(ka_0)]$. However, in the case of the attractive Coulomb field it is the short-range potential which is responsible for the dominant energy dependence of η, since the attractive Coulomb field pulls the scattered particle into the neighborhood of the origin. Of course, the Born-approximation result in the repulsive-Coulomb-field case is only useful as a preliminary orientation except at extremely low energies; it will not normally be sufficient to match the experimental data at other energies. Similar results can be obtained for any $1/r^p$ potential in the presence of the repulsive Coulomb field, where p is an integer ≥ 3. These results are derived in Appendix B. We note that we are now studying the effect of the electric-quadrupole interaction in proton-deuteron scattering. Since this is a three-channel elastic-scattering problem, the knowledge that in the case of the repulsive Coulomb field the Born approximation is exact, to lowest order, both in the k dependence and its coefficient, allows for considerable simplification.

The procedure we use to derive the MERT expansion for the difference phase shift, and the procedure we outline for its use, is applicable to any problem for which a partial-wave analysis is possible. This includes the scattering of electrons by negative or positive ions, and the scattering of protons by light nuclei (since it is applicable to low-energy scattering only, it is not useful for the scattering of protons by heavy nuclei, where the effect is swamped by the very strong Coulomb scattering). In principle, it should be possible to extend the procedure to any problem of *elastic* multichannel scattering (e.g., the scattering of protons by nuclei which have quadrupole moments), and probably even to multichannel inelastic scattering (the latter should be much harder to do, but well worth the effort).

We consider below the single-channel elastic scattering of a particle of charge Ze by a target of charge $Z'e$ in the presence of an additional potential $\mathcal{V}(k)$, where $\mathcal{V}(k)$, the effective generalized optical-model potential, satisfies Eq. (1.3). We have discussed the relevant properties of $\mathcal{V}(k)$ in a previous paper[3] and therefore we need not go into detail here. We merely note that we can set $\mathcal{V}(k)$ equal to two terms:

$$\mathcal{V}(k)=V(k)+U(k)\,,$$

where $V(k)$ is a short-range potential (i.e., one which falls off faster than any power of $1/r$) and where $U(k)$ includes the long-range tail, has at most a $1/r$ singularity at the origin, and is assumed to be a known function. $\mathcal{V}(k)$, $V(k)$, and $U(k)$ are all operators representing nonlocal, energy-dependent potentials, with kernels $\mathcal{V}(k;r,r')$, $V(k;r,r')$, and $U(k;r,r')$, respectively. We assume that $V(k)$ and $U(k)$ satisfy

$$U(k)=U(0)+\sum_{m=1}^{\infty} U_m k^{2m}, \tag{1.4}$$

$$V(k)=V(0)+\sum_{m=1}^{\infty} V_m k^{2m}. \tag{1.5}$$

Although the functions associated with the Coulomb field are well known, we have felt, since there is some difference of notation in the literature [in particular, in the definition of $h_L(n)$, which is difficult to find for all values of L], that a collection of those functions which are relevant to our discussion would be convenient; this will be found in Appendix A. One other point of notation should be noted: we suppress the L dependence of wave functions, phase shifts, and associated functions of k such as $\tilde{C}(k)$ and $\tilde{h}(k)$ [the analogs of $C_L(n)$ and $h_L(n)$] except in the case of those pure Coulomb functions for which L is traditionally carried as a subscript.

2. PRELIMINARIES

Our problem is to determine the phase shift $\eta(k)$ defined by

$$[T(r)+ZZ'e^2/r+V(k)+U(k)-E]u(k,r)=0\,, \quad (2.1)$$

$$u(k,0)=0\,, \quad (2.2)$$

$$u(k,r)\rightarrow F(k)\sin(kr-n\ln 2kr-\tfrac{1}{2}L\pi+\sigma_L+\eta)\,, \quad r\rightarrow\infty \quad (2.3)$$

where

$$T(r)=(\hbar^2/2\mu)(-d^2/dr^2+L(L+1)/r^2)\,,$$
$$E=\hbar^2k^2/(2\mu)\,,$$

$F(k)$ is an arbitrary normalization factor [$F(k)$ conforms to a previous notation[3]; it has nothing whatever to do with the Coulomb function $F_L(k,r)$], $U(k)$ is assumed to be chosen to match the presumably known long-range interaction between the particle and the system, the phase shift $\rho(k)$ associated with $U(k)$ in the presence of the Coulomb potential will be assumed to be determined numerically, and we will develop an expansion for $\cot\delta(k)$, where

$$\delta(k)=\eta(k)-\rho(k)$$

is the phase shift due to the short-range potential $V(k)$ in the presence of $U(k)$ and the Coulomb potential. While $\eta(k)$ can be a rapidly varying function of the energy because of the long-range potential, $\delta(k)$ will be relatively slowly varying. Hence only a few terms in the expansion of $\cot\delta(k)$ will have to be retained, and these coefficients can be determined from an analysis of the experimental data. We begin by introducing several long-range functions.

A. Long-Range Functions

We consider the three solutions $f(k,r)$, $g(k,r)$, and $v(k,r)$ of

$$[T(r)+ZZ'e^2/r+U(k)-E]\times\{f(k,r),g(k,r),v(k,r)\}=0\,. \quad (2.4)$$

Clearly only two of these three functions, which differ in their boundary conditions, are independent. The boundary conditions on the regular solution, $f(k,r)$, are

$$f(k,0)=0\,,$$

$$f(k,r)\rightarrow\sin(kr-n\ln 2kr-\tfrac{1}{2}L\pi+\sigma_L+\rho)\,, \quad r\rightarrow\infty\,, \quad (2.5)$$

and we define $g(k,r)$ as that solution which satisfies the boundary condition

$$g(k,r)\rightarrow\cos(kr-n\ln 2kr-\tfrac{1}{2}L\pi+\sigma_L+\rho)\,, \quad r\rightarrow\infty\,. \quad (2.6)$$

Since $U(k)$, by construction, has at most a $1/r$ singularity at the origin, the regular solution $f(k,r)$ will behave as r^{L+1} in the neighborhood of the origin. We define the function $\tilde{C}(k)$, the analog of the Coulomb function $C_L(n)$, by

$$\tilde{C}(k)\equiv\lim_{r\to 0}\left(\frac{f(k,r)}{k^{L+1}r^{L+1}}\right). \quad (2.7)$$

It follows that as $k\rightarrow 0$, $f(k,r)$ approaches $\tilde{C}(k)k^{L+1}$ times a function of r. Since $\tilde{C}(k)$ is fixed by the asymptotic condition (2.5), the quantity $\tilde{C}(k)k^{L+1}$ might either vanish or become infinite at $k=0$, so that setting $k=0$ in $f(k,r)$ might not yield a zero-energy solution. Let $f_0(r)$ be a solution of

$$[T(r)+ZZ'e^2/r+U(0)]f_0(r)=0\,, \quad (2.8)$$

which is regular at the origin. Then as k approaches zero $f(k,r)$ must become proportional to $\tilde{C}(k)k^{L+1}f_0(r)$, and we fix the normalization of $f_0(r)$ by choosing, for any fixed r,

$$f_0(r)=\lim_{k\to 0}\left(\frac{f(k,r)}{k^{L+1}\tilde{C}(k)}\right). \quad (2.9)$$

The irregular function $g(k,r)$ has a $1/r^L$ singularity at the origin, so that, using the asymptotic conditions, Eqs. (2.5) and (2.6), and the Wronskian relationship (see Sec. 3 below for the definition of $W_r[X,Y]$)

$$W_r[f(k,r),g(k,r)]=\text{const}=-k\,,$$

we find, with Eq. (2.7) in the form

$$f(k,r)\rightarrow\tilde{C}(k)k^{L+1}r^{L+1}\,, \quad r\rightarrow 0\,,$$

that

$$g(k,r)\rightarrow[(2L+1)k^L\tilde{C}(k)r^L]^{-1}\,, \quad r\rightarrow 0\,. \quad (2.10)$$

Clearly we can now use this result to determine a convenient zero-energy irregular solution $g_0(r)$: It is uniquely defined by the differential equation

$$[T(r)+ZZ'e^2/r+U(0)]g_0(r)=0 \quad (2.11)$$

and the limiting condition

$$g_0(r)=\lim_{k\to 0}[k^L\tilde{C}(k)g(k,r)]. \quad (2.12)$$

The functions $v(k,r)$ and $v(0,r)$ satisfy, for all r, differential equations which are independent of the short-range potentials $V(k)$ and $V(0)$, but we define them according to boundary conditions which depend on these potentials, i.e., we define them to be the asymptotic forms of $u(k,r)$ and $u(0,r)$. Hence we defer the definition of $v(k,r)$ and $v(0,r)$ until we have discussed the boundary conditions on the scattering functions $u(k,r)$ and $u(0,r)$.

B. The Scattering Functions

We have defined $u(k,r)$, to within a normalization factor, by Eqs. (2.1) and (2.2). We shall fix the normalization of $u(k,r)$ by fixing the zero-energy scattering solution and then requiring that $u(0,r)$ be this solution, i.e., that $u(0,r)$ satisfy the differential equation

$$[T(r)+ZZ'e^2/r+V(0)+U(0)]u(0,r)=0, \quad (2.13)$$

and the conditions

$$u(0,0)=0, \quad u(0,r)=\lim_{k\to 0} u(k,r). \quad (2.14)$$

Since $V(0)$ is by definition a short-range function, $u(0,r)$ must approach a linear combination of $f_0(r)$ and of $g_0(r)$. We fix the normalization of $u(0,r)$ by imposing the asymptotic condition

$$u(0,r)\to v(0,r), \quad r\to\infty, \quad (2.15)$$

where

$$v(0,r)=(-1/A)f_0(r)+g_0(r); \quad (2.16)$$

consequently, $v(0,r)$ satisfies the differential equation

$$[T(r)+ZZ'e^2/r+U(0)]v(0,r)=0. \quad (2.17)$$

We choose $v(k,r)$ to be that linear combination of $f(k,r)$ and $g(k,r)$ which satisfies

$$u(k,r)\to v(k,r), \quad r\to\infty. \quad (2.18)$$

Comparing Eq. (2.18) with the asymptotic forms (2.3), (2.5), and (2.6), we now find that

$$v(k,r)=F(k)\cos\delta(k)[f(k,r)+g(k,r)\tan\delta(k)]. \quad (2.19)$$

We now fix $F(k)$ by noting that Eqs. (2.14), (2.15), and (2.18) result in

$$v(0,r)=\lim_{k\to 0} v(k,r). \quad (2.20)$$

Hence Eqs. (2.16), (2.19), (2.20), (2.9), and (2.12) result in

$$\lim_{k\to 0}[F(k)\cos\delta(k)k^{L+1}\tilde{C}(k)]=-1/A, \quad (2.21)$$

$$\lim_{k\to 0}\left[\frac{F(k)\sin\delta(k)}{k^L\tilde{C}(k)}\right]=1, \quad (2.22)$$

and since Eq. (2.20) is the only requirement on $F(k)$ we can choose

$$F(k)=k^L\tilde{C}(k)/\sin\delta(k) \quad (2.23)$$

for *all* k. Replacing $F(k)$ in Eq. (2.19) and (2.21) by the right-hand side of Eq. (2.23) we finally arrive at

$$\lim_{k\to 0}[k^{2L+1}\tilde{C}^2(k)\cot\delta(k)]=-1/A, \quad (2.24)$$

$$v(k,r)=k^L\tilde{C}(k)[\cot\delta(k)f(k,r)+g(k,r)]. \quad (2.25)$$

3. THE MODIFIED EFFECTIVE-RANGE THEORY (MERT) EXPANSION

Having introduced the various functions, we are now prepared to derive the MERT expansion. We denote the Wronskian of X and Y by

$$W_a[X,Y]=W_a[X(r),Y(r)]=[X(a)Y'(a)-X'(a)Y(a)],$$

where the prime indicates differentiation with respect to r, and we define

$$W_{ba}[X,Y]\equiv W_b[X,Y]-W_a[X,Y].$$

Multiplying Eq. (2.1) for $u(k,r)$ *by* $u(0,r)$ and Eq. (2.13) for $u(0,r)$ by $u(k,r)$, subtracting, integrating between the arbitrary limits a and b, and using the Hermiticity of $\mathcal{V}(k)$ to invert some terms in the integrand, we obtain

$$W_{ba}[u(k,r),u(0,r)]=k^2\int_a^b u(k,r)u(0,r)dr$$
$$-(2\mu/\hbar^2)\int_a^b u(k,r)[V(k)-V(0)]u(0,r)dr$$
$$-(2\mu/\hbar^2)\int_a^b u(k,r)[U(k)-U(0)]u(0,r)dr. \quad (3.1)$$

Proceeding in precisely the same way with Eqs. (2.4) and (2.17) for $v(k,r)$ *and* $v(0,r)$ we find

$$W_{ba}[v(k,r),v(0,r)]=k^2\int_a^b v(k,r)v(0,r)dr$$
$$-(2\mu/\hbar^2)\int_a^b v(k,r)[U(k)-U(0)]v(0,r)dr. \quad (3.2)$$

Substracting Eq. (3.1) from Eq. (3.2) and letting $b\to\infty$ we find

$$W_a[u(k,r),u(0,r)]-W_a[v(k,r),v(0,r)]$$
$$=k^2\int_a^\infty [v(k,r)v(0,r)-u(k,r)u(0,r)]dr+R(k,a), \quad (3.3)$$

where

$$R(k,a)\equiv(2\mu/\hbar^2)\int_a^\infty u(k,r)[V(k)-V(0)]u(0,r)dr$$
$$+(2\mu/\hbar^2)\int_a^\infty \{u(k,r)[U(k)-U(0)]u(0,r)$$
$$-v(k,r)[U(k)-U(0)]v(0,r)\}dr,$$

the Wronskians with respect to b canceling since $u(k,r)$ and $u(0,r)$ approach $v(k,r)$ and $v(0,r)$, respectively. The integrals in Eq. (3.3) are finite, since all of the integrands are short-range functions. We now want to let $a\to 0$, but other than for $L=0$ we are not yet in a position to do so throughout, since for $L>0$ $v(k,r)$ and $v(0,r)$ are singular at the origin. We can, however, let a go to zero in all nonsingular terms. To begin with, since

$$u(0,0)=u(k,0)=0,$$

we see that as $a \to 0$

$$W_a[u(k,r),u(0,r)] \to 0. \tag{3.4}$$

Next, we use Eqs. (2.25) and (2.16) to express $v(k,r)$ and $v(0,r)$ in terms of $f(k,r)$ and $g(k,r)$, and $f_0(r)$ and $g_0(r)$, and replace $v(k,r)$ and $v(0,r)$ on the left-hand side of Eq. (3.3) by these expressions. Equations (2.7), (2.9), (2.10), and (2.12) result in

$$W_a[f(k,r),g_0(r)] \to -k^{L+1}\tilde{C}(k),$$
$$W_a[g(k,r),f_0(r)] \to [k^L\tilde{C}(k)]^{-1},$$
$$W_a[f(k,r),f_0(r)] \to 0,$$

as $a \to 0$. Putting these limits, together with the limit (3.4), into Eq. (3.3), and letting $a \to 0$ for nonsingular terms we find

$$k^{2L+1}\tilde{C}^2(k)\cot\delta(k) - k^L\tilde{C}(k)W_a[g(k,r),g_0(r)] = -1/A + k^2\int_a^\infty [v(k,r)v(0,r) - u(k,r)u(0,r)]dr + R(k,a). \tag{3.5}$$

Up to this point, we have made no specific assumption about the behavior of $U(k)$ for r small. Since all that is required of $U(k)$ is that it contain all long-range terms in the equivalent one-body potential, we can simplify the analysis by assuming that $U(k)=0$ for $r<d$, where d is arbitrary. [More precisely, the kernel $U(k;r,r')=0$ for $r<d$.]

It follows that

$$f_0(r) = \bar{F}_L(0,r), \qquad r<d, \tag{3.6a}$$

$$f(k,r) = [\tilde{C}(k)/C_L(n)]F_L(k,r), \quad r<d, \tag{3.6b}$$

$$g_0(r) = \bar{G}_L(0,r) + \bar{E}_0\bar{F}_L(0,r), \qquad r<d, \tag{3.6c}$$

$$g(k,r) = [C_L(n)/\tilde{C}(k)]G_L(k,r) + [\tilde{E}(k)/C_L(n)]F_L(k,r), \quad r<d, \tag{3.6d}$$

where $F_L(k,r)$, $G_L(k,r)$, $\bar{F}_L(0,r)$ and $\bar{G}_L(0,r)$ are the Coulomb wave functions given in Appendix A, $\tilde{E}(k)$ is defined by the last equation and is to be determined numerically, and

$$\lim_{k\to 0}[k^{2L+1}\tilde{C}(k)\tilde{E}(k)] = \bar{E}_0. \tag{3.7}$$

We can now use Eqs. (A10), (A11), and (A12) to evaluate $\tilde{C}(k)k^LW_a[g(k,r),g_0(r)]$. Putting the result into Eq. (3.5) and subtracting the divergent terms from both sides of the resulting equation [see the discussion below Eq. (A12)], we can now let $a \to 0$ and find

$$\tilde{C}^2(k)k^{2L+1}\cot\delta(k) + \tilde{h}(k) = -1/A + k^2\int_0^{a_0/2}\left\{v(k,r)v(0,r) - u(k,r)u(0,r) + (1-\delta_{L0})\left(\frac{1}{k^2r}\left[2nk^{2L+1}\frac{C_L^2(n)}{C_0^2(n)} - \frac{\epsilon}{\Gamma^2(2L+2)}\left(\frac{2}{a_0}\right)^{2L+1}\right] - \phi_L(k,r)\right)\right\}dr$$
$$+ k^2\int_{a_0/2}^\infty [v(k,r)v(0,r) - u(k,r)u(0,r) - (1-\delta_{L0})\phi_L(k,r)]dr + R(k,0), \tag{3.8}$$

where

$$\tilde{h}(k) = k^{2L+1}\tilde{C}(k)\tilde{E}(k) - \bar{E}_0 + h_L(n),$$

$h_L(n)$ is given in Appendix A [Eqs. (A9)], δ_{L0} is a Kronecker-delta function, and ϵ and $\phi_L(k,r)$ are defined in Appendix A.

To obtain the MERT expansion, we keep only terms up to k^2 in Eq. (3.8). Then, using Eqs. (1.4) and (1.5), we find

$$\tilde{C}^2(k)k^{2L+1}\cot\delta(k) + \tilde{h}(k) = -1/A + \tfrac{1}{2}r_0k^2 + \cdots, \tag{3.9}$$

where

$$\tfrac{1}{2}r_0 = \int_0^{a_0/2}\left\{v^2(0,r) - u^2(0,r) + (1-\delta_{L0})\left[\frac{\epsilon a_0^2}{\Gamma^2(2L+2)}\left(\frac{2}{a_0}\right)^{2L+1}\frac{L(2L+1)(L+1)}{6r} - \phi_L(0,r)\right]\right\}dr$$
$$+ \int_{a_0/2}^\infty [v^2(0,r) - u^2(0,r) - (1-\delta_{L0})\phi_L(0,r)]dr + (2\mu/\hbar^2)\int_0^\infty u(0,r)V_1u(0,r)dr$$
$$+ (2\mu/\hbar^2)\int_0^\infty [u(0,r)U_1u(0,r) - v(0,r)U_1v(0,r)]dr. \tag{3.10}$$

The only reason for obtaining the expression (3.10) is to show that r_0 is finite, which we now do. The first two integrals are finite by construction. The third integral is finite because $u(0,0)$ vanishes, and because the integrand is a short-range function. The last integral is finite because $u(0,r)$ approaches $v(0,r)$ asymptotically in such a way that the integrand becomes a short-range function, and because we have required U_1 to vanish for

$r<d$. Consequently, one can use Eq. (3.9) to match the data, determining A and r_0 in the process, and forgetting the complicated expression (3.10) entirely.

4. THE BORN APPROXIMATION

The Born approximation phase shift η_B for scattering by a non-Coulombic long-range potential U in the presence of a Coulomb potential will now be obtained both because the leading term of η_B is, as will be shown below, exactly equal to the leading term of η when the Coulomb field is repulsive and because η_B serves as a preliminary orientation in studying the effect of U. In particular, if there are special effects at low energies due to the long-range tail of the potential, these must appear in the Born approximation, since at sufficiently great distances the long-range potential, which falls off as some power of $1/r$, is arbitrarily weak. As a concrete example, we consider here the electric dipole polarization potential $U(r)$ in the presence of the Coulomb field. The long-range form of $U(r)$ is

$$U(r)=-\beta^2r^{-4}\hbar^2/(2\mu). \quad (4.1)$$

Since we are looking for special effects due to the long-range tail, and since the lowest order contribution to η_B from the short-range part of the potential has an easily recognizable energy dependence, it will be sufficient to find η_B for an unrealistic potential which has the form of Eq. (4.1) all the way into the origin. This simplifies the calculations but forces us to consider $L\geq 1$ only. We will, however, be able to get some qualitative ideas about the $L=0$ phase shift based on the results for $L\geq 1$, and we will derive the exact result for the $L=0$ phase shift in the limit of very low energies in the case of the repulsive Coulomb field, in Appendix B.

In the absence of a Coulomb field, if $U(r)$ is given by Eq. (4.1) for all r, we find[2]

$$\tan\eta_B=\pi(\beta k)^2[(2L-1)(2L+1)(2L+3)]^{-1}, \quad (L\geq 1,\ \text{no Coulomb field}) \quad (4.2)$$

and this is quite different from the lowest order energy dependence (k^{2L+1}) for a short-range potential. Since if we cut off $U(r)$ beyond some large value of r, however large, it is still true that for sufficiently small k, η_B has the k^{2L+1} dependence of a short-range potential, we conclude that the k^2 dependence of Eq. (4.2) is due solely to the long-range $1/r^4$ behavior of $U(r)$. Further, Eq. (4.2) must be exactly the lowest order term for the *true* electric dipole polarization potential, even though we have put in the wrong short-range potential, since any short-range potential will contribute terms of order k^{2L+1} and higher. The $L=0$ case is more complicated, since in the absence of the centrifugal barrier any short-range potential must contribute a significant part of the phase shift. Of course, special effects due to the long-range potential must appear for $L=0$ also.[1,2]

In the presence of a Coulomb field the results depend on the relative strengths of the Coulomb potential, the centrifugal barrier, and the kinetic energy, and on whether the Coulomb field is attractive or repulsive. It will again suffice to study the problem with the unrealistic potential $U(r)$, i.e., we consider the scattering function $w(k,r)$ which satisfies

$$\left[-\frac{d^2}{dr^2}+\frac{L(L+1)}{r^2}+\frac{2nk}{r}-\frac{\beta^2}{r^4}-k^2\right]w(k,r)=0, \quad (4.3)$$

where

$$n=-1/ka_0,\quad \text{attractive Coulomb field},$$
$$=+1/ka_0,\quad \text{repulsive Coulomb field},$$

and where, again, we must exclude $L=0$. If $w(k,r)$ is normalized so that

$$w(k,r)\to F_L(k,r)+G_L(k,r)\tan\eta(k),\quad r\to\infty,$$

where $F_L(k,r)$ and $G_L(k,r)$ are the Coulomb functions defined in Appendix A, and $\eta(k)$ is the phase shift relative to Coulomb scattering, then the Born approximation phase shift $\eta_B(k)$ is given by

$$\tan\eta_B(k)=(\beta^2/k)\int_0^\infty F_L^2(k,r)r^{-4}dr,\quad L\geq 1. \quad (4.4)$$

This integral can be evaluated by integrating by parts twice and using the following properties of the Coulomb wave functions[6]:

$$\frac{dF_L(k,r)}{dr}=\left[\frac{L+1}{r}+\frac{nk}{L+1}\right]F_L(k,r)-\left[k^2+\frac{1}{a_0^2(L+1)^2}\right]^{1/2}F_{L+1}(k,r), \quad (4.5a)$$

$$\frac{dF_L(k,r)}{dr}=-\left[\frac{L}{r}+\frac{nk}{L}\right]F_L(k,r)+\left[k^2+\frac{1}{a_0^2L^2}\right]^{1/2}F_{L-1}(k,r),\quad L\neq 0, \quad (4.5b)$$

$$\int_0^\infty F_L(k,r)F_{L'}(k,r)r^{-2}dr=k[(L-L')(L+L'+1)]^{-1}\times\sin[\tfrac{1}{2}(L-L')\pi-(\sigma_L-\sigma_{L'})], \quad (4.6)$$

$$\int_0^\infty F_L^2(k,r)r^{-2}dr=k(2L+1)^{-1}\chi_L(n), \quad (4.7)$$

where

$$\chi_L(n)=\frac{\pi}{2}-\sum_{m=L+1}^{\infty}\frac{n}{m^2+n^2}=\frac{\pi}{2}(1-\coth\pi n)+\frac{1}{2n}+\sum_{m=1}^{L}\frac{n}{m^2+n^2}. \quad (4.8)$$

Equations (4.5) and (4.6) follow from the differential equations and asymptotic forms that $F_L(k,r)$ and

$F_{L'}(k,r)$ satisfy, and Eq. (4.7) follows from Eq. (4.6) and from

$$\frac{d\sigma_L}{dL}=\mathrm{Im}\Psi(L+1+in)=\sum_{m=L+1}^{\infty}\frac{n}{m^2+n^2},$$

where $\Psi(z)$ is the logarithmic derivative of the gamma function $\Gamma(z)$ and Im denotes the imaginary part. The result of integrating Eq. (4.4) is

$$\tan\eta_B=\frac{\beta^2\{2[3+k^2a_0{}^2L(L+1)]\chi_L(n)-3n^{-1}(2L+1)\}}{a_0{}^2L(L+1)(2L+1)(2L-1)(2L+3)},\quad L\geq 1. \tag{4.9}$$

We will find the lowest order energy term of Eq. (4.9) in two limits.

If $Lka_0\ll 1$, then $|n|=(ka_0)^{-1}$ is very large compared to L, and we can write

$$\begin{aligned}\sum_{m=1}^{L}\frac{n}{m^2+n^2}&=\frac{1}{n}\sum_{m=1}^{L}\left(1-\frac{m^2}{n^2}+\frac{m^4}{n^4}-\cdots\right)\\&=\frac{L}{n}\left[1-\frac{(L+1)(2L+1)}{6n^2}\right.\\&\qquad\left.+\frac{(L+1)(2L+1)(3L^2+3L-1)}{30n^4}-\cdots\right].\end{aligned} \tag{4.10}$$

We have also, for large $|n|$,

$$\begin{aligned}1-\coth\pi n&=2+\mathcal{O}(e^{-2\pi/ka_0}), &&\text{attractive Coulomb field},\\&=\mathcal{O}(e^{-2\pi/ka_0}), &&\text{repulsive Coulomb field}.\end{aligned}$$

Putting this last result, together with the expression given by Eq. (4.10) for the sum, into Eq. (4.8) in the second form, we find

$$\tan\eta_B=\frac{\pi(\beta/a_0)^2[6+2L(L+1)(ka_0)^2]}{L(L+1)(2L+1)(2L-1)(2L+3)}-\left(\frac{\beta}{a_0}\right)^2\frac{(ka_0)^5}{15}+\cdots+\mathcal{O}(e^{-2\pi/ka_0}),$$

$$(Lka_0\ll 1,\ \text{attractive Coulomb field});\tag{4.11}$$

$$\tan\eta_B=\left(\frac{\beta}{a_0}\right)^2\frac{(ka_0)^5}{15}+\cdots+\mathcal{O}(e^{-2\pi/ka_0}),$$

$$(Lka_0\ll 1,\ \text{repulsive Coulomb field}),\tag{4.12}$$

where in both Eq. (4.11) and Eq. (4.12) the ellipsis stands for terms of higher order than the fifth in (ka_0). Since at low energies the phase shift due to a short-range potential in the presence of a repulsive Coulomb field is of order $\exp(-2\pi/ka_0)$, Eq. (4.12) shows that special effects due to the long-range $1/r^4$ potential dominate over effects due to the short-range potential. Hence, the leading term in Eq. (4.12) is probably exact, regardless of the true short range. On the other hand, at low energies the phase shift due to a short-range potential in the presence of an *attractive* Coulomb field begins with a term independent of the energy, as does Eq. (4.11). Hence special effects due to the long-range $1/r^4$ potential do not dominate, and the leading term in Eq. (4.11) cannot be exact, since we have not taken account of the correct short-range potential.

To emphasize the meaning of the above results, we give an argument which, though only a qualitative classical argument, is not without physical content. We note first that $Lka_0\ll 1$ implies (since we have restricted our attention to $L\geq 1$) a very large value of $|n|$, hence a strong Coulomb field as compared to the centrifugal barrier. The result, in the case of the attractive Coulomb field, is that the Coulomb field pulls the particle in toward the origin at least as far as the point $r=R_{Lc}$ at which the Coulomb potential just equals the centrifugal barrier, where

$$R_{Lc}=\tfrac{1}{2}L(L+1)a_0.$$

But if the Coulomb field is not present, the particle penetrates only as far as $r=R_{Lk}$, where

$$R_{Lk}=k^{-1}[L(L+1)]^{1/2}.$$

Since $Lka_0\ll 1$ implies $R_{Lc}\ll R_{Lk}$, the effect of the strongly attractive Coulomb field is that the short-range contribution of the potential dominates over its long-range contribution. This result is certainly true when $L=0$ also, since there is then no centrifugal barrier to interfere with the action of the attractive Coulomb field.

In the case of the repulsive Coulomb field, both the Coulomb potential and the centrifugal potential are barriers, hence the long-range contribution of the $1/r^4$ potential to the phase shift must dominate over its short-range contribution. However, since the k^5 dependence of Eq. (4.12) is quite different from the k^2 dependence of Eq. (4.2), and since L does not even appear in the coefficient of the k^5 term, the centrifugal barrier is not only dominated by the strongly repulsive Coulomb potential but is irrelevant with regard to the determination of the leading term in η. To see this more clearly, we compare the two barriers at $r=R_{ck}$, where

$$R_{ck}=2/(a_0k^2)$$

is the turning point, the point at which the energy is equal to the Coulomb potential. If $Lka_0\ll 1$, then the centrifugal barrier is negligible compared to the Coulomb barrier at $r=R_{ck}$, and therefore it is the Coulomb barrier which prevents the particle from penetrating any further than $r=R_{ck}$. Having noticed this, we can actually use it to rederive the k^5 dependence. If we assume that the contribution to the phase shift is negligible for $r<R_{ck}$, we have

$$\tan\eta_B=(\beta^2/k)\int_{R_{ck}}^{\infty} F_L{}^2(k,r)r^{-4}dr. \tag{4.13}$$

For very small energies R_{ck} is a large number. Assuming it to be sufficiently large in $r\geq R_{ck}$ for $F_L(k,r)$ to be approximated by its asymptotic form,

$$F_L(k,r)\cong\sin(kr-n\ln 2kr-\tfrac{1}{2}L\pi+\sigma_L),$$

and approximating the integral in Eq. (4.13) by using the average value $(\sin^2\theta)_{\mathrm{av}}=\frac{1}{2}$, we find

$$\tan\eta_B\approx\tfrac{1}{2}\beta^2k^{-1}\int_{R_{ck}}^{\infty}\frac{dr}{r^4}=\left(\frac{\beta}{a_0}\right)^2\frac{(ka_0)^5}{48}, \tag{4.14}$$

which, like Eq. (4.12), has a k^5 dependence and no L dependence. Since for $L\geq 1$ and a strong Coulomb barrier, the centrifugal barrier is irrelevant, to lowest order, we suspect that Eq. (4.12) may actually give the lowest order energy dependence for $L=0$ also, for a potential which has the r^{-4} long-range behavior but is finite at the origin. We will prove in Appendix B that Eq. (4.12) is in fact valid for $L=0$.

Equation (4.14) is not exact, since it differs from the correct result [the leading term in Eq. (4.12)] by the numerical coefficient. But we could not expect it to be exact, since in reality it is not sufficient to approximate η_B by merely using the asymptotic form of $F_L(k,r)$ in Eq. (4.13). (It would suffice to use the W. K. B. approximation. See Appendix B.) The reason we have gone through the argument, however, is that it provides some physical insight.

The other limit that we want to mention is $Lka_0\gg 1$. Here

$$|n|\sum_{m=L+1}^{\infty}\frac{1}{m^2+n^2}<|n|\sum_{m=L+1}^{\infty}\frac{1}{m^2}<|n|\sum_{m=L}^{\infty}\frac{1}{m(m+1)}=\frac{1}{Lka_0}\ll 1,$$

and Eq. (4.8) in the first form, for either Coulomb field, becomes

$$\chi_L(n)\cong\tfrac{1}{2}\pi.$$

Hence Eq. (4.9) results in

$$\tan\eta_B\approx\beta^2\pi k^2[(2L-1)(2L+1)(2L+3)]^{-1},\quad (Lka_0\gg 1,\text{ either Coulomb field}).$$

But this is precisely the same as Eq. (4.2); hence to lowest order the Coulomb field plays no role. Since $Lka_0\gg 1$ implies a relatively large L, i.e., a large centrifugal barrier, and a relatively small n, i.e., a weak Coulomb field, this is precisely what we should expect. Of course, if we look sufficiently far out, the Coulomb field, however weak, eventually dominates over the centrifugal barrier. However, the region where a weak Coulomb field dominates over the centrifugal barrier is only important for very small values of k, sufficiently small to violate the condition $Lka_0\gg 1$.

APPENDIX A: COULOMB FUNCTIONS

The Coulomb wave functions satisfy the differential equation

$$\{T(r)+ZZ'e^2/r-E\}\{F_L(k,r),G_L(k,r)\}=0.$$

The Coulomb wave function which is regular at the origin is normalized to unit amplitude asymptotically. With this normalization, it is found to satisfy

$$F_L(k,r)\to\sin(kr-n\ln 2kr-\tfrac{1}{2}L\pi+\sigma_L),\quad r\to\infty.$$

The irregular Coulomb wave function is chosen to satisfy the asymptotic condition

$$G_L(k,r)\to\cos(kr-n\ln 2kr-\tfrac{1}{2}L\pi+\sigma_L),\quad r\to\infty.$$

These functions can be shown to have the following series expansions (undefined coefficients will be found in Ref. 6; their $a_j{}^L$ corresponds to our $B_j{}^L$, and their η corresponds to our n):

$$F_L(k,r)=C_L(n)\sum_{j=L+1}^{\infty}A_j{}^L(n)(kr)^j, \tag{A1}$$

$$G_L(k,r)=[C_L(n)(2L+1)]^{-1}\times\sum_{j=-L}^{\infty}B_j{}^L(n)(kr)^j+[2n/C_0{}^2(n)]\times F_L(k,r)[\ln 2kr+q_L(n)/p_L(n)], \tag{A2}$$

where

$$n=\epsilon(ka_0)^{-1},$$
$\epsilon=+1$ for the repulsive Coulomb field,
$\quad=-1$ for the attractive Coulomb field,
$$a_0=|\hbar^2(\mu ZZ'e^2)^{-1}|,$$
$$C_0{}^2(n)=2\pi n/(e^{2\pi n}-1),$$
$$C_L(n)=2^L\Gamma^{-1}(2L+2)|\Gamma(L+1+in)|e^{-\frac{1}{2}n\pi},$$
$$A_{L+1}{}^L=B_{-L}{}^L=1,\quad B_{L+1}{}^L=0.$$

The zero-energy solutions are, for the attractive Coulomb field,

$$\bar{F}_L(0,r)=\tfrac{1}{2}[(2L+1)!](a_0/2)^{L+1}\{xJ_{2L+1}(x)\}, \tag{A3}$$

$$\bar{G}_L(0,r)=-\tfrac{1}{2}[\pi/(2L+1)!](2/a_0)^L\{xN_{2L+1}(x)\}, \tag{A4}$$

and for the repulsive Coulomb field,

$$\bar{F}_L(0,r)=\tfrac{1}{2}[(2L+1)!](a_0/2)^{L+1}\times(i)^{-(2L+1)}\{xJ_{2L+1}(ix)\}, \tag{A5}$$

$$\bar{G}_L(0,r)=\tfrac{1}{2}(-1)^{L+1}[\pi/(2L+1)!]\times(2/a_0)^L\{xH_{2L+1}{}^{(1)}(ix)\}, \tag{A6}$$

where

$$x=(8r/a_0)^{1/2},$$

$J_{2L+1}(x)$ is the Bessel function of order $2L+1$,
$N_{2L+1}(x)$ is the Neumann function of order $2L+1$,

and

$H_{2L+1}^{(1)}(ix)$ is the Hankel function of the first kind of order $2L+1$.

[We note that the $\bar{F}_L(0,r)$ and the $\bar{G}_L(0,r)$ are all real functions. We note further that they are usually written without the bars; however, since they are not the functions $F_L(k,r)$ and $G_L(k,r)$ with $k=0$, we use the bars to preserve this distinction.]

Substituting the series expansions for $J_m(x)$, $N_m(x)$, and $H_m^{(1)}(x)$ into Eqs. (A3) to (A6), one finds for either Coulomb field:

$$\bar{F}_L(0,r)=\sum_{j=L+1}^{\infty}\alpha_j^L r^j, \tag{A7}$$

$$\bar{G}_L(0,r)=\sum_{j=-L}^{L}\beta_j^L r^j+\epsilon(2/a_0)^{2L+1}\Gamma^{-2}(2L+2)\bar{F}_L(0,r)\times[\ln(2r/a_0)+2\gamma-\sum_{s=1}^{2L+1}(1/s)]+\cdots, \tag{A8}$$

where

$$\alpha_j^L=\epsilon^{j+L+1}\Gamma(2L+2)(2/a_0)^{j-L-1}\times[\Gamma(j-L)\Gamma(j+L+1)]^{-1},$$

$$\beta_j^L=(-\epsilon)^{j+L}\Gamma(L+1-j)(2/a_0)^{j+L}\times[\Gamma(L+1+j)\Gamma(2L+2)]^{-1},$$

$\gamma=0.5772\ldots$ (Euler's constant),

and terms in $\bar{G}_L(0,r)$ which have been dropped are of order r^{L+2} or higher.

The function $h_L(n)$ which occurs in any effective-range theory expansion for scattering in the presence of a Coulomb field is given by:

$$h_0(n)=\epsilon(2/a_0)\left[-\ln(\epsilon n)+\mathrm{Re}\frac{\Gamma'(in)}{\Gamma(in)}\right], \tag{A9a}$$

$$h_L(n)=[k^L/(2L+1)]\sum_{j=-L+1}^{L}(1-2j)\beta_j^L B_{1-j}^L k^{1-j}+[2nk^{2L+1}C_L^2(n)/C_0^2(n)-\epsilon(2/a_0)^{2L+1}\Gamma^{-2}(2L+2)]\times\left[2\gamma-\sum_1^{2L}\frac{1}{s}\right]+2nk^{2L+1}C_L^2(n)/C_0^2(n)\times[Q_L(n)+r_L(n)/p_L(n)]. \tag{A9b}$$

The Coulomb wave functions satisfy the following Wronskian relationships, as $a\to 0$,

$$W_a[F_L(k,r),\bar{G}_L(0,r)]\to -k^{L+1}C_L(n),$$
$$W_a[G_L(k,r),\bar{F}_L(0,r)]\to [k^L C_L(n)]^{-1}, \tag{A10}$$
$$W_a[F_L(k,r),\bar{F}_L(0,r)]\to 0.$$

The $L=0$ irregular Coulomb wave functions satisfy

$$C_0(n)W_0[\bar{G}_0(0,r),G_0(k,r)]=h_0(n), \tag{A11}$$

and for a and L both nonzero, neglecting some terms which vanish at $a=0$,

$$k^L C_L(n)W_a[\bar{G}_L(0,r),G_L(k,r)]=\frac{k^L}{2L+1}\sum_{j,l=-L}^{L}\beta_j^L B_l^L k^l a^{l+j-1}(l-j)+\left[2nk^{2L+1}\frac{C_L^2(n)}{C_0^2(n)}-\left(\frac{2}{a_0}\right)^{2L+1}\frac{\epsilon}{\Gamma^2(2L+2)}\right]\times\left[\ln\left(\frac{2a}{a_0}\right)+2\gamma-\sum_{s=1}^{2L}\frac{1}{s}\right]+2nk^{2L+1}\frac{C_L^2(n)}{C_0^2(n)}\left[Q_L(n)+\frac{r_L(n)}{p_L(n)}\right], \tag{A12}$$

where

$$Q_L(n)=-\ln(\epsilon n)+\mathrm{Re}\frac{\Gamma'(in)}{\Gamma(in)}+\sum_{s=1}^{L}\frac{s}{s^2+n^2}.$$

The left-hand side of Eq. (A12) involves: (1) terms which diverge as $1/a^m$, where m ranges from 1 to $(2L+1)$. These terms, which can be written as

$$1/a^m=m\int_a^\infty r^{-(m+1)}dr,$$

will be lumped together as

$$k^2\int_a^\infty \phi_L(k,r)dr,$$

(2) terms which are independent of a; the sum of these terms is equal to $h_L(n)$, given by Eq. (A9b), (3) terms which vanish as $a\to 0$, and (4) a divergent term, $\ln(2a/a_0)$, which can be written as

$$\ln(2a/a_0)=-\int_a^{a_0/2}dr/r.$$

APPENDIX B: NOTES ON THE BORN APPROXIMATION

We prove here that Eq. (4.12) gives the lowest order energy dependence of the phase shift in the presence of the repulsive Coulomb field for $L=0$ as well as for $L\geqslant 1$. Since we are only interested in the leading term for small energies, we shall, throughout the derivation, ignore terms of order $\exp[-2\pi/ka_0]$. The integral we want to evaluate is

$$\tan\eta_B(k)=(\beta^2/k)\int_R^\infty F_0^2(k,r)r^{-4}dr,$$

where $R\neq 0$ but is otherwise arbitrary. Integrating by parts once, and using Eq. (4.5a), we find

$$\tan\eta_B(k)\cong\beta^2\left\{2n\int_R^\infty r^{-3}F_0^2(k,r)dr - 2[1+n^2]^{1/2}\int_R^\infty r^{-3}F_0(k,r)F_1(k,r)dr\right\}, \quad \text{(B1)}$$

where we have dropped the integrated term since for small k it is irrelevant:

$$(\beta^2/k)F_0^2(k,R)/R^3 \to \beta^2 kC_0^2/R \to 2\pi e^{-2\pi/ka_0}\beta^2/(Ra_0).$$

The second integral in Eq. (B1) can be integrated by parts, and using Eqs. (4.5) and (4.7), we find

$$\int_R^\infty r^{-3}F_0(k,r)F_1(k,r)dr \cong \tfrac{1}{2}k[1+n^2]^{1/2}\int_R^\infty [F_0^2(k,r)-F_1^2(k,r)]r^{-2}dr \cong [6n(1+n^2)^{1/2}]^{-1}k^2, \quad \text{(B2)}$$

where we have again dropped terms of order $e^{-2\pi/ka_0}$. To evaluate the first integral in Eq. (B1), we first consider $L\neq 0$ and integrate by parts, using Eq. (4.5a), to find

$$L\int_R^\infty r^{-3}F_L^2(k,r)dr=\left[\frac{F_L^2}{2r^2}\right]_R^\infty-\frac{nk}{L+1}\int_R^\infty r^{-2}F_L^2(k,r)dr +k[1+n^2/(L+1)^2]^{1/2}\int_R^\infty r^{-2}F_L(k,r)F_{L+1}(k,r)dr. \quad \text{(B3)}$$

Both integrals on the right-hand side of Eq. (B3) can be integrated for all values of L. But when $L=0$ each side of the equation vanishes identically. However, we can use the equation to evaluate the integral on the left when $L=0$ by integrating the right-hand side for $L\neq 0$, taking the derivative with respect to L of both sides of the equation, and finally letting $L=0$. The result is

$$\int_R^\infty r^{-3}F_0^2(k,r)dr\cong k^2[3n\chi_0(n)+n\,\mathrm{Im}\Psi'(1+in)-\tfrac{1}{2}],$$

where $\Psi'(z)$ is the derivative with respect to z of $\Psi(z)$, and where terms we have dropped are of order $\exp(-2\pi/ka_0)$. Since

$$\chi_0(n)=\tfrac{1}{2}\pi(1-\coth\pi n)+1/2n\cong 1/2n$$

and

$$\mathrm{Im}\Psi'(1+in)=-\frac{1}{n}+\frac{1}{6n^3}+\frac{1}{30n^5}+\cdots,$$

we obtain

$$\int_R^\infty r^{-3}F_0^2(k,r)dr\cong(k^2/6)[n^{-2}+n^{-4}/5].$$

Putting this result, together with Eq. (B2), into Eq. (B1), we find that to lowest order

$$\tan\eta_B(k)\cong(\beta/a_0)^2((ka_0)^5/15),$$

which is what we set out to prove.

We note that the finite integrals from zero to infinity discussed in Sec. 4 can also be extracted from the paper by Alder *et al.*[8] In addition, their WKB results can be used to obtain $\tan\eta_B$ to lowest order in the energy for any $1/r^p$ potential, where p is an integer. (In the following, all equation numbers with Roman numerals refer to equations in Ref. 8.) Assuming that the potential is given, for all r, by

$$U(r)=-(\hbar^2/2\mu)(\beta_p^{p-2}/r^p),$$

where β_p has the dimensions of length, we find, for the orbital angular momentum L,

$$\tan\eta_B=k\beta_p^{p-2}M_{L,L}^{-p},$$

where

$$M_{L,L}^{-p}=k^{-2}\int_0^\infty F_L(k,r)r^{-p}F_L(k,r)dr.$$

We have introduced the notation $M_{L,L}^{-p}$ in order to make contact with Eq. (II B.46); note that we have $k_i=k_f=k$ and $l_i=l_f=L$. In the WKB approximation, which is good to lowest order in the energy, $M_{L,L}^{-p}$ is given by Eq. (II B.98); for our purposes, this expression simplifies enormously, since $\xi=0$ for $k_i=k_f$ [see Eq. (II B.95), noting that we use the symbol n where these authors use η], and since $\epsilon\cong 1$ [see Eq. (II B.96)] for $ka_0L\ll 1$. There results

$$M_{L,L}^{-p}\cong\frac{k^{p-3}}{2n^{p-1}}\int_0^\infty[\cosh\omega+1]^{-p+1}d\omega.$$

The transformation $x=e^\omega+1$ reduces the integral to a trivial form, and we find

$$\tan\eta_B\cong\left(\frac{\beta_p}{a_0}\right)^{p-2}(ka_0)^{2p-3} \times\left[(p-2)!2^{-p+1}\sum_{m=0}^{p-2}\frac{(-1)^{p-1-m}2^m}{m!(p-2-m)!(m-2p+3)}\right].$$

For $p=4$, this reproduces the lowest order term as given by Eq. (4.12). As two further examples, we find

$$\tan\eta_B\cong(\beta_5/a_0)^3(ka_0)^7/35,\quad p=5,$$

and

$$\tan\eta_B\cong 4(\beta_6/a_0)^4(ka_0)^9/315,\quad p=6.$$

Note added in proof. Recently, some consideration has been given by Bassel, Drisko, Satchler, Lee, Jr., Schiffer, and Zeidman to long-range polarization potential effects in deuteron scattering by ^{40}Ca, and in the competing deuteron-stripping reaction, at 7–12 MeV. As was to be expected, the polarization potential was found to have little effect at these energies. See Phys. Rev. **136**, B960 and B971 (1964).

[8] K. Alder, A. Bohr, T. Huus, B. Mottelson, and A. Winther, Rev. Mod. Phys. 28, 432 (1956).