Internal Conversion Coefficients: General Formulation for all Shells and Application to Low-Energy Transitions

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Exact analytical results (in the case of a point nucleus with no screening) for the internal conversion coefficients for any shell are presented. These results simplify considerably at threshold values of the gamma-ray energy. Numerical results, at threshold, are obtained for the K, $L(L_{\text{I}}, L_{\text{II}}, L_{\text{III}})$ and $M(M_{\text{I}}, M_{\text{II}}, M_{\text{III}}, M_{\text{IV}}, M_{\text{V}})$ shells, for 19 values of Z in the range $5 \leqslant Z \leqslant 95$, and for the first five electric and magnetic multipoles. The results for the K shell agree with those obtained by Spinrad. It is shown that the theshold results are actually correct to order P (momentum of electron in units of its mass). The effects of finite nuclear size are also considered.

INTRODUCTION

CALCULATIONS of internal conversion coefficients (hereafter referred to as ICC's) based on a point nucleus with no screening have been performed by Rose *et al.*[1,2] Subsequent tabulations of ICC's by Sliv *et al.*[3] and by Rose[4] included both the static effect of the nuclear size and the effect of screening. A detailed discussion of nuclear-structure effects, as well as a comprehensive bibliography, is given in the review article of Church and Weneser.[5]

Sliv *et al.* and Rose both consider the nuclear charge as being uniformly distributed in a sphere of radius $R=1.2A^{1/3}\times10^{-13}$ cm. They differ only in their treatment of the so-called penetration terms[5]; Sliv *et al.* evaluated the penetration terms by assuming a nuclear model in which the nuclear transition current lies on the spherical nuclear surface whereas Rose set the penetration terms equal to zero. A discussion of nuclear-structure effects in ICC's, from a very general viewpoint, is given by Green and Rose.[6] In most cases the ICC's for a finite-size nucleus are smaller than those calculated for a point nucleus with larger reductions occurring for larger Z values, particularly for $M1$ transitions. In general, penetration terms are at most only a few percent of the principal term, the exceptions occurring when the gamma-ray matrix element is inhibited while the penetration terms are allowed.[5]

Thus we expect that the results of Sliv *et al.* and Rose should agree to within a few percent; however, as pointed out by Church and Weneser,[5] there are unexplained disagreements between the two tabulations. A further cause of concern is the disagreement between the tabulated values and various experimental results. For example, McGowan and Stelson[7] measured K-shell ICC's for a number of pure $E2$ and mixed $E2+M1$ transitions in several rare-earth nuclei, at gamma-ray energies ≤ 140 keV, and obtained results 20% higher than the tabulated values of Sliv *et al.* This is very surprising particularly for $E2$ transitions because the results of Sliv *et al.* (which include nuclear size effects) for $E2$ transitions agree to within 2% with the earlier results of Rose *et al.*[1] for the case of a point nucleus. This indicates that finite nuclear size has a relatively small influence on ICC's for $E2$ transitions. The latter fact also seems to rule out the possibility that the discrepancy is due to penetration effects (which Church and Weneser[5,8] showed could play an important role in $M1$ transitions). A detailed account of the various discrepancies between experiment and theory, as well as a complete list of references, may be found in the paper of Frey, Hamilton, and Hultberg[9] and in the review article of Ramaswamy.[10] In fact, Ramaswamy points out that the observed deviations from theory in the case of unhindered $E2$ transitions pose a challenge to theoreticians. The question is whether the existing theoretical calculations are correct or perhaps whether the deviations may be correlated with nuclear deformations.

Results of recent measurements of ICC's at low gamma-ray energies also appear to be in disagreement

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[1] M. E. Rose, G. H. Goertzel, B. I. Spinrad, H. Harr, and P. Strong, Phys. Rev. **83**, 79 (1951).

[2] M. E. Rose, in *Beta- and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (Interscience Publishers, Inc., New York, 1955).

[3] L. A. Sliv and I. M. Band, *Coefficients of Internal Conversion of Gamma Radiation, Part I—K Shell and Part II—L Shell* (Physico-Technical Institute, Academy of Science, Leningrad, U. S. S. R., 1956) [issued in U. S. as Rept. 571CC K1 and Rept. 581CC L1, Physics Dept., University of Illinois, Urbana (unpublished)].

[4] M. E. Rose, *Internal Conversion Coefficients* (Interscience Publishers, Inc., New York, 1958).

[5] E. L. Church and J. Weneser, Ann. Rev. Nucl. Sci. **10**, 193 (1960).

[6] T. A. Green and M. E. Rose, Phys. Rev. **110**, 105 (1958).

[7] F. K. McGowan and P. M. Stelson, Phys. Rev. **107**, 1674 (1957).

[8] E. L. Church and J. Weneser, Phys. Rev. **104**, 1382 (1956).

[9] W. F. Frey, J. H. Hamilton, and S. Hultberg, Arkiv Fysik **21**, 383 (1962).

[10] M. K. Ramaswamy, Current Sci. (India) 32, 249 (1963).

with the results of Sliv *et al.* and Rose.[11] Furthermore, it is also desired to obtain information on ICC's for electron momenta p in the range 0 to a minimum of 0.32 (the latter value[12] corresponding to the lowest transition energy $\omega=0.05$ considered by both Sliv *et al.*[3] and Rose[4]). Thus we are provided with a strong motivation for re-examining and extending ICC tabulations. We consider it important to separate the effect on the ICC's due to the pure Coulomb field, nuclear size, and screening; and as an initial step in that direction, we calculate analytically the ICC's for all shells and for all values of the gamma-ray energy neglecting completely screening and nuclear-size effects. The radial integrals appearing in the calculation were evaluated analytically (previous analytic evaluations of radial integrals were confined to the K shell[1]) and the results expressed in the form of double sums which are most amenable to numerical evaluation. As pointed out by Rose *et al.*,[1] it is more difficult to obtain numerical results at low energies due to the occurrence of slowly convergent series. In view of this and because of the fact that the discrepancies between theory and experiment occur at low gamma-ray energies, we will focus our attention on the low-energy region. In particular, we will apply our results to the calculation of ICC's, for the K, L, and M shells, at threshold values of the gamma-ray energy. Previous calculations of threshold values of ICC's have been confined to the K shell.[13,14] We next show that all threshold results are actually correct to order p^2. In other words, the momentum spectrum of the ICC's, to order p, is parabolic, a fact which should prove extremely useful in interpolating to gamma-ray energies away from threshold. As will be noticed, the expansion is facilitated by factoring out the indicial ω dependence arising from the Hankel function but the range of validity of our approximation can only be stated in terms of the as yet uncomputed coefficient of the p^2 term. Finally, the effects of finite nuclear size are considered and a preliminary discussion of screening effects is included.

GENERAL INFORMATION

Our notation generally follows that used by Rose[2,15,16] and by Akhiezer and Berestetski.[17] The ICC's will be denoted by $\beta_L^{(\lambda)}$ where $\lambda=0$, 1 refer to magnetic and electric radiation, respectively. The expression for an angular-momentum eigenstate of the electron is[15]

$$\psi_\kappa=\begin{bmatrix} g_\kappa(r)\Omega_\kappa(\hat{r}) \\ if_\kappa(r)\Omega_{-\kappa}(\hat{r})\end{bmatrix}, \tag{1}$$

where $\kappa=\mp(j+\frac{1}{2})$ for $j=l\pm\frac{1}{2}$. The radial functions for the continuum-state electron will be designated by g_κ and f_κ while $g_{\kappa'}$ and $f_{\kappa'}$ will refer to the bound-state electron. Then the $\beta_L^{(\lambda)}$ can be written in terms of radial integrals as follows[2]:

$$\beta_L^{(0)}=\frac{\alpha\omega}{16\pi}\frac{1}{L(L+1)(2L+1)}\sum_\kappa B_{\kappa\kappa'}|R_5+R_6|^2, \tag{2}$$

$$\beta_L^{(1)}=\frac{\alpha\omega}{16\pi}\frac{L}{(L+1)(2L+1)} \times\sum_\kappa C_{\kappa\kappa'}\left|\frac{\kappa'-\kappa}{L}(R_3+R_4)+(R_4-R_3)+R_1+R_2\right|^2, \tag{3}$$

where α is the fine-structure constant, ω is the transition energy and the values of κ in Eq. (3) are the negatives of the values of κ in Eq. (2).[12] Values of $B_{\kappa\kappa'}$ and $C_{\kappa\kappa'}$ for various values of κ and κ' are tabulated by Rose.[18] The radial integrals are given by

$$R_1=\int_0^\infty g_\kappa g_{\kappa'}G_L r^2dr, \tag{4a}$$

$$R_2=\int_0^\infty f_\kappa f_{\kappa'}G_L r^2dr, \tag{4b}$$

$$R_3=i\int_0^\infty f_\kappa g_{\kappa'}G_{L-1} r^2dr, \tag{4c}$$

$$R_4=i\int_0^\infty g_\kappa f_{\kappa'}G_{L-1} r^2dr, \tag{4d}$$

$$R_5=\int_0^\infty f_\kappa g_{\kappa'}G_L r^2dr, \tag{4e}$$

$$R_6=\int_0^\infty g_\kappa f_{\kappa'}G_L r^2dr, \tag{4f}$$

where

$$G_L(\omega r)=(2\pi)^{3/2}i^L\frac{1}{(\omega r)^{1/2}}H_{L+1/2}^{(1)}(\omega r), \tag{5}$$

and $H_{L+1/2}^{(1)}(\omega r)$ is the Hankel function of the first kind.

The exact continuum-state wave functions are given by[15]

$$f_\kappa=i(w-1)^{1/2}\frac{e^{\pi\nu/2}}{(\pi p)^{1/2}}\frac{|\Gamma(\gamma+i\nu)|}{2\Gamma(2\gamma+1)}(2pr)^{\gamma}\frac{1}{r}I_-, \tag{6}$$

[11] J. W. Mihelich and E. Funk (private communication).

[12] In our units $\hbar=c=m=1$, where m refers to the mass of the electron.

[13] B. I. Spinrad, Phys. Rev. **98**, 1302 (1955).

[14] R. F. O'Connell, Nucl. Phys. **45**, 142 (1963); R. C. Young, Phys. Rev. **115**, 577 (1959).

[15] M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1960).

[16] M. E. Rose, *Multipole Fields* (John Wiley & Sons, Inc., New York, 1955).

[17] A. E. Akhiezer and V. B. Berestetski, Atomic Energy Commission Report AEC-tr-2876 (unpublished). This translation is unfortunately marred by many misprints; however, a revised and enlarged 2nd Russian edition of this book is now in press (John Wiley & Sons, New York).

[18] See Ref. 2. A typographical error in the tabulated values of $B_{\kappa\kappa'}$ (at the values $\kappa'=-2$ and $\kappa=-L+1$) appearing in this reference on page XV is corrected. The corrected value is $6L(L-1)(L+1)^2/(2L-1)$.

and

$$g_\kappa=(w+1)^{1/2}\frac{e^{\pi\nu/2}}{(\pi p)^{1/2}}\frac{|\Gamma(\gamma+i\nu)|}{2\Gamma(2\gamma+1)}(2pr)^{\gamma}\frac{1}{r}I_+, \quad (7)$$

where

$$I_\pm=\{e^{-ipr+i\eta}(\gamma+i\nu)\times{}_1F_1(\gamma+1+i\nu;2\gamma+1;2ipr)\pm\text{c.c.}\}, \quad (8)$$

${}_1F_1$ is the confluent hypergeometric function, $\Gamma(Z)$ is the gamma function and c.c. denotes complex conjugate. The energy and momentum of the electron are denoted by w and p, respectively;

$$\nu=\alpha Zw/p,\quad \gamma=(\kappa^2-\alpha^2Z^2)^{1/2},$$

and

$$e^{2i\eta}=\left(-\kappa+\frac{i\alpha Z}{p}\Big/\gamma+\frac{i\alpha Zw}{p}\right).$$

The exact bound-state wave functions are given by[15]

$$f_{\kappa'}=-\frac{2^{1/2}\lambda^{5/2}}{\Gamma(2\gamma'+1)}\left[\frac{\Gamma(2\gamma'+n'+1)}{n'!\rho(\rho-\lambda\kappa')}\right]^{1/2}\times(1-W_B)^{1/2}(2\lambda)^{\gamma'-1}r^{\gamma'-1}e^{-\lambda r}J_-, \quad (9)$$

and

$$g_{\kappa'}=-\frac{2^{1/2}\lambda^{5/2}}{\Gamma(\gamma'+1)}\left[\frac{\Gamma(2\gamma'+n'+1)}{n'!\rho(\rho-\lambda\kappa')}\right]^{1/2}\times(1+W_B)^{1/2}(2\lambda)^{\gamma'-1}r^{\gamma'-1}e^{-\lambda r}J_+, \quad (10)$$

where

$$J_\pm=\{n'\,{}_1F_1(-n'+1;2\gamma'+1;2\lambda r)\pm[\kappa'-(\rho/\lambda)]\,{}_1F_1(-n';2\gamma'+1;2\lambda r)\}, \quad (11)$$

$$\rho=\alpha Z,\quad \gamma'=(\kappa'^2-\alpha^2Z^2)^{1/2},$$
$$n'=n-|\kappa'|\quad (n=1,2,3,\cdots),$$
$$W_B=\{1+(\alpha^2Z^2/(\gamma'+n')^2)\}^{-1/2}\ \text{and}\ \lambda=(1-W_B^2)^{1/2}. \quad (12)$$

Writing

$$G_L(\omega r)=8\pi\sum_{k=0}^{L}\frac{(-1)^k(L+k)!}{k!(L-k)!}\frac{e^{i\omega r}}{(2i\omega r)^{k+1}}, \quad (13)$$

we find that the evaluation of R_i ($i=1,\cdots 6$) reduces essentially to the determination of one type of integral. This integral, which we call $I(\omega,n')$, is written as follows:

$$I(\omega,n')=e^{i\eta}(\gamma+i\nu)\int_0^\infty r^{(\gamma+\gamma'-k-1)}e^{-(\lambda+ip-i\omega)r}\,{}_1F_1(\gamma+1+i\nu;2\gamma+1;2ipr)\,{}_1F_1(-n';2\gamma'+1;2\lambda r)dr. \quad (14)$$

Using the series expansion for the bound-state hypergeometric function,

$${}_1F_1(-n';2\gamma'+1;2\lambda r)=\Gamma(2\gamma'+1)\sum_{t=0}^{n'}\frac{n'!}{t!(n'-t)!}(-1)^t\frac{(2\lambda r)^t}{\Gamma(2\gamma'+t+1)}, \quad (15)$$

and making use of the formula

$$\int_0^\infty e^{-\mu r}r^n\,{}_1F_1(a;c;br)dr=\Gamma(n+1)\mu^{-n-1}\,{}_1F_1\left(a,n+1;c;\frac{b}{\lambda}\right),\quad \text{Re}\mu>0,\ n>-1, \quad (16)$$

gives the result

$$I(\omega,n')=e^{i\eta}\Gamma(2\gamma'+1)(\gamma+i\nu)\sum_{t=0}^{n'}(-1)^t\frac{n'!}{t!(n'-t)!}\times\frac{(2\lambda)^t\Gamma(\gamma+\gamma'+t-k)}{\Gamma(2\gamma'+t+1)(\lambda+ip-i\omega)^{\gamma+\gamma'+t-k}}\,{}_2F_1\left(\gamma+1+i\nu;\gamma+\gamma'+t-k;2\gamma+1;\frac{2ip}{\lambda+ip-i\omega}\right). \quad (17)$$

We now set

$$A_+=(w+1)^{1/2}\{n'[I(\omega,n'-1)+I^*(-\omega,n'-1)]\}, \quad (18a)$$

$$B_+=(w+1)^{1/2}\{[\kappa'-(\rho/\lambda)]\times[I(\omega,n')+I^*(-\omega,n')]\}, \quad (18b)$$

$$A_-=(w-1)^{1/2}\{n'[I(\omega,n'-1)-I^*(-\omega,n'-1)]\}, \quad (18c)$$

$$B_-=(w-1)^{1/2}\{[\kappa'-(\rho/\lambda)]\times[I(\omega,n')-I^*(-\omega,n')]\}, \quad (18d)$$

$$a=\frac{(L+k)!}{k!(L-k)!}\frac{i^k}{(2\omega)^k} \quad (19)$$

and

$$d_1=i\pi 2^{3/2}\frac{1}{\omega}(1+W_B)^{1/2}\frac{e^{\pi\nu/2}|\Gamma(\gamma+i\nu)|(2p)^\gamma}{(\pi p)^{1/2}\Gamma(2\gamma+1)}\frac{\lambda^{5/2}}{\Gamma(2\gamma'+1)}\times\left[\frac{\Gamma(2\gamma'+n'+1)}{n'!\rho(\rho-\lambda\kappa')}\right]^{1/2}(2\lambda)^{\gamma'-1}. \quad (20)$$

Thus, we can write the radial integrals in the following form:

$$R_1=d_1\sum_{k=0}^{L}a(A_++B_+), \quad (21a)$$

$$R_2=i(1-W_B/1+W_B)^{1/2}d_1\sum_{k=0}^{L}a(A_- - B_-)\,, \qquad (21b)$$

$$R_5=id_1\sum_{k=0}^{L}a(A_- + B_-)\,, \qquad (21c)$$

$$R_6=(1-W_B/1+W_B)^{1/2}d_1\sum_{k=0}^{L}a(A_+ - B_+)\,, \qquad (21d)$$

$$R_3=-d_1\sum_{k=0}^{L}(L-k/L+k)a(A_- + B_-)\,, \qquad (21e)$$

$$R_4=i(1-W_B/1+W_B)^{1/2}d_1 \times\sum_{k=0}^{L}(L-k/L+k)a(A_+ - B_+)\,. \qquad (21f)$$

Now making use of the well-known relations between hypergeometric functions,[19] we can obtain the following result:

$$I(\omega,n')\pm I^*(-\omega,n') = e^{-i\eta}\Gamma(2\gamma'+1)\sum_{t=0}^{n'}(-1)^t n'!/t!(n'-t)! \times\frac{(2\lambda)^t\Gamma(\gamma+\gamma'+t-k)}{\Gamma(2\gamma'+t+1)(\lambda+ip-i\omega)^{\gamma+\gamma'+t-k}}\left\{e^{2i\eta}(\gamma+i\nu)\times{}_2F_1\left(\gamma+1+i\nu,\gamma+\gamma'+t-k;2\gamma+1;\frac{2ip}{\lambda+ip-i\omega}\right)\pm(\gamma-i\nu)\,{}_2F_1\left(\gamma+i\nu,\gamma+\gamma'+t-k;2\gamma+1;\frac{2ip}{\lambda+ip-i\omega}\right)\right\}. \qquad (22)$$

TABLE I. Threshold values of internal conversion coefficients for the K shell.[a]

ω (mc^2 units)	Z	Magnetic multipoles $\beta_1^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
6.6582(−4)	5	7.3842(2)	1.9916(6)	1.4231(9)	4.6148(11)	8.5015(13)
2.6660(−3)	10	1.8654(2)	1.2528(5)	2.2304(7)	1.8029(9)	8.2815(10)
6.0085(−3)	15	8.4374(1)	2.5016(4)	1.9681(6)	7.0360(7)	1.4301(9)
1.0707(−2)	20	4.8653(1)	8.0366(3)	3.5281(5)	7.0462(6)	8.0057(7)
1.6781(−2)	25	3.2163(1)	3.3574(3)	9.3350(4)	1.1826(6)	8.5300(6)
2.4256(−2)	30	2.3255(1)	1.6592(3)	3.1623(4)	2.7516(5)	1.3646(6)
3.3164(−2)	35	1.7940(1)	9.2216(2)	1.2714(4)	8.0207(4)	2.8877(5)
4.3547(−2)	40	1.4554(1)	5.5943(2)	5.7978(3)	2.7574(4)	7.4951(4)
5.5451(−2)	45	1.2307(1)	3.6338(2)	2.9130(3)	1.0752(4)	2.2723(4)
6.8936(−2)	50	1.0786(1)	2.4948(2)	1.5808(3)	4.6307(3)	7.7825(3)
8.4072(−2)	55	9.7641(0)	1.7939(2)	9.1362(2)	2.1612(3)	2.9401(3)
1.0094(−1)	60	9.1137(0)	1.3424(2)	5.5659(2)	1.0779(3)	1.2037(3)
1.1964(−1)	65	8.7668(0)	1.0405(2)	3.5467(2)	5.6837(2)	5.2684(2)
1.4030(−1)	70	8.6982(0)	8.3257(1)	2.3501(2)	3.1427(2)	2.4394(2)
1.6305(−1)	75	8.9214(0)	6.8624(1)	1.6120(2)	1.8103(2)	1.1848(2)
1.8808(−1)	80	9.4977(0)	5.8191(1)	1.1408(2)	1.0807(2)	5.9950(1)
2.1560(−1)	85	1.0564(1)	5.0747(1)	8.3088(1)	6.6587(1)	3.1418(1)
2.4589(−1)	90	1.2405(1)	4.5546(1)	6.2182(1)	4.2205(1)	1.6973(1)
2.7928(−1)	95	1.5647(1)	4.2155(1)	4.7788(1)	2.7450(1)	9.4129(0)
Electric multipoles						
ω (mc^2 units)	Z	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
6.6582(−4)	5	3.0303(4)	1.9665(7)	5.0366(9)	7.2708(11)	6.8538(13)
2.6660(−3)	10	1.8891(3)	3.0526(5)	1.9451(7)	6.9868(8)	1.6391(10)
6.0085(−3)	15	3.7155(2)	2.6503(4)	7.4446(5)	1.1792(7)	1.2202(8)
1.0707(−2)	20	1.1686(2)	4.6433(3)	7.2530(4)	6.3911(5)	3.6803(6)
1.6781(−2)	25	4.7501(1)	1.1923(3)	1.1744(4)	6.5284(4)	2.3729(5)
2.4256(−2)	30	2.2695(1)	3.8908(2)	2.6129(3)	9.9106(3)	2.4594(4)
3.3164(−2)	35	1.2117(1)	1.4950(2)	7.2166(2)	1.9691(3)	3.5185(3)
4.3547(−2)	40	7.0138(0)	6.4608(1)	2.3278(2)	4.7461(2)	6.3449(2)
5.5451(−2)	45	4.3171(0)	3.0482(1)	8.4291(1)	1.3213(2)	1.3605(2)
6.8936(−2)	50	2.7883(0)	1.5380(1)	3.3350(1)	4.1099(1)	3.3372(1)
8.4072(−2)	55	1.8720(0)	8.1763(0)	1.4143(1)	1.3963(1)	9.1298(0)
1.0094(−1)	60	1.2974(0)	4.5289(0)	6.3419(0)	5.1055(0)	2.7457(0)
1.1964(−1)	65	9.2332(−1)	2.5919(0)	2.9801(0)	1.9940(0)	9.0409(−1)
1.4030(−1)	70	6.7204(−1)	1.5234(0)	1.4613(0)	8.3230(−1)	3.2809(−1)
1.6305(−1)	75	4.9865(−1)	9.1632(−1)	7.4918(−1)	3.7479(−1)	1.3311(−1)
1.8808(−1)	80	3.7624(−1)	5.6423(−1)	4.0549(−1)	1.8525(−1)	6.1204(−2)
2.1560(−1)	85	2.8806(−1)	3.5793(−1)	2.3616(−1)	1.0232(−1)	3.1902(−2)
2.4589(−1)	90	2.2341(−1)	2.3780(−1)	1.5174(−1)	6.3586(−2)	1.8516(−2)
2.7928(−1)	95	1.7526(−1)	1.7061(−1)	1.0968(−1)	4.3946(−2)	1.1633(−2)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

[19] *Higher Transcendental Functions, Bateman Manuscript Project*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1.

Defining

$$B_{\pm}=[\kappa'-(\rho/\lambda)]\sum_{t=0}^{n'}c^{\pm}, \tag{23}$$

and using Eq. (22), we can show that

$$A_{\pm}=\sum_{t=0}^{n'}(n'-t)c^{\pm}. \tag{24}$$

This formula enables us to write down the results for the radial integrals in their most elegant form

$$R_1=d_1\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^+c^+, \tag{25a}$$

$$R_2=i\left(\frac{1-W_B}{1+W_B}\right)^{1/2}d_1\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^-c^-, \tag{25b}$$

$$R_5=id_1\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^+c^-, \tag{25c}$$

$$R_6=\left(\frac{1-W_B}{1+W_B}\right)^{1/2}d_1\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^-c^+, \tag{25d}$$

$$R_3=-d_1\sum_{k=0}^{L}a(L-k/L+k)\sum_{t=0}^{n'}d^+c^-, \tag{25e}$$

$$R_4=i\left(\frac{1-W_B}{1+W_B}\right)^{1/2}d_1\sum_{k=0}^{L}a(L-k/L+k)\sum_{t=0}^{n'}d^-c^+, \tag{25f}$$

where

$$c^{\pm}=(w\pm1)^{1/2}e^{-i\eta}\Gamma(2\gamma'+1)(-1)^t\frac{n'!}{t!(n'-t)!}\frac{(2\lambda)^t\Gamma(\gamma+\gamma'+t-k)}{\Gamma(2\gamma'+t+1)(\lambda+ip-i\omega)^{\gamma+\gamma'+t-k}}$$
$$\times\{e^{2i\eta}(\gamma+i\nu)\,{}_2F_1(\gamma+1+i\nu,\gamma+\gamma'+t-k;2\gamma+1;(2ip/\lambda+ip-i\omega))$$
$$\pm(\gamma-i\nu)\,{}_2F_1(\gamma+i\nu,\gamma+\gamma'+t-k;2\gamma+1;(2ip/\lambda+ip-i\omega))\}, \tag{26}$$

$$d^{\pm}=(n'-t)\pm[\kappa'-(\rho/\lambda)], \tag{27}$$

and a and d_1 are given by Eqs. (19) and (20), respectively.

These results, combined with Eqs. (2) and (3), give us exact analytic expressions for the ICC's for all shells and for all transition energies in the case of a point nucleus with no screening.

APPLICATION TO THRESHOLD

The foregoing expressions simplify considerably at threshold, i.e., when p goes to zero. We find that

$$\lim_{p\to0}c^-=i\alpha ZQ\times{}_1F_1(\gamma+\gamma'+t-k;2\gamma+1;(-2\alpha Z/\lambda-i\omega)), \tag{28}$$

and

$$\lim_{p\to0}c^+=\frac{\gamma-\kappa}{i\alpha Z}\lim_{p\to0}c^- +(2i/\lambda-i\omega)(\gamma+\gamma'+t-k/2\gamma+1)c^0, \tag{29}$$

where

$$Q=\sqrt{2}\Gamma(2\gamma'+1)(-1)^t\times\frac{n'!}{t!(n'-t)!}\frac{(2\lambda)^t\Gamma(\gamma+\gamma'+t-k)}{\Gamma(2\gamma'+t+1)(\lambda-i\omega)^{\gamma+\gamma'+t-k}}, \tag{30}$$

and

$$c^0=i\alpha ZQ\,{}_1F_1(\gamma+\gamma'+t-k+1;2\gamma+2;-2\alpha Z/\lambda-i\omega). \tag{31}$$

Defining the quantities

$$d_2=\frac{i\pi}{\Gamma(2\gamma+1)}2^{\gamma+\gamma'+3/2}(1+W_B)^{1/2}\frac{\lambda}{\omega}\left(\frac{\alpha Z}{\lambda}\right)^{\gamma-1/2}\times\left(1-\frac{i\omega}{\lambda}\right)^{-(\gamma+\gamma')}\left[\frac{\Gamma(2\gamma'+n'+1)}{n'!\rho(\rho-\lambda\kappa')}\right]^{1/2}, \tag{32}$$

$$T=i\alpha Z(-1)^t\frac{n'!}{(n'-t)!t!}2^{t-k}\frac{\Gamma(\gamma+\gamma'+t-k)}{\Gamma(2\gamma'+t+1)}\times\lambda^k\left(1-\frac{i\omega}{\lambda}\right)^{k-t}, \tag{33}$$

$$b=T\,{}_1F_1\left(\gamma+\gamma'+t-k;2\gamma+1;\frac{-2\alpha Z/\lambda}{1-(i\omega/\lambda)}\right), \tag{34}$$

$$l=T\,{}_1F_1\left(\gamma+\gamma'+t-k+1;2\gamma+2;\frac{-2\alpha Z/\lambda}{1-(i\omega/\gamma)}\right), \tag{35}$$

and

$$h=\frac{\gamma-\kappa}{i\alpha Z}b+\frac{2i}{\lambda-i\omega}\frac{\gamma+\gamma'+t-k}{2\gamma+1}l, \tag{36}$$

enables us to write down the following relatively simple expressions for the radial integrals at threshold:

$$R_1=d_2\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^+h, \tag{37a}$$

$$R_2=i(\omega/\lambda)d_2\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^-b, \tag{37b}$$

$$R_3=-d_2\sum_{k=0}^{L}(L-k/L+k)a\sum_{t=0}^{n'}d^+b, \tag{37c}$$

$$R_4=i(\omega/\lambda)d_2\sum_{k=0}^{L}(L-k/L+k)a\sum_{t=0}^{n'}d^-h, \tag{37d}$$

$$R_5=id_2\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^+b, \tag{37e}$$

$$R_6=(\omega/\lambda)d_2\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^-h. \tag{37f}$$

TABLE II. Threshold values of internal conversion coefficients for the L_I subshell.[a]

ω (mc^2 units)	Z	$\beta_1^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
			Magnetic multipoles			
1.6647(−4)	5	5.9024(3)	2.5498(8)	2.9147(12)	1.5120(16)	4.4558(19)
6.6672(−4)	10	1.4935(3)	1.6049(7)	4.5677(10)	5.9027(13)	4.3348(16)
1.5033(−3)	15	6.7745(2)	3.2083(6)	4.0305(9)	2.3013(12)	7.4711(14)
2.6803(−3)	20	3.9221(2)	1.0324(6)	7.2253(8)	2.3014(11)	4.1709(13)
4.2040(−3)	25	2.6063(2)	4.3219(5)	1.9118(8)	3.8555(10)	4.4283(12)
6.0825(−3)	30	1.8967(2)	2.1413(5)	6.4764(7)	8.9503(9)	7.0530(11)
8.3258(−3)	35	1.4747(2)	1.1939(5)	2.6039(7)	2.6020(9)	1.4848(11)
1.0947(−2)	40	1.2075(2)	7.2699(4)	1.1876(7)	8.9177(8)	3.8304(10)
1.3960(−2)	45	1.0321(2)	4.7429(4)	5.9675(6)	3.4652(8)	1.1532(10)
1.7385(−2)	50	9.1584(1)	3.2727(4)	3.2392(6)	1.4865(8)	3.9187(9)
2.1244(−2)	55	8.4093(1)	2.3670(4)	1.8728(6)	6.9075(7)	1.4674(9)
2.5562(−2)	60	7.9767(1)	1.7831(4)	1.1415(6)	3.4284(7)	5.9489(8)
3.0372(−2)	65	7.8143(1)	1.3925(4)	7.2790(5)	1.7983(7)	2.5757(8)
3.5713(−2)	70	7.9143(1)	1.1239(4)	4.8276(5)	9.8864(6)	1.1785(8)
4.1630(−2)	75	8.3075(1)	9.3550(3)	3.3154(5)	5.6595(6)	5.6492(7)
4.8181(−2)	80	9.0772(1)	8.0216(3)	2.3498(5)	3.3561(6)	2.8174(7)
5.5437(−2)	85	1.0396(2)	7.0848(3)	1.7148(5)	2.0529(6)	1.4534(7)
6.3487(−2)	90	1.2616(2)	6.4516(3)	1.2865(5)	1.2911(6)	7.7167(6)
7.2445(−2)	95	1.6512(2)	6.0715(3)	9.9179(4)	8.3275(5)	4.1989(6)
			Electric multipoles			
ω (mc^2 units)	Z	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
1.6647(−4)	5	7.1057(4)	1.0341(8)	5.5569(12)	3.0602(16)	7.5041(19)
6.6672(−4)	10	4.4320(3)	1.6581(6)	2.1845(10)	2.9925(13)	1.8281(16)
1.5033(−3)	15	8.7253(2)	1.5200(5)	8.6171(8)	5.2024(11)	1.4043(14)
2.6803(−3)	20	2.7477(2)	2.8748(4)	8.7612(7)	2.9403(10)	4.4278(12)
4.2040(−3)	25	1.1185(2)	8.1493(3)	1.4999(7)	3.1725(9)	3.0252(11)
6.0825(−3)	30	5.3524(1)	3.0030(3)	3.5773(6)	5.1558(8)	3.3695(10)
8.3258(−3)	35	2.8621(1)	1.3328(3)	1.0745(6)	1.1123(8)	5.2571(9)
1.0947(−2)	40	1.6590(1)	6.8056(2)	3.8277(5)	2.9542(7)	1.0495(9)
1.3960(−2)	45	1.0222(1)	3.8799(2)	1.5558(5)	9.2023(6)	2.5288(8)
1.7385(−2)	50	6.6049(0)	2.4188(2)	7.0277(4)	3.2527(6)	7.0670(7)
2.1244(−2)	55	4.4320(0)	1.6240(2)	3.4631(4)	1.2742(6)	2.2264(7)
2.5562(−2)	60	3.0658(0)	1.1612(2)	1.8364(4)	5.4372(5)	7.7422(6)
3.0372(−2)	65	2.1739(0)	8.7649(1)	1.0372(4)	2.4942(5)	2.9246(6)
3.5713(−2)	70	1.5727(0)	6.9383(1)	6.1896(3)	1.2175(5)	1.1851(6)
4.1630(−2)	75	1.1563(0)	5.7315(1)	3.8794(3)	6.2733(4)	5.0996(5)
4.8181(−2)	80	8.6091(−1)	4.9234(1)	2.5412(3)	3.3896(4)	2.3112(5)
5.5437(−2)	85	6.4707(−1)	4.3883(1)	1.7335(3)	1.9105(4)	1.0955(5)
6.3487(−2)	90	4.8934(−1)	4.0554(1)	1.2281(3)	1.1184(4)	5.3990(4)
7.2445(−2)	95	3.7104(−1)	3.8892(1)	9.0200(2)	6.7759(3)	2.7520(4)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

Substitution of these results in Eqs. (2) and (3) enables the ICC's at threshold to be calculated for any shell (point nucleus, no screening model).

Numerical results were obtained with the use of an IBM 7040 computer. The hypergeometric series which occur were summed until the absolute values of two successive partial sums differed by less than 10^{-8}. The computations were performed to 8 significant figures and we have rounded off to give five significant figures in the ICC tables. The results are presented in Tables I–IX for the K, $L(L_I,L_{II},L_{III})$ and $M(M_I,M_{II},M_{III},M_{IV},M_V)$ shells, for 19 values of Z in the range $5\leq Z\leq 95$, and for the first five electric and magnetic multipoles. It is seen that the results for the K shell agree with those obtained by Spinrad.[13]

LOW-ENERGY TRANSITIONS

An explicit calculation of c^- and c^+ to order p show that they do not contain terms of order p. This leads to the conclusion that the threshold ICC results are actually correct to order p^2. Thus, the momentum spectrum of the ICC's, to order p, is parabolic, a fact which should prove extremely useful in interpolating to gamma-ray energies away from threshold. As already mentioned, we see that the expansion is facilitated by factoring out the indicial ω dependence arising from the Hankel function [see Eq. (13)] but the range of validity of our approximation can only be stated in terms of the as yet uncomputed coefficient of the p^2 term. In a further publication, we hope to calculate the ICC's to order p^2. This should provide us with very accurate low-energy results. It will then be possible to compare these results in the low-energy region with the M shell results of Rose[4] (which refer to a point nucleus and unscreened electrons) and, after screening and nuclear size have been taken into account (see outline of procedure in following sections), with the K- and L-shell results of Sliv *et al.*,[3] and Rose.[4] In the special

TABLE III. Threshold values of internal conversion coefficients for the L_{II} subshell.[a]

ω (mc^2 units)	Z	$\beta_1^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
			Magnetic multipoles			
1.6647(−4)	5	5.6446(2)	1.3544(7)	1.0326(11)	3.8810(14)	8.7178(17)
6.6672(−4)	10	1.4280(2)	8.5188(5)	1.6167(9)	1.5134(12)	8.4702(14)
1.5033(−3)	15	6.4745(1)	1.7008(5)	1.4242(8)	5.8889(10)	1.4567(13)
2.6803(−3)	20	3.7464(1)	5.4634(4)	2.5472(7)	5.8730(9)	8.1078(11)
4.2040(−3)	25	2.4879(1)	2.2821(4)	6.7195(6)	9.8043(8)	8.5745(10)
6.0825(−3)	30	1.8090(1)	1.1276(4)	2.2679(6)	2.2661(8)	1.3591(10)
8.3258(−3)	35	1.4051(1)	6.2662(3)	9.0776(5)	6.5536(7)	2.8445(9)
1.0947(−2)	40	1.1492(1)	3.8008(3)	4.1184(5)	2.2323(7)	7.2881(8)
1.3960(−2)	45	9.8094(0)	2.4685(3)	2.0570(5)	8.6124(6)	2.1769(8)
1.7385(−2)	50	8.6913(0)	1.6946(3)	1.1088(5)	3.6646(6)	7.3308(7)
2.1244(−2)	55	7.9670(0)	1.2185(3)	6.3606(4)	1.6872(6)	2.7171(7)
2.5562(−2)	60	7.5429(0)	9.1185(2)	3.8425(4)	8.2868(5)	1.0888(7)
3.0372(−2)	65	7.3739(0)	7.0686(2)	2.4258(4)	4.2958(5)	4.6531(6)
3.5713(−2)	70	7.4509(0)	5.6576(2)	1.5908(4)	2.3306(5)	2.0979(6)
4.1630(−2)	75	7.8010(0)	4.6651(2)	1.0787(4)	1.3145(5)	9.8927(5)
4.8181(−2)	80	8.4994(0)	3.9581(2)	7.5372(3)	7.6657(4)	4.8436(5)
5.5437(−2)	85	9.7028(0)	3.4544(2)	5.4125(3)	4.6018(4)	2.4473(5)
6.3487(−2)	90	1.1732(1)	3.1033(2)	3.9875(3)	2.8332(4)	1.2692(5)
7.2445(−2)	95	1.5292(1)	2.8756(2)	3.0109(3)	1.7838(4)	6.7246(4)
			Electric multipoles			
ω (mc^2 units)	Z	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
1.6647(−4)	5	6.5203(4)	1.8008(10)	7.1409(14)	7.0160(18)	3.0683(22)
6.6672(−4)	10	4.0791(3)	2.8302(8)	2.7944(12)	6.8386(15)	7.4518(18)
1.5033(−3)	15	8.0711(2)	2.5093(7)	1.0937(11)	1.1824(14)	5.6944(16)
2.6803(−3)	20	2.5598(2)	4.5288(6)	1.0998(10)	6.6304(12)	1.7822(15)
4.2040(−3)	25	1.0517(2)	1.2089(6)	1.8557(9)	7.0805(11)	1.2057(14)
6.0825(−3)	30	5.0903(1)	4.1405(5)	4.3463(8)	1.1358(11)	1.3264(13)
8.3258(−3)	35	2.7594(1)	1.6869(5)	1.2770(8)	2.4116(10)	2.0383(12)
1.0947(−2)	40	1.6255(1)	7.8144(4)	4.4312(7)	6.2838(9)	3.9955(11)
1.3960(−2)	45	1.0203(1)	3.9988(4)	1.7467(7)	1.9137(9)	9.4215(10)
1.7385(−2)	50	6.7346(0)	2.2165(4)	7.6164(6)	6.5885(8)	2.5673(10)
2.1244(−2)	55	4.6297(0)	1.3126(4)	3.6053(6)	2.5038(8)	7.8552(9)
2.5562(−2)	60	3.2915(0)	8.2222(3)	1.8271(6)	1.0319(8)	2.6417(9)
3.0372(−2)	65	2.4070(0)	5.4083(3)	9.8098(5)	4.5512(7)	9.6061(8)
3.5713(−2)	70	1.8028(0)	3.7155(3)	5.5355(5)	2.1254(7)	3.7288(8)
4.1630(−2)	75	1.3782(0)	2.6559(3)	3.2625(5)	1.0423(7)	1.5290(8)
4.8181(−2)	80	1.0724(0)	1.9701(3)	1.9986(5)	5.3314(6)	6.5674(7)
5.5437(−2)	85	8.4726(−1)	1.5146(3)	1.2679(5)	2.8286(6)	2.9332(7)
6.3487(−2)	90	6.7815(−1)	1.2064(3)	8.3070(4)	1.5495(6)	1.3538(7)
7.2445(−2)	95	5.4879(−1)	9.9699(2)	5.6113(4)	8.7310(5)	6.4209(6)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

case of magnetic-dipole K-shell conversion, Olsson and Hultberg[20] have carried out a program similar to that proposed here for all shells and for all transitions. They expanded their results about $w-1$ (which, in their case, is equivalent to an expansion about ω because they neglected the binding energy) and obtained results to first order in $w-1$. They included nuclear size effects but neglected screening and then made detailed comparisons with the work of Rose and Sliv.

The results also provide information on ICC's for electron momenta p in the range 0 to a minimum of 0.32 (the latter value[12] corresponding to the lowest transition energy $\omega=0.05$ considered by both Sliv *et al.*[3] and Rose[4]). The results may also be looked upon as an end in themselves in that we are calculating the separate effects on the ICC's due to the pure Coulomb field, nuclear size and screening. Another application is that the general results [Eqs. (25a) to (25f)] are in a form very suitable for numerical computation if we wished, for example, to calculate ICC's for the M shell in the range $Z\leq 65$ to supplement the present M subshell values[4]; the fact that values of individual M-subshell coefficients are available only for $Z\geq 65$ (with screening and nuclear size neglected[4]) made it necessary for Chu and Perlman[21] in their paper on empirical screening corrections (see next section) to extrapolate to the Z values required as well as interpolate for transition energy. A further point to be noticed is that our results give in fact upper limits for ICC values.

[20] P. O. M. Olsson and S. Hultberg, Arkiv Fysik **15**, 361 (1959).

SCREENING EFFECTS

It would seem that the simplest and most effective means of including screening effects is to replace Z in our results by $Z_{\rm eff}<Z$. Chu and Perlman[21] discuss this method in detail for the M subshells and quote values for $Z_{\rm eff}$. However, this method suffers from the

[21] Y. Y. Chu and M. L. Perlman, Phys. Rev. **135**, B319 (1964).

TABLE IV. Threshold values of internal conversion coefficients for the L_{III} subshell.[a]

ω (mc^2 units)	Z	$\beta_1^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
				Magnetic multipoles		
1.6641(−4)	5	2.3788(2)	1.9315(8)	2.9202(13)	7.1607(17)	6.4357(21)
6.6582(−4)	10	5.9584(1)	1.2186(7)	4.5968(11)	2.8132(15)	6.3125(18)
1.4987(−3)	15	2.6569(1)	2.4458(6)	4.0870(10)	1.1086(14)	1.1032(17)
2.6660(−3)	20	1.5015(1)	7.9152(5)	7.4053(9)	1.1255(13)	6.2813(15)
4.1687(−3)	25	9.6668(0)	3.3384(5)	1.9865(9)	1.9225(12)	6.8403(14)
6.0085(−3)	30	6.7624(0)	1.6694(5)	6.8444(8)	4.5712(11)	1.1241(14)
8.1871(−3)	35	5.0116(0)	9.4125(4)	2.8077(8)	1.3674(11)	2.4565(13)
1.0707(−2)	40	3.8757(0)	5.8078(4)	1.3107(8)	4.8444(10)	6.6193(12)
1.3571(−2)	45	3.0975(0)	3.8480(4)	6.7639(7)	1.9553(10)	2.0950(12)
1.6781(−2)	50	2.5414(0)	2.7030(4)	3.7831(7)	8.7560(9)	7.5342(11)
2.0341(−2)	55	2.1305(0)	1.9955(4)	2.2614(7)	4.2690(9)	3.0067(11)
2.4256(−2)	60	1.8186(0)	1.5388(4)	1.4299(7)	2.2350(9)	1.3086(11)
2.8529(−2)	65	1.5765(0)	1.2343(4)	9.4923(6)	1.2433(9)	6.1301(10)
3.3164(−2)	70	1.3851(0)	1.0270(4)	6.5763(6)	7.2910(8)	3.0594(10)
3.8169(−2)	75	1.2313(0)	8.8493(3)	4.7337(6)	4.4783(8)	1.6137(10)
4.3547(−2)	80	1.1062(0)	7.8936(3)	3.5279(6)	2.8669(8)	8.9382(9
4.9305(−2)	85	1.0032(0)	7.2939(3)	2.7149(6)	1.9051(8)	5.1722(9)
5.5451(−2)	90	9.1769(−1)	6.9956(3)	2.1530(6)	1.3099(8)	3.1135(9)
6.1992(−2)	95	8.4615(−1)	6.9899(3)	1.7569(6)	9.2949(7)	1.9430(9)
				Electric multipoles		
ω (mc^2 units)	Z	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
1.6641(−4)	5	1.3037(5)	1.8265(10)	1.4289(15)	1.4046(19)	6.1457(22)
6.6582(−4)	10	8.1490(3)	2.8723(8)	5.5968(12)	1.3722(16)	1.4979(19)
1.4987(−3)	15	1.6101(3)	2.5496(7)	2.1944(11)	2.3818(14)	1.1517(17)
2.6660(−3)	20	5.0968(2)	4.6093(6)	2.2121(10)	1.3433(13)	3.6365(15)
4.1687(−3)	25	2.0888(2)	1.2332(6)	3.7443(9)	1.4449(12)	2.4881(14)
6.0085(−3)	30	1.0081(2)	4.2362(5)	8.8032(8)	2.3385(11)	2.7755(13)
8.1871(−3)	35	5.4462(1)	1.7324(5)	2.5983(8)	5.0180(10)	4.3365(12)
1.0707(−2)	40	3.1959(1)	8.0631(4)	9.0628(7)	1.3237(10)	8.6664(11)
1.3571(−2)	45	1.9978(1)	4.1503(4)	3.5932(7)	4.0882(9)	2.0893(11)
1.6781(−2)	50	1.3127(1)	2.3171(4)	1.5769(7)	1.4298(9)	5.8379(10)
2.0341(−2)	55	8.9822(0)	1.3844(4)	7.5162(6)	5.5298(8)	1.8371(10)
2.4256(−2)	60	6.3551(0)	8.7662(3)	3.8373(6)	2.3236(8)	6.3747(9)
2.8529(−2)	65	4.6248(0)	5.8427(3)	2.0763(6)	1.0467(8)	2.3998(9)
3.3164(−2)	70	3.4475(0)	4.0791(3)	1.1809(6)	5.0020(7)	9.6779(8)
3.8169(−2)	75	2.6239(0)	2.9735(3)	7.0151(5)	2.5147(7)	4.1387(8)
4.3547(−2)	80	2.0336(0)	2.2592(3)	4.3297(5)	1.3210(7)	1.8614(8)
4.9305(−2)	85	1.6014(0)	1.7883(3)	2.7648(5)	7.2097(6)	8.7433(7)
5.5451(−2)	90	1.2791(0)	1.4765(3)	1.8203(5)	4.0690(6)	4.2645(7)
6.1992(−2)	95	1.0345(0)	1.2752(3)	1.2322(5)	2.3650(6)	2.1489(7)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

defect of being based on a comparison of experimental results for particular elements with theoretical results obtained by extrapolation of the unscreened M shell results of Rose.[4] We are at present developing an analytic method for the calculation of Z_{eff} for all shells and for all Z (because Z_{eff} for a particular shell should also be a function of Z and ω). It is hoped that a table of values of Z_{eff} will provide a simple means of treating screening effects for any problem. Details of this work will appear in a future publication.

NUCLEAR-SIZE EFFECTS

Church and Weneser[5] have emphasized that the important change in going from the point nucleus to the finite nucleus is in dropping the singular Dirac Coulomb wave functions in the region where they make no physical sense (i.e., within the nuclear charge) and, that once this singular contribution is removed, the nuclear region is no longer very important. So, to get an upper limit of the effects of finite nuclear size, we evaluate the radial integrals between the limits R and ∞, where R is the nuclear radius. Thus, $I(\omega,n')$ is replaced by $I'(\omega,n')$, where $I'(\omega,n')$ is obtained by replacing the lower limit 0 of the integral in Eq. (14) by R. Then, using the series expansion for the continuum-state hypergeometric function as well as the expansion given by Eq. (15) for the bound-state hypergeometric function, we can write

$$I'(\omega,n')=I(\omega,n')-e^{i\eta}\frac{(\gamma+i\nu)}{\Gamma(\gamma+1+i\nu)}\Gamma(2\gamma'+1)\Gamma(2\gamma+1)\sum_{t=0}^{n'}\frac{n'!}{t!(n'-t)!}\frac{(-2\lambda)^t}{\Gamma(2\gamma'+t+1)}$$
$$\times\sum_{l=0}^{\infty}\frac{\Gamma(\gamma+1+i\nu+l)}{\Gamma(2\gamma+1+l)}\frac{(2ip)^l}{l!}\frac{R^{(\gamma+\gamma'+t-k+l)}}{\gamma+\gamma'+t-k+l}\,{}_1F_1[\gamma+\gamma'+t-k+l;\gamma+\gamma'+t-k+l+1;-(\lambda+ip-i\omega)R]$$
$$\equiv I(\omega,n')-\Delta I(\omega,n'). \quad (38)$$

If we now restrict ourselves to threshold, we find

$$\Delta I(\omega,n')=e^{i\eta}(\gamma+i\nu)\Gamma(2\gamma'+1)\Gamma(2\gamma+1)R^{\gamma+\gamma'-k}\sum_{t=0}^{n'}\frac{n'!}{t!(n'-t)!}\frac{(-2\lambda R)^t}{\Gamma(2\gamma'+t+1)}\sum_{l=0}^{\infty}\frac{(-2R\alpha Z)^l}{\Gamma(2\gamma+1+l)l!}\frac{1}{\delta}\left\{1-\frac{ipl}{2\alpha Z}(2\gamma+1+l)\right\}$$
$$\times\{{}_1F_1(\delta;\delta+1;\bar{R})-(\delta/\delta+1)ipR\,{}_1F_1(\delta+1;\delta+2;\bar{R})\}, \quad (39)$$

where

$$\bar{R}\equiv-(\lambda-i\omega)R \quad\text{and}\quad \delta\equiv\gamma+\gamma'+t-k+l.$$

In a procedure similar to that used for the point nucleus and using an obvious notation, we obtain

$$\Delta A_+=(w+1)^{1/2}\{n'[\Delta I(\omega,n'-1)+\Delta I^*(-\omega,n'-1)]\}$$
$$=\Lambda\sqrt{2}\sum_{t=0}^{n'}(n'-t)F(n',t)\sum_{l=0}^{\infty}G(l)$$
$$\times\{(\gamma-\kappa)F_1+2\alpha ZF_3\}, \quad (40a)$$

and, in a similar manner,

$$\Delta B_+=\Lambda\sqrt{2}(\kappa'-(\rho/\lambda))\sum_{t=0}^{n'}F(n',t)$$
$$\times\sum_{l=0}^{\infty}G(l)\{(\gamma-\kappa)F_1+2\alpha ZF_3\}, \quad (40b)$$

$$\Delta A_-=\Lambda\sqrt{2}i\alpha Z\sum_{t=0}^{n'}(n'-t)F(n',t)\sum_{l=0}^{\infty}G(l)F_1, \quad (40c)$$

$$\Delta B_-=\Lambda\sqrt{2}i\alpha Z(\kappa'-(\rho/\lambda))\sum_{t=0}^{n'}F(n',t)\sum_{l=0}^{\infty}G(l)F_1, \quad (40d)$$

TABLE V. Threshold values of internal conversion coefficients for the $M_{\rm I}$ subshell.[a]

ω (mc^2 units)	Z	$\beta_1^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
		Magnetic multipoles				
6.6761(−4)	15	2.2916(3)	5.5007(7)	3.5035(11)	1.0142(15)	1.6693(18)
1.1896(−3)	20	1.3291(3)	1.7750(7)	6.3052(10)	1.0194(14)	9.3782(16)
1.8645(−3)	25	8.8529(2)	7.4573(6)	1.6767(10)	1.7189(13)	1.0036(16)
2.6951(−3)	30	6.4612(2)	3.7113(6)	5.7156(9)	4.0227(12)	1.6145(15)
3.6848(−3)	35	5.0411(2)	2.0804(6)	2.3153(9)	1.1809(12)	3.4397(14)
4.8383(−3)	40	4.1446(2)	1.2748(6)	1.0653(9)	4.0936(11)	8.9993(13)
6.1608(−3)	45	3.5593(2)	8.3770(5)	5.4077(8)	1.6117(11)	2.7536(13)
7.6587(−3)	50	3.1755(2)	5.8285(5)	2.9697(8)	7.0190(10)	9.5322(12)
9.3396(−3)	55	2.9338(2)	4.2556(5)	1.7398(8)	3.3179(10)	3.6455(12)
1.1213(−2)	60	2.8022(2)	3.2400(5)	1.0764(8)	1.6789(10)	1.5134(12)
1.3289(−2)	65	2.7667(2)	2.5608(5)	6.9796(7)	8.9996(9)	6.7302(11)
1.5581(−2)	70	2.8267(2)	2.0948(5)	4.7169(7)	5.0697(9)	3.1731(11)
1.8104(−2)	75	2.9965(2)	1.7700(5)	3.3084(7)	2.9825(9)	1.5731(11)
2.0878(−2)	80	3.3103(2)	1.5436(5)	2.4010(7)	1.8237(9)	8.1471(10)
2.3925(−2)	85	3.8383(2)	1.3894(5)	1.7993(7)	1.1546(9)	4.3849(10)
2.7273(−2)	90	4.7232(2)	1.2926(5)	1.3910(7)	7.5491(8)	2.4421(10)
3.0961(−2)	95	6.2801(2)	1.2462(5)	1.1094(7)	5.0882(8)	1.4029(10)
		Electric multipoles				
ω (mc^2 units)	Z	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
6.6761(−4)	15	1.4867(3)	1.6356(6)	4.5844(9)	1.8052(11)	1.6872(16)
1.1896(−3)	20	4.6833(2)	3.1045(5)	4.9414(8)	1.1151(10)	4.9756(14)
1.8645(−3)	25	1.9067(2)	8.8573(4)	9.1422(7)	2.0013(9)	3.1284(13)
2.6951(−3)	30	9.1245(1)	3.2942(4)	2.4035(7)	6.4965(8)	3.1709(12)
3.6848(−3)	35	4.8789(1)	1.4812(4)	8.1090(6)	2.7362(8)	4.4929(11)
4.8383(−3)	40	2.8280(1)	7.6941(3)	3.2998(6)	1.3173(8)	8.2610(10)
6.1608′−3)	45	1.7423(1)	4.4800(3)	1.5533(6)	6.9180(7)	1.9045(10)
7.6587(−3)	50	1.1253(1)	2.8633(3)	8.2101(5)	3.8802(7)	5.4385(9)
9.3396(−3)	55	7.5473(0)	1.9779(3)	4.7656(5)	2.2955(7)	1.9052(9)
1.1213(−2)	60	5.2167(0)	1.4592(3)	2.9867(5)	1.4206(7)	8.0023(8)
1.3289(−2)	65	3.6957(0)	1.1394(3)	1.9949(5)	9.1394(6)	3.8828(8)
1.5581(−2)	70	2.6683(0)	9.3509(2)	1.4059(5)	6.0842(6)	2.0924(8)
1.8104(−2)	75	1.9574(0)	8.0240(2)	1.0376(5)	4.1757(6)	1.2131(8)
2.0878(−2)	80	1.4529(0)	7.1736(2)	7.9738(4)	2.9464(6)	7.4041(7)
2.3925(−2)	85	1.0879(0)	6.6680(2)	6.3559(4)	2.1330(6)	4.6927(7)
2.7273(−2)	90	8.1832(−1)	6.4408(2)	5.2417(4)	1.5821(6)	3.0626(7)
3.0961(−2)	95	6.1652(−1)	6.4737(2)	4.4681(4)	1.2016(6)	2.0480(7)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

where

$$F_1={}_1F_1(\delta;\delta+1;\bar{R}),\quad F_2={}_1F_1(\delta+1;\delta+2;\bar{R}),$$

$$F_3=\frac{aR}{a+1}F_2+\frac{l}{2}\frac{(2\gamma+1+l)}{\alpha Z}F_1,$$

$$F(n',t)=\frac{n'!}{t!(n'-t)!}\frac{(-2\lambda R)^t}{\Gamma(2\gamma'+t+1)},$$

$$G(l)=\frac{(\alpha Z)^l}{\Gamma(2\gamma+1+l)}\frac{(-2R)^l}{l!}\frac{1}{\delta},$$

and

$$\Lambda=\Gamma(2\gamma'+1)\Gamma(2\gamma+1)R^{\gamma+\gamma'-k}.$$

We finally obtain

$$\Delta R_1=d_1\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^+\Delta c^+,\tag{41a}$$

$$\Delta R_2=i(\omega/\lambda)d_1\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^-\Delta c^-,\tag{41b}$$

$$\Delta R_3=-d_1\sum_{k=0}^{L}a(L-k/L+k)\sum_{t=0}^{n'}d^+\Delta c^-,\tag{41c}$$

$$\Delta R_4=i(\omega/\lambda)d_1\sum_{k=0}^{L}a(L-k/L+k)\sum_{t=0}^{n'}d^-\Delta c^+,\tag{41d}$$

$$\Delta R_5=id_1\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^+\Delta c^-,\tag{41e}$$

$$\Delta R_6=(\omega/\lambda)d_1\sum_{k=0}^{L}a\sum_{t=0}^{n'}d^-\Delta c^+,\tag{41f}$$

where the similarity with Eq. (25) is obvious. In the limit $R\rightarrow\infty$, we find that $\Delta R_i\rightarrow R_i$, as they should.

TABLE VI. Threshold values of internal conversion coefficients for the M_{II} subshell.[a]

ω (mc^2 units)	Z	Magnetic multipoles $\beta^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
6.6761(−4)	15	2.4610(2)	2.5349(6)	8.4911(9)	1.4377(13)	1.4861(16)
1.1896(−3)	20	1.4264(2)	8.1687(5)	1.5258(9)	1.4428(12)	8.3368(14)
1.8645(−3)	25	9.4930(1)	3.4263(5)	4.0496(8)	2.4282(11)	8.9064(13)
2.6951(−3)	30	6.9216(1)	1.7017(5)	1.3772(8)	5.6694(10)	1.4296(13)
3.6848(−3)	35	5.3938(1)	9.5151(4)	5.5631(7)	1.6596(10)	3.0380(12)
4.8383(−3)	40	4.4286(1)	5.8138(4)	2.5512(7)	5.7343(9)	7.9246(11)
6.1608(−3)	45	3.7976(1)	3.8081(4)	1.2903(7)	2.2493(9)	2.4166(11)
7.6587(−3)	50	3.3824(1)	2.6399(4)	7.0558(6)	9.7550(8)	8.3341(10)
9.3396(−3)	55	3.1193(1)	1.9195(4)	4.1142(6)	4.5898(8)	3.1739(10)
1.1213(−2)	60	2.9736(1)	1.4548(4)	2.5321(6)	2.3106(8)	1.3116(10)
1.3289(−2)	65	2.9299(1)	1.1441(4)	1.6325(6)	1.2316(8)	5.8036(9)
1.5581(−2)	70	2.9869(1)	9.3073(3)	1.0963(6)	6.8953(7)	2.7213(9)
1.8104(−2)	75	3.1590(1)	7.8172(3)	7.6365(5)	4.0295(7)	1.3413(9)
2.0878(−2)	80	3.4816(1)	6.7723(3)	5.5005(5)	2.4461(7)	6.9032(8)
2.3925(−2)	85	4.0270(1)	6.0522(3)	4.0884(5)	1.5366(7)	3.6907(8)
2.7273(−2)	90	4.9426(1)	5.5862(3)	3.1326(5)	9.9628(6)	2.0411(8)
3.0961(−2)	95	6.5543(1)	5.3395(3)	2.4742(5)	6.6547(6)	1.1638(8)

ω (mc^2 units)	Z	Electric multipoles $\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
6.6761(−4)	15	1.5473(3)	4.7629(8)	8.2491(12)	3.5697(16)	7.0408(19)
1.1896(−3)	20	4.8980(2)	8.6119(7)	8.3222(11)	2.0117(15)	2.2182(18)
1.8645(−3)	25	2.0071(2)	2.3045(7)	1.4102(11)	2.1620(14)	1.5135(17)
2.6951(−3)	30	9.6828(1)	7.9170(6)	3.3206(10)	3.4960(13)	1.6827(16)
3.6848(−3)	35	5.2270(1)	3.2372(6)	9.8195(9)	7.4943(12)	2.6190(15)
4.8383(−3)	40	3.0645(1)	1.5061(6)	3.4335(9)	1.9750(12)	5.2117(14)
6.1608(−3)	45	1.9123(1)	7.7460(5)	1.3655(9)	6.0947(11)	1.2506(14)
7.6587(−3)	50	1.2537(1)	4.3186(5)	6.0155(8)	2.1304(11)	3.4771(13)
9.3396(−3)	55	8.5494(0)	2.5745(5)	2.8812(8)	8.2381(10)	1.0887(13)
1.1213(−2)	60	6.0215(0)	1.6250(5)	1.4798(8)	3.4632(10)	3.7581(12)
1.3289(−2)	65	4.3556(0)	1.0782(5)	8.0678(7)	1.5621(10)	1.4077(12)
1.5581(−2)	70	3.2194(0)	7.4805(4)	4.6325(7)	7.4836(9)	5.6509(11)
1.8104(−2)	75	2.4239(0)	5.4076(4)	2.7849(7)	3.7777(9)	2.4071(11)
2.0878(−2)	80	1.8520(0)	4.0633(4)	1.7450(7)	1.9969(9)	1.0794(11)
2.3925(−2)	85	1.4319(0)	3.1702(4)	1.1360(7)	1.0999(9)	5.0636(10)
2.7273(−2)	90	1.1166(0)	2.5687(4)	7.6682(6)	6.2901(8)	2.4719(10)
3.0961(−2)	95	8.7545(−1)	2.1654(4)	5.3619(6)	3.7247(8)	1.2506(10)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

TABLE VII. Threshold values of internal conversion coefficients for the M_{III} subshell.[a]

ω (mc^2 units)	Z	$\beta_1^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
		Magnetic multipoles				
6.6627(−4)	15	9.9776(1)	4.9498(7)	4.1936(12)	5.7553(16)	2.8978(20)
1.1854(−3)	20	5.6428(1)	1.6041(7)	7.6040(11)	5.8443(15)	1.6495(19)
1.8540(−3)	25	3.6368(1)	6.7784(6)	2.0420(11)	9.9869(14)	1.7960(18)
2.6731(−3)	30	2.5474(1)	3.3974(6)	7.0441(10)	2.3757(14)	2.9505(17)
3.6437(−3)	35	1.8906(1)	1.9207(6)	2.8939(10)	7.1100(13)	6.4454(16)
4.7672(−3)	40	1.4647(1)	1.1888(6)	1.3533(10)	2.5205(13)	1.7361(16)
6.0450(−3)	45	1.1729(1)	7.9048(5)	6.9972(9)	1.0180(13)	5.4923(15)
7.4791(−3)	50	9.6453(0)	5.5750(5)	3.9222(9)	4.5623(12)	1.9742(15)
9.0714(−3)	55	8.1065(0)	4.1341(5)	2.3502(9)	2.2263(12)	7.8742(14)
1.0824(−2)	60	6.9391(0)	3.2037(5)	1.4901(9)	1.1667(12)	3.4250(14)
1.2740(−2)	65	6.0344(0)	2.5836(5)	9.9212(8)	6.4972(11)	1.6033(14)
1.4822(−2)	70	5.3197(0)	2.1623(5)	6.8957(8)	3.8145(11)	7.9963(13)
1.7073(−2)	75	4.7472(0)	1.8750(5)	4.9810(8)	2.3460(11)	4.2146(13)
1.9496(−2)	80	4.2821(0)	1.6840(5)	3.7263(8)	1.5039(11)	2.3326(13)
2.2096(−2)	85	3.9009(0)	1.5676(5)	2.8794(8)	1.0008(11)	1.3487(13)
2.4877(−2)	90	3.5856(0)	1.5153(5)	2.2935(8)	6.8928(10)	8.1114(12)
2.7842(−2)	95	3.3234(0)	1.5269(5)	1.8803(8)	4.8998(10)	5.0575(12)
		Electric multipoles				
ω (mc^2 units)	Z	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
6.6627(−4)	15	3.0948(3)	4.7777(8)	1.6500(13)	7.1405(16)	1.4073(20)
1.1854(−3)	20	9.7966(2)	8.6596(7)	1.6646(12)	4.0241(15)	4.4308(18)
1.8540(−3)	25	4.0159(2)	2.3246(7)	2.8209(11)	4.3251(14)	3.0210(17)
2.6731(−3)	30	1.9382(2)	8.0180(6)	6.6413(10)	6.9930(13)	3.3551(16)
3.6437(−3)	35	1.0473(2)	3.2947(6)	1.9634(10)	1.4987(13)	5.2140(15)
4.7672(−3)	40	6.1464(1)	1.5420(6)	6.8613(9)	3.9474(12)	1.0354(15)
6.0450(−3)	45	3.8421(1)	7.9873(5)	2.7262(9)	1.2170(12)	2.4780(14)
7.4791(−3)	50	2.5242(1)	4.4909(5)	1.1993(9)	4.2477(11)	6.8658(13)
9.0714(−3)	55	1.7273(1)	2.7040(5)	5.7320(8)	1.6389(11)	2.1399(13)
1.0824(−2)	60	1.2217(1)	1.7268(5)	2.9351(8)	6.8680(10)	7.3444(12)
1.2740(−2)	65	8.8876(0)	1.1616(5)	1.5933(8)	3.0841(10)	2.7309(12)
1.4822(−2)	70	6.6208(0)	8.1902(4)	9.0944(7)	1.4686(10)	1.0861(12)
1.7073(−2)	75	5.0352(0)	6.0339(4)	5.4231(7)	7.3527(9)	4.5733(11)
1.9496(−2)	80	3.8978(0)	4.6363(4)	3.3610(7)	3.8442(9)	2.0216(11)
2.2096(−2)	85	3.0650(0)	3.7139(4)	2.1558(7)	2.0869(9)	9.3162(10)
2.4877(−2)	90	2.4431(0)	3.1048(4)	1.4262(7)	1.1706(9)	4.4497(10)
2.7842(−2)	95	1.9709(0)	2.7168(4)	9.7040(6)	6.7557(8)	2.1921(10)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

Taking $R=1.2A^{1/3}\times10^{-13}$ cm (which in our units is $3.1A^{1/3}\times10^{-3}$), we see that the dominant contribution to the l summation in Eq. (39) comes from the $l=0$ term. This is particularly true for small Z. For the purpose of getting a clear insight into the effect of R we shall write down just the dominant contribution to the radial integrals. Thus

$$\Delta R_1 \simeq \sqrt{2}(\gamma-\kappa)d_1d^+a_L(R^y/y)\,, \tag{42a}$$

$$\Delta R_2 \simeq -(\alpha Z/\gamma-\kappa)(\omega/\lambda)(d^-/d^+)\Delta R_1\,, \tag{42b}$$

$$\Delta R_3 \simeq 0\,, \tag{42c}$$

$$\Delta R_4 \simeq 0\,, \tag{42d}$$

$$\Delta R_5 \simeq -(\alpha Z/\gamma-\kappa)\Delta R_1\,, \tag{42e}$$

$$\Delta R_6 \simeq (\omega/\lambda)(d^-/d^+)\Delta R_1\,, \tag{42f}$$

where $y=\gamma+\gamma'-L$ and

$$a_L=((2L)!/L!)(i^L/\omega^L)2^L. \tag{43}$$

Because of the factor R^y/y we can see immediately that the effect of finite nuclear size increases with increasing R and increasing Z (recalling the dependence of γ and γ' on Z).

However, in calculating the effects of finite nuclear size, we have used the exact Eq. (41) to calculate the quantity

$$D=(\beta_L^{(\lambda)}-\beta_{L,N}^{(\lambda)}/\beta_L^{(\lambda)})\times100\,, \tag{44}$$

where $\beta_{L,N}^{(\lambda)}$ refers to the ICC's calculated using a finite nuclear radius R. The following results were obtained (all numbers are rounded off):

(a) Magnetic Transitions

(i) D increases with increasing Z. For instance, for the K shell and for $L=1$, D increased from 0.1 at $Z=5$ to 42 at $Z=95$.

(ii) D decreases with increasing L but not in a linear fashion. To illustrate the trend, the values of D corresponding to $L=1\cdots5$ for the K shell at $Z=95$ are 42, 20, 16, 14, and 13.

(iii) The values of D for the $L_{\rm I}$ and $M_{\rm I}$ subshells are,

TABLE VIII. Threshold values of internal conversion coefficients for the M_{IV} subshell.[a]

ω (mc^2 units)	Z	$\beta_1^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
				Magnetic multipoles		
1.1854(−3)	20	1.2565(1)	3.8294(5)	1.0327(10)	5.5091(13)	1.1579(17)
1.8540(−3)	25	8.0998(0)	1.6157(5)	2.7707(9)	9.4031(12)	1.2589(16)
2.6731(−3)	30	5.6749(0)	8.0835(4)	9.5483(8)	2.2336(12)	2.0647(15)
3.6437(−3)	35	4.2136(0)	4.5606(4)	3.9179(8)	6.6735(11)	4.5013(14)
4.7672(−3)	40	3.2656(0)	2.8162(4)	1.8296(8)	2.3611(11)	1.2096(14)
6.0450(−3)	45	2.6165(0)	1.8677(4)	9.4450(7)	9.5152(10)	3.8167(13)
7.4791(−3)	50	2.1527(0)	1.3135(4)	5.2848(7)	4.2536(10)	1.3678(13)
9.0714(−3)	55	1.8104(0)	9.7103(3)	3.1605(7)	2.0699(10)	5.4375(12)
1.0824(−2)	60	1.5508(0)	7.5006(3)	1.9995(7)	1.0814(10)	2.3565(12)
1.2740(−2)	65	1.3496(0)	6.0284(3)	1.3281(7)	6.0021(9)	1.0988(12)
1.4822(−2)	70	1.1907(0)	5.0276(3)	9.2071(6)	3.5109(9)	5.4560(11)
1.7073(−2)	75	1.0635(0)	4.3442(3)	6.6320(6)	2.1507(9)	2.8622(11)
1.9496(−2)	80	9.6028(−1)	3.8877(3)	4.9465(6)	1.3728(9)	1.5761(11)
2.2096(−2)	85	8.7574(−1)	3.6058(3)	3.8099(6)	9.0946(8)	9.0627(10)
2.4877(−2)	90	8.0595(−1)	3.4734(3)	3.0242(6)	6.2326(8)	5.4189(10)
2.7842(−2)	95	7.4801(−1)	3.4880(3)	2.4702(6)	4.4072(8)	3.3575(10)

ω (mc^2 units)	Z	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
				Electric multipoles		
1.1854(−3)	20	6.8275(2)	1.3959(6)	4.5617(11)	1.4672(16)	8.8226(19)
1.8540(−3)	25	2.8033(2)	3.6728(5)	7.7368(10)	1.5948(15)	6.1211(18)
2.6731(−3)	30	1.3558(2)	1.2369(5)	1.8236(10)	2.6147(14)	6.9466(17)
3.6437(−3)	35	7.3438(1)	4.9462(4)	5.3986(9)	5.6983(13)	1.1079(17)
4.7672(−3)	40	4.3224(1)	2.2468(4)	1.8897(9)	1.5307(13)	2.2682(16)
6.0450(−3)	45	2.7109(1)	1.1277(4)	7.5224(8)	4.8274(12)	5.6230(15)
7.4791(−3)	50	1.7879(1)	6.1429(3)	3.3163(8)	1.7292(12)	1.6219(15)
9.0714(−3)	55	1.2283(1)	3.5896(3)	1.5888(8)	6.8707(11)	5.2912(14)
1.0824(−2)	60	8.7283(0)	2.2344(3)	8.1578(7)	2.9761(11)	1.9119(14)
1.2740(−2)	65	6.3819(0)	1.4761(3)	4.4418(7)	1.3869(11)	7.5307(13)
1.4822(−2)	70	4.7814(0)	1.0339(3)	2.5438(7)	6.8834(10)	3.1943(13)
1.7073(−2)	75	3.6586(0)	7.6870(2)	1.5225(7)	3.6093(10)	1.4449(13)
1.9496(−2)	80	2.8518(0)	6.0810(2)	9.4749(6)	1.9867(10)	6.9163(12)
2.2096(−2)	85	2.2593(0)	5.1350(2)	6.1048(6)	1.1420(10)	3.4812(12)
2.4877(−2)	90	1.8161(0)	4.6439(2)	4.0588(6)	6.8271(9)	1.8331(12)
2.7842(−2)	95	1.4789(0)	4.5128(2)	2.7766(6)	4.2298(9)	1.0054(12)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

surprisingly, of the same order as the corresponding values for the K shell (in fact the M_I values are slightly greater than the L_I values which, in turn, are slightly greater than the K values). This indicates that D is very insensitive to changes in the shell designation n' but depends primarily, as we shall see, on the value of κ' or, in other words, on the total angular momentum of the electron.

(iv) The values of D for L_{II} and M_{II} are of the same order (those for M_{II} being slightly greater). Compared to the D values for L_I and M_I at $L=1$, we notice a decrease by a factor of 10^{-3} at $Z=5$ to a factor of 0.5 at $Z=95$. There is a further decrease by a factor of 2 as we go to $L=5$.

(v) The values of D for the L_{III} and M_{III} shells are of the same order but in no case is D greater than 23. Lower values still are generally obtained for the M_{IV} and M_V shells.

(b) Electric Transitions

(i) D is generally much smaller than the corresponding magnetic value.

(ii) D increases with increasing Z. For instance, for the K shell and for $L=1$, D increased from 0.2×10^{-4} at $Z=5$ to 0.3 at $Z=95$.

(iii) In contrast to the magnetic case, D generally increases with increasing L. To illustrate the trend, the values of D corresponding to $L=1\cdots5$ for the K shell at $Z=95$ are 0.3, 3, 9, 11, and 12. For the K shell, D always increases with increasing L but for the other shells D sometimes decreases with increasing L.

(iv) The values of D for K, L_I and M_I are of the same order.

(v) The values of D for L_{II} and M_{II} are of the same order and never greater than 6.

(vi) The values of D for L_{III} and M_{III} are of the same order and never greater than 1. Lower values still are generally obtained for the M_{IV} and M_V shells.

To examine the sensitivity of D on the assumed nuclear radius we considered different values ranging from R to $10R$ (Church and Weneser[5] assert that there is an appreciable difference between the finite-size and point-nucleus wave functions up to a radius of the order of $10R$). For the purpose of illustration, we will compare the values of D at R and $10R$. The general

TABLE IX. Threshold values of internal conversion coefficients for the M_V subshell.[a]

Magnetic multipoles

ω (mc^2 units)	Z	$\beta_1^{(0)}$	$\beta_2^{(0)}$	$\beta_3^{(0)}$	$\beta_4^{(0)}$	$\beta_5^{(0)}$
1.1840(−3)	20	8.2771(0)	5.9681(4)	3.2511(10)	3.2708(15)	4.3646(19)
1.8506(−3)	25	5.3112(0)	2.4564(4)	8.7509(9)	5.6103(14)	4.7730(18)
2.6660(−3)	30	3.6998(0)	1.1916(4)	3.0281(9)	1.3408(14)	7.8839(17)
3.6304(−3)	35	2.7285(0)	6.4771(3)	1.2489(9)	4.0352(13)	1.7332(17)
4.7444(−3)	40	2.0980(0)	3.8275(3)	5.8691(8)	1.4397(13)	4.7028(16)
6.0085(−3)	45	1.6658(0)	2.4116(3)	3.0535(8)	5.8579(12)	1.5002(16)
7.4231(−3)	50	1.3567(0)	1.5987(3)	1.7247(8)	2.6471(12)	5.4428(15)
8.9891(−3)	55	1.1280(0)	1.1045(3)	1.0433(8)	1.3037(12)	2.1932(15)
1.0707(−2)	60	9.5416(−1)	7.8970(2)	6.6930(7)	6.9022(11)	9.6473(14)
1.2578(−2)	65	8.1891(−1)	5.8127(2)	4.5210(7)	3.8874(11)	4.5713(14)
1.4602(−2)	70	7.1162(−1)	4.3864(2)	3.1991(7)	2.3104(11)	2.3098(14)
1.6781(−2)	75	6.2513(−1)	3.3824(2)	2.3629(7)	1.4401(11)	1.2345(14)
1.9115(−2)	80	5.5440(−1)	2.6583(2)	1.8174(7)	9.3665(10)	6.9350(13)
2.1607(−2)	85	4.9584(−1)	2.1246(2)	1.4542(7)	6.3320(10)	4.0731(13)
2.4256(−2)	90	4.4679(−1)	1.7238(2)	1.2105(7)	4.4352(10)	2.4905(13)
2.7064(−2)	95	4.0534(−1)	1.4177(2)	1.0500(7)	3.2106(10)	1.5799(13)

Electric multipoles

ω (mc^2 units)	Z	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\beta_4^{(1)}$	$\beta_5^{(1)}$
1.1840(−3)	20	1.0210(3)	1.6226(6)	6.9235(11)	1.4760(16)	1.3228(20)
1.8506(−3)	25	4.1848(2)	4.2626(5)	1.1824(11)	1.6089(15)	9.1753(18)
2.6660(−3)	30	2.0198(2)	1.4313(5)	2.8109(10)	2.6474(14)	1.0409(18)
3.6304(−3)	35	1.0913(2)	5.6940(4)	8.4084(9)	5.7937(13)	1.6592(17)
4.7444(−3)	40	6.4044(1)	2.5647(4)	2.9800(9)	1.5639(13)	3.3946(16)
6.0085(−3)	45	4.0036(1)	1.2704(4)	1.2038(9)	4.9603(12)	8.4081(15)
7.4231(−3)	50	2.6307(1)	6.7828(3)	5.3993(8)	1.7882(12)	2.4227(15)
8.9891(−3)	55	1.7999(1)	3.8486(3)	2.6392(8)	7.1559(11)	7.8926(14)
1.0707(−2)	60	1.2733(1)	2.2965(3)	1.3871(8)	3.1243(11)	2.8469(14)
1.2578(−2)	65	9.2647(0)	1.4297(3)	7.7607(7)	1.4688(11)	1.1190(14)
1.4602(−2)	70	6.9044(0)	9.2280(2)	4.5881(7)	7.3606(10)	4.7341(13)
1.6781(−2)	75	5.2531(0)	6.1456(2)	2.8504(7)	3.9004(10)	2.1346(13)
1.9115(−2)	80	4.0695(0)	4.2063(2)	1.8537(7)	2.1717(10)	1.0178(13)
2.1607(−2)	85	3.2035(0)	2.9492(2)	1.2587(7)	1.2641(10)	5.0991(12)
2.4256(−2)	90	2.5576(0)	2.1125(2)	8.9141(6)	7.6606(9)	2.6698(12)
2.7064(−2)	95	2.0679(0)	1.5426(2)	6.5877(6)	4.8168(9)	1.4545(12)

[a] The power of 10 by which the numbers should be multiplied is given by the number in parentheses.

trend, for both magnetic and electric transitions, in going from R to $10R$, is an increase in D by factors of the order of 10 for $Z=5$ to factors of the order of 2 to 5 for $Z=95$.

The conclusions reached by Church and Weneser[5] for the K shell agree, at least qualitatively, with those enumerated above. To take one example, for $Z\simeq 82$, Church and Weneser show that the M_I coefficient for the K shell has a D value of about 30 whereas our corresponding figure is about 25. Our results also give support to their statement that the ICC results are quite insensitive to uncertainties in nuclear radius and shape.

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