Nuclear States with High Angular Momenta According to the Single-Particle Model

DANIEL SPERBER
Physics Division, IIT Research Institute, Chicago, Illinois

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An expression for the minimum rotational energy of a nucleus is obtained assuming that any observable quantity may be represented by a sum of contributions from the individual nucleons. The expression is evaluated explicitly using the Fermi-gas model with two different potentials, and using the shell model. Computation of selected effective moments of inertia indicate that the moment of inertia of the nucleus is of the same order of magnitude as that of a rigid body.

I. INTRODUCTION

IN this paper a calculation of the minimal energy of a nucleus having a specified angular momentum is presented. The scope of this calculation is, however, limited to nuclei which are spherically symmetric when in their ground state. In addition, Coulomb forces and pairing effects[1,2] are not included. The latter effect is of minor importance for states with high angular momenta.[3,4] Neglecting the electrostatic energy is an unrealistic assumption except for the lightest nuclei. The consequence of the inclusion of the electrostatic energy in heavy nuclei in calculating the deformation is an increase in the deformation. From the spin dependence of the rotational energy effective moments of inertia have been calculated.

It should be stressed that the rotational motion studied in this paper differs from the rotational motion studied by Bohr and Mottelson.[5] The pioneering work of Rainwater[6] and Bohr and Mottelson,[5,7] which was soon followed by a number of other papers, concerned itself with the calculation of moments of inertia of nuclei having many nucleons outside of closed shells, i.e., nuclei highly deformed even when in their ground state. In the present paper, moments of inertia are calculated only for nuclei which are spherical when in their ground state.

The importance of nuclear states with high angular momentum was realized from experimental data obtained with heavy-ion accelerations. When uranium is bombarded by 100-MeV oxygen nuclei, states with angular momentum as high as 60 units of $\hbar$ are obtained. It therefore is important to study the rotational states with high angular momentum. To carry out such a study it is necessary to adopt a model for the nucleus. The simplest model to use is the liquid-drop model. Rotating liquid drops were studied in various degrees of sophistication by Poincaré,[8] Lord Rayleigh,[9] Appel,[10] Pik Pichak,[11] Hiskes,[12] Beringer and Knox,[13] Sperber,[14] Cohen, Plasil, and Swiatecki,[15] Carlson and Pao Lu,[16] and Chandrasekhar.[17] In each of these papers the nucleus was considered as an incompressible rotating liquid drop with surface tension. An alternative treatment which may be more realistic, is to consider the nucleons as a Fermi gas moving in a potential. Such a model is considered in the present paper. In these calculations all observable quantities are calculated as sums of contributions from single particles. Calculations based on the Fermi-gas model are presented in Secs. II and III of this paper. In addition a calculation based on the shell model is presented in Sec. IV.

It should be stressed that the validity of this calculation is limited to those values of the angular momentum for which stable configurations exist. Carlson and Pao Lu[16] have shown that when the parameter λ^2 becomes larger than 0.28, chargeless nuclei become secularly unstable. For values of $\lambda^2 > 0.28$ the present calculation refers to secularly unstable configurations. The question of their significance, if any, would require an analysis of the dynamics of small oscillations of the system.

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II. THE SHAPE OF THE OCCUPIED REGION IN MOMENTUM SPACE, AND THE SHAPE OF DEFORMED NUCLEI

In a previous paper by the present author[14] a nucleus with high angular momentum was viewed as a rotating incompressible liquid drop. In the present treatment, that of the Fermi gas, the incompressibility condition is relaxed. One of the consequences of high angular momentum according to this latter model is the monotonic increase of the density of nuclear matter with the distance from the center of the nucleus. However, the energy associated with nuclear deformation is larger by an order of magnitude than the energy associated with the change in density.

For the case of nucleons moving independently, inside the nucleus and for a nucleus having no net angular momentum, the Fermi-gas model predicts a spherical shape.[18–21] The density of nuclear matter, in this case, is spherically symmetric. Also, the occupied region in momentum space is spherically symmetric, with the center of the sphere at zero momentum. The variation of density of nuclear matter (or the Fermi radius p_F) with the space coordinates depends on the single-particle potential. For a spherically symmetric potential this density depends only on the distance from the center of the nucleus.

Now the shapes of the nuclei and the shapes of the occupied region in momentum space are calculated for nonvanishing angular momenta. The shape of the occupied region in momentum space is calculated for a spherically symmetric potential. This shape will be obtained by minimizing the total energy subject to the constraints of a specified number of particles and specified angular momentum. First, all observable quantities, which are due to sums of contribution from single particles, are expressed as integrals over phase space. Accordingly the following expressions are obtained for the number of particles A, the kinetic energy E_k, the potential energy E_p and the angular momentum I:

$$A=\frac{4}{h^3}\int\int\int\int\int\int dp_x dp_y dp_z dx dy dz, \quad (1)$$

$$E_k=\frac{4}{2Mh^3}\int\int\int\int\int\int (p_x^2+p_y^2+p_z^2) \times dp_x dp_y dp_z dx dy dz, \quad (2)$$

$$E_p=\frac{4}{h^3}\int\int\int\int\int\int V(|\mathbf{r}|) dp_x dp_y dp_z dx dy dz, \quad (3)$$

$$I\hbar=\frac{4}{h^3}\int\int\int\int\int\int [xp_y-yp_x] dp_x dp_y dp_z dx dy dz. \quad (4)$$

(The z direction is chosen as the direction of the angular momentum.) To calculate the shape of the occupied region in momentum space a knowledge of p_z as a function of the other variables on the boundary of the occupied region in momentum space is necessary. This dependence is obtained by using Eqs. (1) through (4) and demanding that:

$$\delta(E_k+E_p-\alpha'^2A-2\beta'I) =\delta\int\int\int\int\int\left\{(p_x^2+p_y^2)p_z+\frac{p_z^2}{3}+2MV(|\mathbf{r}|)p_z -2MV\alpha'^2p_z-\frac{2\beta' M}{\hbar}[xp_y-yp_x]p_z\right\} \times dp_x dp_y dx dy dz=0. \quad (5)$$

Here α'^2 and β' are Lagrange multipliers. Applying the Euler-Lagrange equations to momentum space one obtains:

$$\left(p_x+\frac{\beta' M}{\hbar}y\right)^2+\left(p_y-\frac{\beta' M}{\hbar}x\right)^2+p_z^2 =2M\alpha'^2+2MV(|r|)-\frac{\beta'^2M^2}{\hbar^2}(x^2+y^2)=p_F^2. \quad (6)$$

The occupied region in momentum space consists of spheres with a center at $p_{x,c}$, $p_{y,c}$ such that:

$$p_{x,c}=-(\beta' M/\hbar)y,\quad p_{y,c}=(\beta' M/\hbar)x. \quad (7)$$

It will be convenient to introduce cylindrical coordinates in momentum space, such that $p_{\parallel}$ is the linear momentum in the direction of the unit vector $\boldsymbol{\phi}$ indicating the increase of the aximuthal angle ϕ in coordinate space and $p_\perp$ is perpendicular to $p_{\parallel}$. Using these coordinates the equation of the boundary of momentum space becomes

$$\left(p_{\parallel}-\frac{\beta' M}{\hbar}\rho\right)^2+p_\perp^2=2M\alpha'^2+2MV(|r|)-\frac{\beta'^2M^2}{\hbar^2}\rho^2. \quad (8)$$

It can easily be seen that the boundary of the occupied region in momentum space will be of similar shape for a potential proportional to the surface having the form

$$E_p=\sigma\int\int\rho_m(z,\phi) \times\left[1+\left(\frac{\partial\rho_m}{\partial z}\right)^2+\left(\frac{1}{\rho_m}\right)^2\left(\frac{\partial\rho_m}{\partial\phi}\right)^2\right]^{1/2} dz d\phi. \quad (9)$$

In Eq. (9) ρ_m is the value of ρ on the boundary of the nucleus. In a similar way it can be shown (by solving

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the Euler-Lagrange equation in coordinate space) that the nuclei under consideration are cylindrically symmetric.

III. CALCULATION OF SHAPE AND ROTATIONAL ENERGY BASED ON A FERMI-GAS MODEL

In the present section the minimal rotational energy of a Fermi-gas nucleus is calculated. The calculation is performed for two different potentials: (a) A square-well potential supplemented by a potential proportional to the surface area, (b) a harmonic-oscillator potential.

A. Square-Well Potential Supplement by Surface Potential

The accumulating evidence for a surface potential as a supplement to the volume potential[22–28] suggests the present calculation. After integrating $p_\perp$ in cylindrical coordinates from the axis of the occupied region of momentum space to the points on its boundary, one obtains for the observable quantities:

$$A=\frac{8\pi^2}{h^3}\int\int\int p_\perp^2 dp_{\shortparallel}\rho d\rho dz, \tag{10}$$

$$E_k=\frac{8\pi^2}{2h^3M}\int\int\int\left(\frac{p_\perp^2}{2}+p_{\shortparallel}^2\right)p_\perp^2 dp_{\shortparallel}\rho d\rho dz, \tag{11}$$

$$E_v=\frac{-8\pi^2V_0}{h^3}\int\int\int p_\perp^2 dp_{\shortparallel}\rho d\rho dz, \tag{12}$$

$$E_s=2\pi\sigma\int\rho_m\left[1+\left(\frac{d\rho_m}{dz}\right)^2\right]^{1/2}dz, \tag{13}$$

$$I\hbar=\frac{8\pi^2}{h^3}\int\int\int p_\perp^2 p_{\shortparallel}dp_{\shortparallel}\rho^2 d\rho dz. \tag{14}$$

The potential energy E_p in Eq. (3) is replaced by the volume energy E_v and the surface energy E_s. The equation for the boundary of the occupied region in momentum space becomes

$$p_\perp^2+(p_{\shortparallel}-\beta\rho)^2=\alpha^2+\rho^2\beta^2. \tag{15}$$

Here the new Lagrange multipliers are defined as

$$\alpha^2=2M[\alpha'^2+V_0],\quad \beta=M\beta'/\hbar. \tag{16}$$

For a vanishing angular momentum α becomes the Fermi momentum. The integration over $p_{\shortparallel}$ and ρ in Eqs. (10) through (14) can be carried out explicitly by first integrating $p_{\shortparallel}$ between $\beta\rho-[\alpha^2+(\beta\rho)^2]^{1/2}$ and $\beta\rho+[\alpha^2+(\beta\rho)^2]^{1/2}$ and subsequently integrating ρ from zero to ρ_m. This integration yields

$$A=\frac{64\pi^2}{15h^3\beta^2}\int_0^{z_0}[(\alpha^2+\beta^2\rho_m^2)^{5/2}-\alpha^5]dz, \tag{17}$$

$$E_k=\frac{32\pi^2}{15Mh^3\beta^2}\int_0^{z_0}\left\{\frac{8}{7}[(\alpha^2+\beta^2\rho_m^2)^{7/2}-\alpha^7]-\alpha^2[(\alpha^2+\beta^2\rho_m^2)^{5/2}-\alpha^5]\right\}dz, \tag{18}$$

$$I\hbar=\frac{64\pi^2}{3h^3\beta^3}\int_0^{z_0}\left\{[(\alpha^2+\beta^2\rho_m^2)^{7/2}-\alpha^7]-\frac{\alpha^2}{5}[(\alpha^2+\beta^2\rho_m^2)^{5/2}-\alpha^5]\right\}dz. \tag{19}$$

Using the Euler-Lagrange equation in coordinate space and Eqs. (17) through (19), the differential equation for the meridian cross section of the nucleus becomes

$$\frac{-(Mh^3/2\pi)\sigma\rho_m}{[1+(d\rho_m/dz)^2]^{1/2}}+\frac{8}{105}\frac{1}{\beta^2}[(\alpha^2+\beta^2\rho_m^2)^{7/2}-\alpha^7]=\text{const.} \tag{20}$$

Equation (20) was integrated numerically. This was done by introducing dimensionless variables R and Z such that

$$R=(\beta/\alpha)\rho_m,\quad Z=(\beta/\alpha)z. \tag{21}$$

Here R_e is the highest value of R at the equator. Using Eqs. (17) through (19) one obtains

$$Z=\frac{R_e}{[(1+R_e^2)^{7/2}-1]}\int_R^{R_e}\frac{dR}{\left\{\left[\frac{R}{(1+R^2)^{7/2}-1}\right]^2-\left[\frac{R_e}{(1+R_e^2)^{7/2}-1}\right]^2\right\}^{1/2}}. \tag{22}$$

Equation (22) gives the shape of the nucleus. Similar equations are obtained for its properties by using Eqs. (21)

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and (22).

$$A=\frac{28\pi\sigma M}{\beta^2}\left[\frac{R_e}{[(1+R_e^2)^{7/2}-1]}\right]^{3/2}\int_0^{R_e}\frac{[(1+R^2)^{5/2}-1]dR}{\left\{\left[\frac{R}{(1+R^2)^{7/2}-1}\right]^2-\left[\frac{R_e}{(1+R_e^2)^{7/2}-1}\right]^2\right\}^{1/2}},\tag{23}$$

$$E_k=\frac{14\pi\sigma\alpha^2}{\beta^2}\left[\frac{R_e}{(1+R_e^2)^{7/2}-1}\right]^{3/2}\int_0^{R_e}\frac{\{[8/7(1+R)^{7/2}-1]-[(1+R^2)^{5/2}-1]\}dR}{\left\{\left[\frac{R}{(1+R^2)^{7/2}-1}\right]^2-\left[\frac{R_e}{(1+R_e^2)^{7/2}-1}\right]^2\right\}^{1/2}},\tag{24}$$

$$E_s=\frac{4\pi\sigma\alpha^2}{\beta^2}\int_0^{R_e}\left(\frac{R}{R_e}\right)^2\left[\frac{(1+R_e^2)^{7/2}-1}{(1+R^2)^{7/2}-1}\right]\frac{dR}{\left\{\left[\frac{R}{(1+R^2)^{7/2}-1}\right]^2-\left[\frac{R}{(1+R_e^2)^{7/2}-1}\right]^2\right\}^{1/2}},\tag{25}$$

$$I\hbar=\frac{35\pi\alpha^2M\sigma}{4\beta^3}\left[\frac{R_e}{(1+R_e^2)^{7/2}-1}\right]^{3/2}\int_0^{R_e}\frac{2/7[(1+R_e^2)^{7/2}-1]-\frac{2}{5}[(1+R^2)^{5/2}-1]dR}{\left\{\left[\frac{R}{(1+R^2)^{7/2}-1}\right]^2-\left[\frac{R_e}{(1+R_e^2)^{7/2}-1}\right]^2\right\}^{1/2}}.\tag{26}$$

Using Eqs. (21) and (22) the meridian cross section of a rotating nucleus was calculated (Table I). The rotational energy is the total energy minus the sum of the Fermi energy and surface energy of a nonrotating nucleus. Equations (23) through (26) were used to evaluate the rotational energy (Table II) of some typical nuclei. These results are compared in Table II with similar results based on the liquid-drop model. The calculated rotational energy using the liquid-drop model exceeds the energy based on the Fermi-gas model. This small difference is attributed to compressibility. The nuclear density increases towards the nuclear surface owing to rotation.

The numerical comparison in Table II between a compressible and incompressible fluid can be considered only as illustrative of the effect. The best estimates of nuclear compressibility coefficients are two or three times higher than those obtained from the considerations of the kinetic energy of a Fermi gas upon which the present treatment is based.

TABLE I. Shape of meridian cross section of a rotating bag filled with a Fermi gas of nucleons. The shape of the meridian cross section is described by means of a relation between the dimensionless surface coordinates R/R_e and Z/R_e. The rate of rotation is given by the dimensionless angular-momentum parameter λ^2= (rotational energy)/(surface energy of rigid sphere). For any given choice of this parameter the deformation is most quickly summarized by the ratio $Z(0)/R_e$, i.e., the ratios between the distance from the center to the pole to the distance from the center to the center of the equator.

$\lambda^2=0.000$ $\frac{Z(0)}{R_e}=1.000$		$\lambda^2=0.502$ $\frac{Z(0)}{R_e}=0.813$		$\lambda^2=0.743$ $\frac{Z(0)}{R_e}=0.621$		$\lambda^2=0.864$ $\frac{Z(0)}{R_e}=0.598$	
R/R_e	Z/R_e	R/R_e	Z/R_e	R/R_e	Z/R_e	R/R_e	Z/R_e
0.000	1.000	0.000	0.813	0.000	0.621	0.000	0.598
0.200	0.980	0.281	0.778	0.322	0.607	0.334	0.581
0.400	0.917	0.502	0.696	0.560	0.564	0.576	0.546
0.600	0.800	0.812	0.509	0.840	0.482	0.871	0.471
0.800	0.600	0.945	0.342	1.142	0.265	1.180	0.259
1.000	0.000	1.082	0.000	1.264	0.000	1.312	0.000

B. Harmonic-Oscillator Potential

In the present section the nucleus again is treated as a gas of noninteracting fermions, but this time the nucleons are not constrained by a force acting at a well-defined surface, but rather are forced to move in a

TABLE II. The minimum rotational energy of a nucleus with a specified angular momentum for some typical nuclei. Here $U_R{}^f$ is the rotational energy according to the Fermi-gas model and U_R is the rotational energy according to the liquid-drop model (with energies in MeV and angular momentum in units of $\hbar$). The following constants were used; $r_0=1.216\times10^{-13}$ cm and $4\sigma r_0{}^2=17.80$ MeV.

	$A=100$			$A=150$	
I	$U_R{}^f$	U_R	I	$U_R{}^f$	U_R
0	0.000	0.000	0	0.000	0.000
4	0.252	0.258	5	0.204	0.209
8	1.020	1.030	10	0.825	0.833
12	2.289	2.313	15	1.853	1.874
16	4.052	4.099	20	3.271	3.326
20	6.296	6.376	25	5.122	5.184
	$A=200$			$A=250$	
I	$U_R{}^f$	U_R	I	$U_R{}^f$	U_R
0	0.000	0.000	0	0.000	0.000
7	0.258	0.253	8	0.234	0.228
14	1.000	1.010	16	0.901	0.910
21	2.257	2.271	24	2.147	2.047
28	3.969	4.021	32	2.834	3.636
35	6.208	6.280	40	6.042	5.698

harmonic-oscillator well.[29–32] The potential is defined as:

$$U(x,y,z)=0 \quad \text{for} \quad x^2+y^2+z^2 \geqslant R_0^2,$$
$$U(x,y,z)=-U_0+\tfrac{1}{2}M\omega^2(x^2+y^2+z^2), \quad \text{for} \quad x^2+y^2+z^2 \leqslant R_0^2. \tag{27}$$

Here R_0 is the radius of the nucleus while ω is the characteristic frequency corresponding to a nucleon in a given potential. The frequency ω is defined by the relation

$$\tfrac{1}{2}M\omega^2R_0^2=U_0. \tag{28}$$

The technique utilized in the present calculation is equivalent to the one employed in the previous section. Therefore many intermediate steps are omitted and only final results are quoted for the nuclear properties.

$$A=\frac{64\pi^2 2^{3/2}\alpha^6}{15h^3[M^2\omega^2-\beta^2]M\omega}\int_0^1(1-x^2)^{5/2}dx =\frac{8\pi^3\alpha^6 2^{1/2}}{3h^3[M^2\omega^2-2\beta^2]M\omega}, \tag{29}$$

$$I=\frac{512\pi^3 2^{1/2}\alpha^8}{105h^4[M^2\omega^2-2\beta^2]M\omega}\int_0^1(1-x^2)^{7/2}dx =\frac{16}{3}\frac{\pi^4}{h^4}\frac{\alpha^8 2^{1/2}\beta}{[M^2\omega^2-2\beta^2]^2M\omega}, \tag{30}$$

$$E=\frac{2^{1/2}64\pi^2[4M^2\omega^2+12\beta^2]\alpha^8}{415M^2\omega[M^2\omega^2-2\beta^2]^2h^3}\int_0^1(1-x^2)^{7/2} +\frac{2^{1/2}64\pi^2\alpha^8M^2\omega^2}{15M^2\omega[M^2\omega^2-2\beta^2]^2h^3}\int_0^1(1-x^2)^{5/2}dx =\frac{2^{1/2}8\pi^3\alpha^8[M^2\omega^2+\frac{1}{3}\beta^2]}{3M^2\omega[M^2\omega^2-2\beta^2]^2h^3}. \tag{31}$$

Here α and β are Lagrange multipliers to be chosen so as to obtain the specified number of nucleons (they are analogous to the previous α and β but not equal to them) and specified angular momentum, while the dimensionless parameter x is defined as

$$x=M\omega z/\sqrt{2}\alpha. \tag{32}$$

The energy E in Eq. (31) includes only that part of the energy which increases with deformations. The equation of the meridian cross section of the cylindrically symmetric nucleus becomes

$$z^2+\frac{M^2\omega^2-2\beta^2}{M\omega^2}\rho_m^2=\frac{2\alpha^2}{M^2\omega^2}. \tag{33}$$

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TABLE III. Energy of the lowest state of a rotating nucleus with a specified angular momentum as a function of that angular momentum. The calculation is based on the Fermi-gas model of the nucleus with the nucleons moving in a harmonic-oscillator potential with $R=r_0A^{1/3}$, $r_0=1.216\times10^3$ cm, $\hbar\omega=56.2/A^{1/3}$ MeV. All nuclear rotational energies U_R are in MeV and all angular momenta are in units of $\hbar$.

$A=100$		$A=150$		$A=200$		$A=250$	
I	U_R	I	U_R	I	U_R	I	U_R
0	0.000	0	0.000	0	0.000	0	0.000
4	0.467	5	0.378	7	0.548	8	0.412
8	1.864	10	1.508	14	1.828	16	1.648
12	4.186	15	3.391	21	4.110	24	3.705
16	7.419	20	6.020	28	7.278	32	6.581
20	11.541	25	9.383	35	11.367	40	10.242

Introducing a parameter η such that

$$\eta=\beta/M\omega \tag{34}$$

and using Eqs. (29) through (31), the following parametric dependence between angular momentum and energy is obtained:

$$I=\frac{3^{1/3}}{2^{1/6}}\frac{A^{4/3}\eta}{[1-2\eta^2]^{2/3}}, \tag{35}$$

$$U_R=\frac{3^{1/3}}{2^{1/6}}\left[\frac{1+\eta^2}{[1-2\eta^2]^{2/3}}-1\right]\hbar\omega. \tag{36}$$

For a simpler physical picture, an approximate formula can be obtained which yields an explicit dependence of energy on angular momentum. This simplified expression is limited to η's such that

$$\eta\ll1. \tag{37}$$

Expanding Eqs. (35) and (36) into a power series in η leaving only the lowest powers, and subsequently eliminating η, one obtains:

$$U_R=\frac{5}{2}\frac{2^{1/2}}{3^{1/6}}\frac{\hbar\omega}{A^{4/3}}I^2=\frac{I^2\hbar^2}{2\mathcal{J}_{\text{eff}}}. \tag{38}$$

Therefore,

$$\mathcal{J}_{\text{eff}}=\frac{3^{1/6}\hbar A^{4/3}}{5\times2^{1/6}\omega} \tag{39}$$

may be considered as an effective moment of inertia. Calculations of the lowest energy of a rotating nucleus with a specific angular momentum using Eq. (38) are shown in Table III.

IV. THE CALCULATION OF THE ROTATIONAL ENERGY BASED ON THE SHELL MODEL

In this section a calculation of nuclear rotational energy based on the shell model is presented. Assuming a harmonic-oscillator potential, the single-particle energy including a spin-orbit interaction becomes

$$E=\hbar\omega[2(n-1)+l-\zeta(\mathbf{l}\cdot\mathbf{s})]. \tag{40}$$

Here n is the single-particle principal quantum number, $\mathbf{l}$ and $\mathbf{s}$ are the orbital and spin angular momenta, ζ measures the strength of the spin-orbit interaction. As in the previous sections, the calculation is based on the assumption that all observable quantities of the nucleus as a whole are due to a sum of contributions from individual nucleons. For convenience, an occupation function $\bar{\nu}(n,l,j,m)$ is introduced which states the number of particles in each quantum state; so that the number of nucleons A, the energy E and the angular momentum I become:

$$A=\sum_{n,l,j,m}\bar{\nu}(n,l,j,m)\,, \tag{41}$$

$$E=\hbar\omega\sum_{n,l,j,m_j}[2(n-1)+l-\zeta(\mathbf{l}\cdot\mathbf{s})]\bar{\nu}(n,l,j,m)\,, \tag{42}$$

$$I=\sum_{n,l,j,m}m\bar{\nu}(n,l,j,m)\,. \tag{43}$$

Since this discussion is limited to nuclei with equal numbers of protons and neutrons, as Coulomb forces are neglected, it is convenient to introduce a new occupation function

$$\nu=2\bar{\nu}\,. \tag{44}$$

Here ν can only be zero or one. If n_F, l_F, j_F, and m_F are the highest single-particle quantum numbers for occupied nucleon states in the nucleus, then

$$\nu(n,l,j,m)=1,\quad \text{if}\quad n<n_F,\ j<j_F,\ l<l_F,\ m<m_F\,, \tag{45}$$
$$\nu(n,l,j,m)=0,\ \text{otherwise.}$$

For the quantum numbers l, j, m there is a highest occupied state, characterized by a Fermi radial quantum number n_F so that

$$n_F=n_F(l,j,m;A,I)\,. \tag{46}$$

If n_F is a known function of the other quantum numbers, then the state of the nucleus is determined. The function n_F [Eq. (46)] is determined by using a variational method. To do so, the terms appearing in (41) are split into two groups, (a) those for which $j=l+\frac{1}{2}$ and, (b) those for which $j=l-\frac{1}{2}$. Then the summation over n is carried out so that

$$A=\Big[\sum_{l,j=l+1/2,m}n_F(l,j,m;A,I)+\sum_{l\ j=l-1/2\ m}n_F(l,j,m;A,I)\Big], \tag{47}$$

$$I=\Big[\sum_{l,j=l+1/2,m}\sum mn_F(l,j,m;A,I)+\sum_{l,j=l-1/2,m}mn_F(1,j,m;A,I)\Big], \tag{48}$$

$$E=\hbar\omega\Big[\sum_{l,j=l+1/2,m}n_F{}^2(l,j,m;A,I)-n_F(l,j,m;A,I)+ln_F(l,j,m;A,I)-\tfrac{1}{2}\zeta l+\sum_{l,j=l-1/2,m}n_F{}^2(l,j,m;A,I)-n_F(l,j,m_j;A,I)+ln_F(l,j,m;A,I)+\tfrac{1}{2}\zeta(l+1)\Big]. \tag{49}$$

In order to use the methods of the calculus of variations, the summations (46)–(49) have to be replaced by an integration with n_F becoming a continuous function of its variables; thus, for example, A becomes

$$A=\int\!\!\int n_F(l,\,j=l+\tfrac{1}{2},\,m;A,\,I)dmdj+\int\!\!\int n_F(l,\,j=l-\tfrac{1}{2},\,m;A,\,I)dmdj\,. \tag{50}$$

Similar expressions are obtained for the other observable quantities in the continuous approximation. Since the minimization of the energy is subjected to two subsidiary conditions (a specified number of particles and a specified angular momentum) it is necessary to demand the following:

$$\delta(E-\alpha A-\beta I)=0\,. \tag{51}$$

Again α and β are Lagrange multipliers now to be chosen so as to satisfy Eqs. (47) and (48). In the continuous approximation Eq. (51) yields:

$$n_F(l,\,j=l+\tfrac{1}{2},\,m;A,\,I)=\tfrac{1}{2}[\alpha-(l-1)+\beta]+\tfrac{1}{2}\langle\zeta\rangle l\,,$$
$$n_F(l,\,j=l-\tfrac{1}{2},\,m;A,\,I)=\tfrac{1}{2}[\alpha-(l-1)+\beta]-\tfrac{1}{2}\langle\zeta\rangle(l+1)\,. \tag{52}$$

Equation (52) shows that for the same l, the value of n_F for which $j=l+\frac{1}{2}$ is larger than the value of n_F for which $j=l-\frac{1}{2}$. This is in conformity with the shell model. The constants α and β can be evaluated by introducing Eq. (52) into Eqs. (47) and (48) and solving for α and β. The part of the energy which depends on angular momentum becomes:

$$U_R=\frac{6\hbar\omega I^2}{(l_{F_+}+1)(2l_{F_+}{}^3+10l_{F_+}{}^2+15l_{F_+}+6)+l_{F_-}(l_{F_-}+1)(2l_{F_-}{}^2+2l_{F_-}-1)}\,. \tag{53}$$

Here l_{F_+} is the highest l value for which $j=l+\frac{1}{2}$, and l_{F_-} is the highest l value for which $j=l-\frac{1}{2}$. At first sight it may appear from Eq. (53) that U_R does not depend on the strength of the spin-orbit interaction. However, a more detailed analysis shows that this is not the case since from Eq. (52) it can be seen that the larger the spin-orbit interaction is, the larger l_{F_+} will be and the smaller l_{F_-} will be (for a specified number of

TABLE IV. Energy of the lowest rotational state of a nucleus with specified angular momentum as calculated from the continuous approximation of the shell model with $\hbar\omega=(56.2/A^{1/3})$ MeV. All rotational energies U_R in the table are in MeV and all angular momenta are in units $\hbar$.

$A=100$		$A=150$		$A=200$		$A=250$	
I	U_R	I	U_R	I	U_R	I	U_R
0	0.000	0	0.000	0	0.000	0	0.000
4	0.547	5	0.443	7	0.536	8	0.483
8	2.183	10	0.1776	14	2.141	16	1.931
12	4.904	15	4.814	21	4.814	24	4.340
16	8.690	20	7.051	28	8.524	32	7.708
20	13.517	25	10.990	35	13.313	40	12.018

nucleons). Since U_R depends on l_{F+} and l_{F-} in different ways, the rotational energy depends on the spin-orbit interaction. For a vanishing spin-orbit interaction, Eq. (53) will reduce to

$$U_R=\frac{3}{2}\frac{\hbar\omega I^2}{l_F(l_F+1)^2(l_F+2)}. \tag{54}$$

The quantity $\mathcal{J}_{\rm eff}$ may be obtained from Eq. (53) as:

$$\mathcal{J}_{\rm eff}=\frac{2\hbar}{\omega}(l_{F+}+1)(l_{F+}^3+10l_{F+}^2+15l_{F+}+6) +(l_{F-}+1)(2l_{F-}^2+2l_{F-}-1)l_{F-}. \tag{55}$$

$\mathcal{J}_{\rm eff}$ may be considered as an effective moment of inertia.

As an example of the validity of the present method, it can simply be applied to oxygen-16. For the case of O^{16} the minimum rotational energy with angular momentum of $8\hbar$ can be calculated easily. In this case $l_F=1$ so that $U_R=8\hbar\omega$. Indeed the lowest energy requirement to obtain an angular momentum of $8\hbar$, without violating Pauli's principle, is achieved by elevating four nucleons from the $1p$ to $2p$ state with an energy of $2\hbar\omega$ per nucleon.

TABLE V. The ratio of the effective moment of inertia $\mathcal{J}_{\rm eff}$ to the rigid-body moment of inertia $\mathcal{J}_{\rm rig}$ as a function of λ^2 according to different nuclear models (here λ^2 is the dimensionless measure of angular momentum) for $A=175$ and $r_0=1.216\times10^{-13}$ cm. The angular momentum $I\hbar=50$ corresponds with these constants to $\lambda^2=0.167$. The effective moment of inertia is defined as

$$U_R=\frac{\hbar^2}{\mathcal{J}_{\rm rig}}\frac{\mathcal{J}_{\rm rig}}{\mathcal{J}_{\rm eff}}I^2.$$

The moments of inertia are expressed in a dimensionless form as the ratio between the calculated moment of inertia and the moment of inertia of a rigid sphere.

λ^2	Liquid drop	Bag full of Fermi gas	Fermi gas in oscillator potential	Continuum version of shell model
0.000	1.000	1.015	0.540	0.471
0.148	1.219	1.243	0.604	0.471
0.445	1.498	1.536	0.662	0.471
0.567	1.572	1.621	0.684	0.471
0.750	1.705	1.765	0.710	0.471
1.008	1.862	1.934	0.739	0.471

Table IV lists the minimum rotational nuclear energy as a function of angular momentum for some typical nuclei.

V. DISCUSSION

Expressions for the minimal nuclear rotational energy U_R have been calculated using (a) the Fermi-gas model with two different potentials and, (b) the shell model. It is worthwhile to compare the results obtained by the use of the three different methods. For this purpose the effective moments of inertia are compared in Table V and Fig. 1. In addition these moments of inertia are also compared with calculated moments obtained in a previous paper.[14] It is seen that the calculated moments of inertia using the various nuclear models are of the

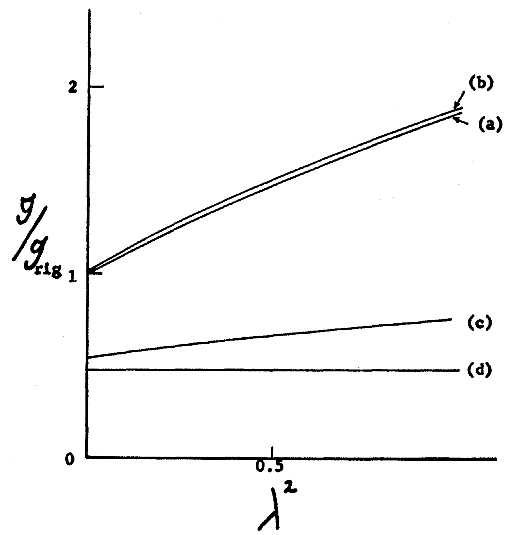

FIG. 1. The ratio of the effective moment of inertia $\mathcal{J}_{\rm eff}$ to the rigid-body moment of inertia as a function of λ^2 according to different models. (Here λ^2 is the dimensionless measure of angular momentum for $A=175$ and $r_0=1{,}216\ 10^{-13}$ cm.) The effective moment of inertia is defined as

$$U_R=\frac{\hbar^2}{2\mathcal{J}_{\rm rig}}\frac{\mathcal{J}_{\rm rig}}{\mathcal{J}_{\rm eff}}I^2.$$

order of magnitude of the moment of inertia of a rigid body.

As mentioned previously each of the calculations is based on a particular set of assumptions, which limits its validity. A comparison between the moments of inertia, as calculated using the different models, allows one to draw some conclusions regarding these assumptions. First a comparison between the results for the liquid-drop model and for a bag full of fermions is made. The only difference between the two calculations is that in the latter the compressibility condition is relaxed. Table II shows that this relaxation decreases the energy. It is also seen that the change of energy due to compressibility is very small. Next a comparison between the two calculations based on the statistical model is considered. In both these calculations the compressibility conditions are relaxed in the same way, but the potentials are very different. From a comparison between the results of the two calculations one can estimate the effect of the potential on the moment of inertia. Table V shows that the moments of inertia for a Fermi gas moving in a harmonic-oscillator potential are smaller that those for a bag full of fermions. Finally a comparison between two calculations assuming a harmonic-oscillator potential is made. Here again the difference stems from the difference in the assumed models. The Fermi-gas model predicts slightly higher moments of inertia which increase with angular momentum. The shell model predicts constant and smaller moments of inertia. An attempt was made to study the moments of inertia as one goes from the extreme case of the liquid-drop model to the extreme case of the shell model. In all calculations interactions are neglected except as they show up in the surface energy. Therefore, all calculations can be considered as single-particle calculations. The two last calculations using a harmonic oscillator could become more realistic if the potential would be allowed to deform. Such calculations with the inclusion of electrostatic forces would approximate very well the behavior of real nuclei. Grover[33] has already concluded that the latter results agree well with moments of inertia derived by him from experimental data.

The moments of inertia calculated from the dependence of the density of levels on angular momentum obtained by Bloch[34] and Ericson and Strutinski[35] indicate similar results to those obtained in this calculation.

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Axially Asymmetric Regions in the *s-d* Shell*

J. Bar-Touv
Department of Nuclear Physics, The Weizmann Institute of Science, Rehovoth, Israel
AND
I. Kelson
Nuclear Structure Laboratory, Yale University, New Haven, Connecticut
(Received 22 January 1965)

The Hartree-Fock problem for even-even nuclei in the *s-d* shell is solved without requiring the intrinsic state to be axially symmetric. Two regions of axial asymmetry are found—around Mg^{24} and around S^{32}. The existence of an energy gap between occupied and nonoccupied single-particle states is established for all cases. The moment-of-inertia tensor for the axially asymmetric cases is computed and provides an improvement over the axially symmetric results of former calculations.

I. INTRODUCTION

THE calculational limitations of the shell model in treating nuclei with many nucleons (more than four) outside closed shells led to the development of the collective picture of nuclei. In this treatment, the nucleus is described as performing some kind of collective motion, such as rotations and vibrations.[1] The coordinates of the individual nucleons are replaced by a smaller number of degrees of freedom, of a more macroscopic nature. (In the unified model, which is an extension of this, some individual nucleons are still considered.)[2] A small number of parameters is sufficient to specify exactly the new Hamiltonian of the system. In the case of a pure axially symmetric rotator, only one such parameter—the moment of inertia—is necessary. The initial justification of such a model was the actual occurence of, say, rotational spectra in certain nuclei. Moreover, the values of the parameters were artificially adjusted to fit those observed spectra. From this point of view, the model provides merely a phenomenological presentation of experimental data. Its qualitative and quantitative relations to the actual many-body problem of the nucleus (or, at least, to the shell-model version of it) are therefore of great interest.

The significance of the "intrinsic" state of the nucleus was long recognized in this connection. The "intrinsic"

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