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Theoretical Aspects of Mixtures of Thermal and Coherent Radiation*

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Some of the statistical properties of a mode of propagation for an electromagnetic field which is a superposition of thermal and coherent radiation are derived. It is found that the electric field and the magnetic field have Gaussian probability densities. The variance of the mixed field is the same as the variance of the thermal part of the field alone, while the average of the mixed field is the same as the average of the coherent field alone. It is pointed out that this result differs from the classical theory in that the zero-point field appears only once in the variance for the mixed field while it appears once in the variance of each of the constituent fields. This result is extended to a more general class of fields, and for that class it is shown that except for the zero-point field, the quantum properties of superposition are the same as the classical properties of superposition of noisy fields. The probability distribution for the number of photons in a mode of mixed radiation is also derived. The results shows that there are fluctuations in the number of photons that arise, because of interference effects.

I. INTRODUCTION

THERE has been considerable discussion of the quantum-mechanical description of the coherence properties of light.[1–4] An outcome of this discussion has been an increased understanding of particular states of harmonic oscillators called "coherent states." The expansion of the density operator in terms of these coherent states is particularly useful for determining the quantum-mechanical description of the superposition of electromagnetic fields.

Section II serves as a technical introduction to the subject matter. The properties of the expansion of the density operator in terms of the "coherent states" that will be of use are discussed briefly. The probability distributions for position and momentum of harmonic oscillators are found in terms of the above-mentioned expansions, and the connection between these quantities and the electric and magnetic fields is also discussed.

In Sec. III we apply the formulas of Sec. II to derive some of the properties of radiation that is a superposition of thermal radiation and coherent radiation. The probability densities for the field components and for the number of photons in a particular mode are derived and discussed.

Some of the properties of radiation that is a superposition of several other fields, are derived in Sec. IV. The results are valid for a moderately general class of fields.

II. P REPRESENTATION AND PROBABILITY DENSITIES

The eigenstates of the photon annihilation operator can be utilized to compute the quantum-mechanical description of the superposition of electromagnetic fields.[1,2] These states have also been described recently in connection with the quantum-mechanical description of coherence.[1,3,4] The eigenstate equation for the annihilation operator of a photon in the kth mode of propagation is

$$a_k|\alpha_k\rangle=\alpha_k|\alpha_k\rangle. \tag{2.1}$$

The range of the eigenvalue α_k is the entire complex plane, and α_k is in general a complex number, since a_k is not a Hermitian operator.

The density operator ρ for the electromagnetic fields that are used in this paper can be expanded in terms of these eigenstates. It is given by

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha, \tag{2.2}$$

where

$$d^2\alpha=d[\mathrm{Im}(\alpha)]d[\mathrm{Re}(\alpha)].$$

It has been shown by Glauber that, owing to the Hermiticity of ρ, the function $P(\alpha)$ will be real, but it can be negative. Even when $P(\alpha)$ is non-negative, it will not be a probability density, because the α states are not independent—another consequence of the non-Hermiticity of a_k. The function $P(\alpha)$ in Eq. (2.2) has been called the "P representation" and the eigenstates of a_k have been called "coherent states" by Glauber.[1] We shall adopt this terminology throughout this paper. A field which is the result of superimposing two fields with the P representations $P_1(\alpha)$ and $P_2(\alpha)$ will have a resultant P representation that is given by[1]

$$P(\alpha)=\int P_1(\alpha')P_2(\alpha-\alpha')d^2\alpha'. \tag{2.3}$$

If a harmonic oscillator is in a state described by the density operator ρ, the probability density for the momentum will be given by

$$\mathrm{Prob}(p)=\langle p|\rho|p\rangle=\int P(\alpha)|\langle\alpha|p\rangle|^2d^2\alpha. \tag{2.4}$$

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[1] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[2] E. C. G. Sudershan, Phys. Rev. Letters **10**, 277 (1963).
[3] C. L. Mehta and E. Wolf, Phys. Rev. **134**, A1149 (1964).
[4] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).

The scalar product $\langle\alpha|p\rangle$ has been computed by Glauber[1] and the result when properly normalized[5] is

$$|\langle\alpha|p_k\rangle|^2=\frac{1}{(\pi\hbar\omega_k)^{1/2}}\exp\left[\frac{-[p_k-(2\hbar\omega_k)^{1/2}\alpha_I]^2}{\hbar\omega_k}\right], \quad (2.5)$$

where ω_k = angular frequency of the kth mode and α_I = imaginary part of α. Therefore, by combining equations (2.4) and (2.5) we obtain

$$\mathrm{Prob}(p_k)=\int\frac{P(\alpha)}{(\pi\hbar\omega_k)^{1/2}}\times\exp\left[\frac{-(p_k-(2\hbar\omega_k)^{1/2}\alpha_I)^2}{\hbar\omega_k}\right]d^2\alpha. \quad (2.6)$$

The quantum-mechanical properties of a radiation field are the same as those of harmonic oscillators.[6] In particular, the electric field can be expanded in the following way:

$$\mathbf{E}(\mathbf{r},t)=C\sum_k p_k(t)\mathbf{U}_k(\mathbf{r}), \quad (2.7)$$

where C is a constant whose magnitude depends on the units of measurement, and $\mathbf{U}_k(\mathbf{r})$ is the normalized spatial part of a modal expansion. Thus, by utilizing Eqs. (2.6) and (2.7) we can obtain the probability density of the electric field. In a similar way, one could obtain the probability density of the magnetic field by finding and utilizing the probability density of the position coordinate q of a harmonic oscillator.[6] The probability density of q can be found from[1,5]

$$\mathrm{Prob}(q_k)=\int\frac{P(\alpha)}{(\pi\hbar/\omega_k)^{1/2}}\times\exp\left[\frac{-[q_k-(2\hbar/\omega_k)^{1/2}\alpha_R]^2}{\hbar/\omega_k}\right]d^2\alpha, \quad (2.8)$$

where α_R = real part of α.

The probability distribution for the number of photons in a particular mode is given by

$$\mathrm{Prob}(n)=\langle n|\rho|n\rangle=\int P(\alpha)|\langle\alpha|n\rangle|^2d^2\alpha. \quad (2.9)$$

The term $|\langle\alpha|n\rangle|^2$ is the probability distribution of the number of photons when the field is in a pure $|\alpha\rangle$ state. This has been found to be[1]

$$|\langle\alpha|n\rangle|^2=(|\alpha|^{2n}/n!)e^{-|\alpha|^2}, \quad (2.10)$$

which is a Poisson distribution whose mean is $|\alpha|^2$. From Eqs. (2.9) and (2.10), we have

$$\mathrm{Prob}(n)=\int P(\alpha)\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}d^2\alpha. \quad (2.11)$$

In this paper we will use Eq. (2.3) to find the P representation for a composite field when the P representation of the component fields is known. Then the probability densities of p, q, and n will be found from Eqs. (2.6), (2.8), and (2.11).

III. THE MIXTURE OF THERMAL AND COHERENT RADIATION

In this section we shall derive some of the properties of radiation that is the superposition of thermal and coherent radiation. We shall consider only a single mode and drop the subscript k. The P representation for an oscillator in a coherent state $|\beta\rangle$ is given by

$$P(\alpha)=\delta^2(\alpha-\beta), \quad (3.1)$$

where

$$\delta^2(\alpha)=\delta[\mathrm{Re}(\alpha)]\delta[\mathrm{Im}(\alpha)].$$

The P representation for a mode of a thermal field[7] whose average number of photons is $\langle n_T\rangle$, is given by[1,3]

$$P(\alpha)=\frac{1}{\pi\langle n_T\rangle}\exp\left[\frac{-|\alpha|^2}{\langle n_T\rangle}\right]. \quad (3.2)$$

Substituting (3.1) and (3.2) into (2.3), one finds that the P representation for the mixed thermal and coherent field is

$$P(\alpha)=\frac{1}{\pi\langle n_T\rangle}\exp\left[\frac{-|\alpha-\beta|^2}{\langle n_T\rangle}\right]. \quad (3.3)$$

The probability densities for p and q are now found by substituting (3.3) into (2.6) and (2.8) and performing the integrations. The results are[5]

$$\mathrm{Prob}(p)=\frac{1}{(2\pi\sigma_p^2)^{1/2}}\exp\left[\frac{-(p-(2\hbar\omega)^{1/2}\beta_I)^2}{2\sigma_p^2}\right], \quad (3.4a)$$

$$\mathrm{Prob}(q)=\frac{1}{(2\pi\sigma_q^2)^{1/2}}\exp\left[\frac{-(q-(2\hbar/\omega)^{1/2}\beta_R)^2}{2\sigma_q^2}\right], \quad (3.4b)$$

where β_I = imaginary part of β,

$$\beta_R=\text{real part of }\beta,$$

$$\sigma_p^2=\hbar\omega(\langle n_T\rangle+\tfrac{1}{2}), \quad (3.5a)$$

$$\sigma_q^2=(\hbar/\omega)(\langle n_T\rangle+\tfrac{1}{2}). \quad (3.5b)$$

The probability densities given by Eqs. (3.4) are Gaussian densities. The average values of p and q are

[5] G. Lachs, Ph.D. dissertation, Syracuse University, 1964 (unpublished).

[6] I. R. Senitzky, Phys. Rev. **119**, 671 (1960).

[7] In this paper the term "thermal field" has been used to describe any field whose P representation is given by Eq. (3.2). In reality, this includes spectrum lines emitted by gas discharges, Gaussian radio noise, Čerenkov radiation, etc., as well as a thermally excited field.

functions of the coherent part of the field $|\beta\rangle$ only, while the variances of p and q are functions of $\langle n_T\rangle$ only. The variances of p and q represent noise, since they result in a stochastic uncertainty in the field quantities **E** and **H**. We have been using the Heisenberg representation when we label the coherent state $|\beta\rangle$. In the Schrödinger representation this becomes $|\beta e^{-i\omega t}\rangle$ and therefore β_I and β_R of Eqs. (3.4) are

$$\beta_R=|\beta|\cos(\omega t+\theta)\,, \tag{3.6a}$$

$$\beta_I=-|\beta|\sin(\omega t+\theta)\,, \tag{3.6b}$$

where

$$\theta=-\tan^{-1}[\mathrm{Im}(\beta)/\mathrm{Re}(\beta)]\,.$$

It can be seen from Eqs. (3.4) and (3.6) that the average values of p and q perform simple harmonic motion with a definite phase. Thus, one might consider the coherent part of the radiation as a deterministic signal. However, even when $\langle n_T\rangle=0$, the variances of p and q will not be zero, owing to the zero-point field. While it is not true that a pure coherent field by itself is a deterministic signal, the fact that the variance of a mixed field is the same as the variance of a pure thermal field allows one to consider the coherent part of a mixed field as the signal part, and its thermal part as random Gaussian noise.

The probability distribution for the number of photons in a mode of mixed radiation can be found by substituting Eq. (3.3) into Eq. (2.11). In order to evaluate this integral we convert the complex variables into polar form.

$$\mathrm{Prob}(n)=\frac{\exp[-|\beta|^2/\langle n_T\rangle]}{\pi\langle n_T\rangle n!}\int_0^\infty\int_0^{2\pi}\{|\alpha|^{2n}e^{-|\alpha|^2 b}\}\times\left\{\exp\left[\frac{-2|\beta|\,|\alpha|}{\langle n_T\rangle}\sin(\chi+\phi)\right]\right\}|\alpha|\;d|\alpha|\;d\phi\,, \tag{3.7}$$

where

$$\chi=\tan^{-1}[\mathrm{Im}(\beta)/\mathrm{Re}(\beta)]\,,$$
$$\phi=\tan^{-1}[\mathrm{Im}(\alpha)/\mathrm{Re}(\alpha)]\,,$$
$$b=1+1/\langle n_T\rangle.$$

The integral over ϕ is well known,[8] and the result can be factored into the following form:

$$\mathrm{Prob}(n)=2\exp\left[\frac{-|\beta|^2}{\langle n_T\rangle+1}\right]\int_0^\infty|\alpha|^{2n}\times\exp\left[\frac{-(|\alpha|^2+|\beta|^2/(\langle n_T\rangle+1))}{\chi_0}\right]\times I_0\left[\frac{|\beta|\,|\alpha|\,\chi_0}{\langle n_T\rangle+1}\right]|\alpha|\;d|\alpha|\,, \tag{3.8}$$

[8] S. O. Rice, in *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover Publications, Inc., New York, 1954), p. 239.

where $\chi_0=\langle n_T\rangle/\langle n_T\rangle+1$, and I_0=modified Bessel function of zero order. The integral represents the $2n$th moment of a modified Rayleigh distribution and this has been evaluated by Rice.[7] The final result is

$$\mathrm{Prob}(n)=\left\{\left(\frac{1}{1+\langle n_T\rangle}\right)\left[\frac{\langle n_T\rangle}{1+\langle n_T\rangle}\right]^n\exp\left[\frac{-|\beta|^2}{\langle n_T\rangle+1}\right]\right\}\times\left\{{}_1F_1\left(-n;1;\frac{-|\beta|^2}{\langle n_T\rangle^2+\langle n_T\rangle}\right)\right\}\,, \tag{3.9}$$

where ${}_1F_1$ is a confluent hypergeometric function. The moments of the probability distribution in Eq. (3.9) are computed in the Appendix. The mean and variance are given by

$$\langle n\rangle=\langle n_T\rangle+\langle n_C\rangle\,, \tag{3.10a}$$

$$\sigma_n^2=2\langle n_C\rangle\langle n_T\rangle+\langle n_C\rangle+\langle n_T\rangle^2+\langle n_T\rangle\,, \tag{3.10b}$$

where $\langle n_C\rangle=|\beta|^2$=average number of photons in the coherent part alone. The general moment can be found by repeated use of the following recursion relations (derived in Appendix).

$$\langle n^{r+1}\rangle=\langle n_T\rangle(\langle n_T\rangle+1)\left(\frac{\partial\langle n^r\rangle}{\partial\langle n_T\rangle}\right)+\langle n^r\rangle(\langle n_T\rangle+\langle n_C\rangle)+\frac{2\langle n_T\rangle+1}{2}|\beta|\frac{\partial\langle n^r\rangle}{\partial|\beta|}\,. \tag{3.11}$$

The positive correlation obtained in the experiments of Hanbury Brown and Twiss[9] is due to the tendency of photons to cluster together.[10] This bunching of photons is due to the variance in the number of photons being greater than that which would be obtained from a Poisson distribution. For a Poisson distribution one has $\sigma_n^2=\langle n\rangle$. It can be seen from Eq. (3.10b) that for mixed radiation the photon bunching will be due to the first and third terms on the right-hand side. The third term represents the photon bunching of the thermal field alone, while the first term represents bunching due to interference effects. Equation (2.10) shows that the probability distribution for the number of photons for a mode in a coherent state is a Poisson distribution. Therefore coherent radiation will yield a null result in the experiment of Hanbury Brown and Twiss. Equation (3.10b) shows, however, that because of interference effects one can increase the correlation obtained in the experiment of Hanbury Brown and Twiss by adding coherent radiation to thermal radiation.

IV. SUPERPOSITION OF QUANTIZED FIELDS

In this section we shall derive the rules of superposition for several moderately general cases. First, we shall consider the superposition of two fields each of which is of the mixed thermal and coherent type. Let

[9] R. Hanbury Brown and R. Q. Twiss, Nature **177**, 27 (1956).
[10] E. M. Purcell, Nature **178**, 1449 (1956).

the P representations of two independent mixed fields be given by

$$P(\alpha')=\frac{1}{\pi\langle n_{T1}\rangle}\exp\left[\frac{-|\alpha'-\beta|^2}{\langle n_{T1}\rangle}\right], \qquad (4.1a)$$

$$P(\alpha'')=\frac{1}{\pi\langle n_{T2}\rangle}\exp\left[\frac{-|\alpha''-\gamma|^2}{\langle n_{T2}\rangle}\right]. \qquad (4.1b)$$

The P-representation for the field that is the superposition of the two mixed fields is obtained by substituting Eqs. (4.1) into (2.3). One obtains[5]

$$P(\alpha)=\frac{1}{\pi\sigma^2}\exp\left[\frac{-|\alpha-(\beta+\gamma)|^2}{\sigma^2}\right], \qquad (4.2a)$$

where

$$\sigma^2=\langle n_{T1}\rangle+\langle n_{T2}\rangle. \qquad (4.2b)$$

By comparing Eqs. (4.2) with Eq. (3.3), one can see that the thermal part adds as the variance and the coherent part adds as complex phasors. It would seem, therefore, that the classical rules of superposition for noisy fields would also hold in the quantum-mechanical description, but this is not quite true.

The variance of p for the $P(\alpha)$ given in (4.2a) is [compare (4.2a) with Eq. (3.3)]

$$\sigma_p^2=\hbar\omega(\langle n_{T1}\rangle+\langle n_{T2}\rangle+\tfrac{1}{2}). \qquad (4.3)$$

The variance for each of the fields whose P representations are given in (4.1) can be found from (3.5a). Clearly, the σ_p^2 for the superimposed field is not equal to the sum of the σ_p^2 of each of the constituent fields. The zero-point contribution $\frac{1}{2}\hbar\omega$ appears only once in the sum field, and it also appears once in each of the constituent fields. Thus, although both P representations in Eqs. (4.1) are real and non-negative, they are still not classical probability distributions. However, if one considered the zero-point field as an independent source of noise (not associated with any particular constituent field) one could treat the superposition of fields classically. The zero-point field would then be considered only once, regardless of whether one were measuring a single field or the superposition of many fields.

We can extend the result for the superposition of mixed thermal and coherent fields to a more general situation. If the P representation for the field which is the result of superimposing several fields is such that

$$P_I(\alpha_I)=\int_{-\infty}^{\infty}P(\alpha)d\alpha_R\geq 0 \qquad (4.4a)$$

and

$$P_R(\alpha_R)=\int_{-\infty}^{\infty}P(\alpha)d\alpha_I\geq 0, \qquad (4.4b)$$

then it will turn out that the zero-point field will enter only once into σ_p^2 and σ_q^2. This can be seen by carrying out the integration over α_R in Eq. (2.6). One obtains

$$\text{Prob}(p)=\int_{-\infty}^{\infty}\frac{P_I(\alpha_I)}{(\pi\hbar\omega)^{1/2}}\times\exp\left[\frac{-(p-(2\hbar\omega)^{1/2}\alpha_I)^2}{\hbar\omega}\right]d\alpha_I. \qquad (4.5)$$

Then, for a field that satisfies the condition expressed by (4.4a), Eq. (4.5) will be similar to a convolution of classical probability densities. $P_1(\alpha_I)$ is not actually a probability density for the same reasons that $P(\alpha)$ was not, but for purposes of evaluating the integral in (4.5), let us consider it as such. Then, from the classical theory of probability, the variance of p for the sum field will be the variance of $P_I(\alpha_I)$ plus $\frac{1}{2}\hbar\omega$. Thus, the zero-point field arises only once, and its source is $|\langle\alpha|p\rangle|^2$ which appears in Eq. (2.4). Equation (4.4a) is a sufficient condition for the variance of $P_I(\alpha_I)$ to be positive, but it is stronger than necessary, and further extension to a more general class of fields should be possible. An analogous result would be obtained for the variance of q.

The conditions expressed by Eqs. (4.4) will include any $P(\alpha)$ that is non-negative and, in particular, it includes the special case of mixed radiation. It has been shown by Glauber[1] that radiation fields produced by current densities and at frequencies where the recoil momentum (due to the emission of a photon) can be neglected have non-negative P representations. In fact, the P representations can be found directly from the probability distribution of the current density.[1] This condition is satisfied to good approximation by radiation fields at frequencies less than or equal to common microwave frequencies.

The results in this section show that for the special cases considered [Eqs. (4.4)] the field will add classically provided that one treats the zero-point field as an independent field.

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APPENDIX: CALCULATION OF THE MOMENTS OF EQUATION (3.9)

In order to simplify the calculation we make the following substitutions:

$$b=\langle n_T\rangle, \qquad (A1a)$$

$$c=|\beta|. \qquad (A1b)$$

The rth moment will be given by

$$\langle n^r\rangle=\sum_n n^r P(n)\,, \tag{A2}$$

$$\langle n^r\rangle=\sum_n \frac{n^r}{1+b}\left(\frac{b}{1+b}\right)^n \exp\left[-\left(\frac{c^2}{b+1}\right)\right] {}_1F_1\left(-n;1;-\frac{c^2}{b^2+b}\right), \tag{A3}$$

$$\begin{aligned}\frac{\partial\langle n^r\rangle}{\partial b}=&-\frac{1}{1+b}\sum_{n=0}^{\infty}\frac{n^r}{(1+b)}\left(\frac{b}{1+b}\right)^n\exp\left[\frac{-c^2}{b+1}\right]{}_1F_1\left(-n;1;-\frac{c^2}{b^2+b}\right)+\frac{1}{b(1+b)}\sum_{n=0}^{\infty}\frac{n^{r+1}}{1+b}\left(\frac{b}{1+b}\right)^n\\&\times\exp\left[-\frac{c^2}{b+1}\right]{}_1F_1\left(-n;1;-\frac{c^2}{b^2+b}\right)+\frac{c^2}{(1+b)^2}\sum_{n=0}^{\infty}\frac{n^{r+1}}{1+b}\left(\frac{b}{1+b}\right)^n\exp\left[-\frac{c^2}{b+1}\right]{}_1F_1\left(-n;1;-\frac{c^2}{b^2+b}\right)\\&-\frac{(2b+1)c^2}{b^2(b+1)^2}\sum_{n=0}^{\infty}\left[\frac{n^{r+1}}{1+b}\left(\frac{b}{1+b}\right)^n{}_1F_1\left[-(n-1);2;-\frac{c^2}{b^2+b}\right]\exp\left[-\frac{c^2}{b+1}\right]\right],\end{aligned} \tag{A4}$$

where we have utilized[11]

$$\frac{d}{dx}\,{}_1F_1(a;c;x)=\frac{a}{c}\,{}_1F_1(a+1,c+1,x)\,. \tag{A5}$$

The summations in the first three terms of Eq. (A4) can easily be substituted for by the use of Eq. (A3). In order to substitute for the last term in Eq. (A4), we note

$$\begin{aligned}\frac{\partial n^r}{\partial c}=&\sum_{n=0}^{\infty}\frac{n^r}{1+b}\left(\frac{b}{1+b}\right)^n\left(\frac{-2c}{1+b}\right)\exp\left[-\frac{c^2}{b+1}\right]{}_1F_1\left(-n;1;-\frac{c^2}{b^2+b}\right)\\&+\sum_{n=0}^{\infty}\frac{n^{r+1}}{1+b}\left(\frac{b}{1+b}\right)^n{}_1F_1\left[-(n-1);2;\frac{-c^2}{b^2+b}\right]\left(\frac{2c}{b(b+1)}\right)e^{(-c^2/b+1)}.\end{aligned} \tag{A6}$$

Combining Eqs. (A3), (A4), and (A6), we obtain

$$\langle n^{r+1}\rangle=b(b+1)\left(\frac{\partial\langle n^r\rangle}{\partial b}\right)+\langle n^r\rangle(b+c^2)+\frac{(2b+1)c}{2}\left(\frac{\partial\langle n^r\rangle}{\partial c}\right). \tag{A7}$$

Substituting Eqs. (A1) into Eq. (A7), we find

$$\langle n^{r+1}\rangle=\langle n_T\rangle(\langle n_T\rangle+1)\left(\frac{\partial\langle n^r\rangle}{\partial\langle n_T\rangle}\right)+\langle n^r\rangle(\langle n_T\rangle+|\beta|^2)+\tfrac{1}{2}(2\langle n_T\rangle+1)|\beta|\left(\frac{\partial\langle n^r\rangle}{\partial|\beta|}\right). \tag{A8}$$

In order to compute $\langle n\rangle$, we substitute into Eq. (A8) $\langle n^0\rangle=1$. This yields

$$\langle n\rangle=\langle n_T\rangle+|\beta|^2. \tag{A9}$$

Next, we substitute Eq. (A9) into Eq. (A8) to obtain $\langle n^2\rangle$

$$\langle n^2\rangle=|\beta|^4+|\beta|^2(4\langle n_T\rangle+1)+2\langle n_T\rangle^2+\langle n_T\rangle. \tag{A10}$$

The fluctuation in n is given by

$$\begin{aligned}(\Delta n)^2&=\langle n^2\rangle-\langle n\rangle^2\\&=|\beta|^2(2\langle n_T\rangle+1)+\langle n_T\rangle^2+\langle n_T\rangle.\end{aligned} \tag{A11}$$

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[11] W. Magnus and F. Oberhettinger, *Formulas and Theorems or the Special Function of Mathematical Physics* (Chelsea Publishing Company, New York, 1949), p. 183.