Motion of a Charge in a Gravitational Field*

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(Received 18 November 1964)

The DeWitt-Brehme equation of motion for a charged particle moving in a Riemannian space-time is examined for the case of absence of an external electromagnetic field by decomposing it into four scalar equations. This is done by introducing four orthonormal vectors along the world line of the particle and decomposing every term of the equation along them. It turns out, for example, that the "tail" has at most two nonvanishing components. We discuss the condition for a geodesic motion. We show that the "tail" and the local radiative damping term are not independent; the vanishing of the latter implies the vanishing of one of the two components of the "tail" and determines the other one to be a constant. The DeWitt-Brehme equation is then examined for the case of slow motion in a Schwarzschild gravitational field by expanding every term of it in a power series in $1/c$ (Einstein-Infeld-Hoffmann method). It is shown that the seventh-order (in $1/c$) force term, which is the first radiative term, is exactly the traditional term $(\frac{2}{3})e^2\partial^3\xi/\partial t^3$ in disagreement with a result obtained recently by DeWitt and DeWitt.

1. INTRODUCTION

THE equation of motion of a particle of charge e and mass m moving in a Riemannian space-time of arbitrary hyperbolic metric, and subjected to an external electromagnetic field $F_{\mu\nu}{}^{\text{ext}}$, was found by DeWitt and Brehme,[1] and is given by[2]

$$m(\delta^2\xi^\mu/\delta s^2)-F^{\text{ext}\mu}{}_\sigma(d\xi^\sigma/ds)=R^\mu+f^\mu. \qquad (1.1)$$

In this equation $\xi^\mu=\xi^\mu(s)$ describes the world-line of the particle, the parameter s being the proper time, R^μ is a local radiative damping force term which looks like that obtained by Dirac[3] for the flat space-time case,[4]

$$R^\mu\equiv\tfrac{2}{3}e^2\left(\frac{\delta^3\xi^\mu}{\delta s^3}-\frac{d\xi^\mu}{ds}\frac{\delta^3\xi^\sigma}{\delta s^3}\frac{d\xi_\sigma}{ds}\right), \qquad (1.2)$$

and f^μ is a nonlocal radiative damping force term involving an integral over the past history of the particle

$$f^\mu\equiv e^2\frac{d\xi^\beta}{ds}\int_{-\infty}^{s} f^\mu{}_{\beta\gamma'}\frac{d\xi^{\gamma'}}{ds'}ds', \qquad (1.3)$$

where the function $f^\mu{}_{\beta\gamma'}$ is the curl of the nonvanishing component of the retarded vector Green's function inside the light cone. This force (1.3) has been called the "*tail*" in Ref. 1. In Eqs. (1.1) and (1.2), and throughout this paper, the symbol $\delta/\delta s$ denotes an absolute derivative:

$$\delta\xi^\mu/\delta s=d\xi^\mu/ds\,,$$
$$\frac{\delta^2\xi^\mu}{\delta s^2}=\frac{d^2\xi^\mu}{ds^2}+\Gamma_{\alpha\beta}{}^\mu\frac{d\xi^\alpha}{ds}\frac{d\xi^\beta}{ds}, \text{ etc.} \qquad (1.4)$$

It should be emphasized that DeWitt-Brehme's work was based on the assumption that the gravitational field itself is given no dynamical properties, that the geometrical structure of space-time is regarded as fixed, and even the Einstein empty-space field equations are not supposed to be satisfied.

Another important assumption was that $e^2\gg m^2$, an assumption which is essentially different from that used in the Einstein-Infeld-Hoffmann (EIH) method. This fact demands some caution when we discuss the behavior of Eq. (1.1) in the limit of small velocities.

In a recent publication by DeWitt and DeWitt[5] the authors continue to discuss Eq. (1.1) by deriving from it the detailed law of motion in the simple case of a radiating charge moving at nonrelativistic velocities in a weak static gravitational field.

The essential point of their paper is that the deviation of the particle motion from a geodesic one, when $F_{\mu\nu}{}^{\text{ext}}=0$, is caused not by the local field of the particle but by a field which originates well outside the classical radius, and which is manifested by the nonlocal term f^μ in Eq. (1.1). So, the authors argued that the effect of the local radiative damping force R^μ depends on the existence of an external electromagnetic (or any other) field, and when $F_{\mu\nu}{}^{\text{ext}}=0$, it may be completely ignored. From this they conclude that to order e^2, the vanishing of the external electromagnetic field implies the vanishing of R^μ.

Assuming the existence of the Einstein gravitational field equations, the spatial component of f^μ was calculated for the special case in which the charged particle moves in the gravitational field produced by a point-like central mass (Schwarzschild field) at small velocity. It was found that f^k can be written as a sum of two force terms, a conservative and a nonconservative one,[6]

$$f^k\equiv f_{\text{C}}{}^k+f_{\text{NC}}{}^k, \qquad (1.5)$$

where

$$f_{\text{C}}{}^k=-(e^2M/r)(1/r)_{,\xi^k}, \qquad (1.6)$$

* Work supported by the Aerospace Research Laboratories of the Office of Aerospace Research, U. S. Air Force.
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[1] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N. Y.) **9**, 220 (1960).
[2] We use units in which the velocity of light c and Newton's gravitational constant G are equal to one. Greek indices run from 0 to 3, Latin from 1 to 3, and $x^0\equiv t$.
[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[4] Our metric is $(+\,-\,-\,-)$.
[5] C. M. DeWitt and B. S. DeWitt, Physics **1**, 3 (1964).
[6] Ordinary partial differentiation is denoted by a comma, $\varphi,_{\xi^k}\equiv\partial\varphi/\partial\xi^k$.

and

$$f_{NC}{}^k = \tfrac{2}{3}e^2 M(1/r)_{,\xi^k\xi^s}\dot{\xi}^s. \tag{1.7}$$

Here M is the mass of the Schwarzschild field, r is defined by $r^2=\xi^s\xi^s$, and a dot denotes time differentiation. The force term (1.7) was also written in terms of Newton's gravitational potential V,

$$f_{NC}{}^k = \tfrac{2}{3}e^2 V_{,ks}\dot{\xi}^s, \tag{1.8}$$

or, to the same accuracy and for arbitrary weak static gravitational field,

$$f_{NC}{}^k = \tfrac{2}{3}e^2 R^k{}_{00s}\dot{\xi}^s, \tag{1.9}$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor.

To this form of the nonconservative force the authors attribute great importance and emphasize its dependence neither on the acceleration nor on the derivative of the acceleration but simply on the *velocity* of the particle and the Riemann tensor. They then state: "It is only the close relation which exists between the Riemann tensor and the particle motion which, in the case of free fall, permits the nonconservative force to be recast in the form

$$f_{NC}{}^k = \tfrac{2}{3}e^2\partial^3\xi^k/\partial t^3\cdots." \tag{1.10}$$

From this the authors conclude that the phenomenon of *preacceleration* does not occur with gravitational forces since the primary expression for the nonconservative force involves $\dot{\xi}^k$ rather than $\partial^3\xi^k/\partial t^3$, and that Eq. (1.10) cannot be used to argue that preacceleration occurs, since Eq. (1.9) shows that it does not.

From the point of view of the slow-motion-approximation method (EIH method[7–10]), the force terms given by Eqs. (1.6) and (1.7) are the sixth- and the seventh-order terms of the tail (1.3), the sixth-order term being the lowest order one.

In a previous paper[11] we have already shown that the tail, in the case of an uncharged finite particle with mass μ moving in an external gravitational field produced by mass M, has the lowest (sixth-order) term[12]

$$-(\mu^2 M/r)(1/r)_{,\xi^k}, \tag{1.11}$$

which is exactly the term (1.6) when we replace μ by e. This term, however, originated from the nonlinearity of the field equations; it is a relativistic correction term (post-Newtonian force term) and, from the point of view of slow motion, it has nothing to do with the radiative damping phenomenon, in spite of the fact that it is proportional to the square of the mass (charge).

That the radiative damping force has the form (1.7), however, seems to be in disagreement with results obtained by the EIH method,[13–15] in which this radiative force was found to have the traditional form (1.10). It seems to us that this disagreement is due to the assumption that the local radiative term R^μ can be completely ignored.

The purpose of this paper is to examine the behavior of the DeWitt-Brehme equation of motion in the absence of an external electromagnetic field, for slow motion and static gravitational field, without ignoring *a priori* any term. As in Ref. 5 we shall assume that the Einstein empty-space field equations are satisfied.

Before going to the limit of slow motion, however, we discuss the DeWitt-Brehme equation by decomposing it into four scalar equations. In particular we discuss the condition for a geodesic motion; it is shown that the vanishing of the tail is both necessary and sufficient, apart from another solution which does not seem to have any physical meaning. This is done in Sec. 2 by introducing four orthogonal unit vectors along the world line of the particle (extension of the Frenet formulas to the Riemann space).

In Sec. 3 we discuss the consequences of the assumption that R^μ can be completely ignored. We find that this assumption determines the tail. We therefore withdraw this assumption in the last section and expand *every* term of the DeWitt-Brehme equation (without external electromagnetic field) into a power series in $1/c$ up to an accuracy of the seventh-order, taking into account the additional condition $m^2 \ll e^2$, thus assessing the force law of motion for a charge moving at nonrelativistic velocity in the Schwarzschild gravitational field. It is shown that this gives the traditional result for the radiative damping force, contrary to the result of Ref. 5.

2. THE GEOMETRICAL MEANING OF THE "TAIL" AND THE CONDITION FOR GEODESIC MOTION

Following Synge and Schild[16] we introduce along the world line of the particle $\xi^\mu(s)$ four orthonormal vectors $\lambda_{(0)}{}^\mu, \cdots, \lambda_{(3)}{}^\mu$. $\lambda_{(0)}{}^\mu$ is called the *unit tangent vector* and is defined by

$$\lambda_{(0)}{}^\mu = d\xi^\mu/ds. \tag{2.1}$$

The other three unit vectors are called the *unit first, second,* and *third normal.* They are related by the

[7] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 66 (1938).
[8] A. Einstein and L. Infeld, Can. J. Math. **1**, 209 (1949).
[9] L. Infeld, Rev. Mod. Phys. **29**, 398 (1957).
[10] L. Infeld and J. Plebanski, *Motion and Relativity* (Pergamon Press, Warsaw, 1960).
[12] In Ref. 11 this term was written with the opposite sign because the tail was written in the left-hand side of the equation of motion rather than in the right-hand side as in Eq. (1.1) in this paper [compare C. M. DeWitt and J. L. Ging, Compt. Rend. **251**, 1868 (1960)].
[11] M. Carmeli, Ann. Phys. (N. Y.) **30**, 168 (1964), Eq. (24).
[13] B. Bertotti, Nuovo Cimento **2**, 231 (1955).
[14] S. Bażański, Acta Phys. Polon. **15**, 363 (1956).
[15] A. Peres, Ann. Phys. (N. Y.) **12**, 86 (1961).
[16] J. L. Synge and A. Schild, *Tensor Calculus* (University of Toronto Press, Toronto, 1949), p. 72.

Frenet formulas

$$\begin{aligned} \delta\lambda_{(0)}{}^{\mu}/\delta s &= K_{(1)}\lambda_{(1)}{}^{\mu}, \\ \delta\lambda_{(1)}{}^{\mu}/\delta s &= K_{(2)}\lambda_{(2)}{}^{\mu}+K_{(1)}\lambda_{(0)}{}^{\mu}, \\ \delta\lambda_{(2)}{}^{\mu}/\delta s &= K_{(3)}\lambda_{(3)}{}^{\mu}-K_{(2)}\lambda_{(1)}{}^{\mu}, \\ \delta\lambda_{(3)}{}^{\mu}/\delta s &= -K_{(3)}\lambda_{(2)}{}^{\mu}, \end{aligned} \tag{2.2}$$

and satisfy the additional relations

$$\begin{aligned} \lambda_{(0)}{}^{\mu}\lambda_{(0)\mu} &= 1, \\ \lambda_{(1)}{}^{\mu}\lambda_{(1)\mu} &= -1, \\ \lambda_{(2)}{}^{\mu}\lambda_{(2)\mu} &= -1, \\ \lambda_{(3)}{}^{\mu}\lambda_{(3)\mu} &= -1. \end{aligned} \tag{2.3}$$

$K_{(1)}, \cdots, K_{(3)}$ are invariants called the *first*, *second*, and *third curvature*. $K_{(i)}=0$ implies $K_{(j)}=0$ for all $3\geqslant j\geqslant i+1$. For example, $K_{(1)}=0$ implies $K_{(2)}=K_{(3)}=0$, i.e., the motion is geodesic.

We now return to the DeWitt-Brehme equation of motion for the case of $F_{\text{ext}}{}^{\mu\nu}=0$. Then we have the equation

$$m\frac{\delta^2\xi^\mu}{\delta s^2}=\tfrac{2}{3}e^2\left(\frac{\delta^3\xi^\mu}{\delta s^3}-\frac{d\xi^\mu}{ds}\frac{\delta^3\xi^\sigma}{\delta s^3}\frac{d\xi_\sigma}{ds}\right)+f^\mu. \tag{2.4}$$

From (2.1) and (2.2) we have

$$\delta^2\xi^\mu/\delta s^2=K_{(1)}\lambda_{(1)}{}^{\mu}, \tag{2.5}$$

$$\delta^3\xi^\mu/\delta s^3=(dK_{(1)}/ds)\lambda_{(1)}{}^{\mu} +K_{(1)}[K_{(2)}\lambda_{(2)}{}^{\mu}+K_{(1)}\lambda_{(0)}{}^{\mu}], \tag{2.6}$$

$$(\delta^3\xi^\sigma/\delta s^3)(d\xi_\sigma/ds)=[K_{(1)}]^2. \tag{2.7}$$

Hence we get for Eq. (2.4)

$$[mK_{(1)}-\tfrac{2}{3}e^2(dK_{(1)}/ds)]\lambda_{(1)}{}^{\mu} -\tfrac{2}{3}e^2K_{(1)}K_{(2)}\lambda_{(2)}{}^{\mu}=f^\mu. \tag{2.8}$$

Furthermore, we write the vector f^μ in terms of its components

$$f^\mu=\sum_{\rho=0}^{3} f_{(\rho)}\lambda_{(\rho)}{}^{\mu}. \tag{2.9}$$

Then Eq. (2.8) gives

$$\begin{aligned} 0&=f_{(0)}, \\ mK_{(1)}-\tfrac{2}{3}e^2(dK_{(1)}/ds)&=f_{(1)}, \\ -\tfrac{2}{3}e^2K_{(1)}K_{(2)}&=f_{(2)}, \\ 0&=f_{(3)}. \end{aligned} \tag{2.10}$$

Thus in the absence of external electromagnetic field the vector f^μ lies in the hyperplane of the two vectors $\lambda_{(1)}{}^{\mu}$ and $\lambda_{(2)}{}^{\mu}$.

Suppose now that the motion is geodesic. Then $K_{(1)}=0$ and from Eq. (2.10) we see that $f^\mu=0$. Thus a geodesic motion implies the vanishing of the tail.

Let us see whether the vanishing of the tail implies a geodesic motion. From Eqs. (2.10), when $f^\mu=0$, we obtain

$$\begin{aligned} a(dK_{(1)}/ds)&=K_{(1)}, \\ K_{(1)}K_{(2)}&=0, \end{aligned} \tag{2.11}$$

where $a=2e^2/3m$ is the "classical radius" of the charge.[17] A possible solution of these equations is

$$K_{(1)}=0, \tag{2.12}$$

which corresponds to a geodesic motion and implies the vanishing of $K_{(2)}$ and $K_{(3)}$.

The most general solution of Eqs. (2.11), however, is given by

$$\begin{aligned} K_{(1)}&=Ke^{(s-s_0)/a}, \\ K_{(2)}&=0, \end{aligned} \tag{2.13}$$

where K is a constant, the value of $K_{(1)}$ at the initial proper time s_0. Of course $K_{(3)}=0$ since $K_{(2)}=0$. The special solution (2.12) corresponds to putting K equal to zero in the general solution (2.13).

Geometrically, the solution (2.13) means that the particle will move in a hyperplane. If the constant K is different from zero then $K_{(1)}$, which is the magnitude of the vector $\delta^2\xi^\mu/\delta s^2$, will increase indefinitely with s. In particular, if one takes $s_0=-\infty$ then $K_{(1)}$ will be infinite for any finite s if $K\neq 0$. This is a runaway solution which does not seem to have any physical meaning. Hence one has to take $K=0$, and thus the curve is a geodesic.

3. THE CONDITION $R^\mu=0$

We now return to the DeWitt-Brehme equation and assume, as in Sec. 2, that there is no external electromagnetic field. Let us find the implication of the assumption made in Ref. 5 that to order e^2 the local force term R^μ may be completely ignored.

From Eq. (2.4) we obtain under this assumption

$$m(\delta^2\xi^\mu/\delta s^2)=f^\mu. \tag{3.1}$$

The decomposition of this equation into its components gives, using Eq. (2.5),

$$\begin{aligned} mK_{(1)}&=f_{(1)}, \\ 0&=f_{(2)}. \end{aligned} \tag{3.2}$$

Thus the assumption that $R^\mu=0$ implies the vanishing of the component of the tail in the direction of $\lambda_{(2)}{}^{\mu}$.

Furthermore, using Eqs. (3.2) in Eqs. (2.10), we obtain the two equations

$$\begin{aligned} dK_{(1)}/ds&=0, \\ K_{(1)}K_{(2)}&=0, \end{aligned} \tag{3.3}$$

which have the solution

$$\begin{aligned} K_{(1)}&=\text{const}, \\ K_{(2)}&=0. \end{aligned} \tag{3.4}$$

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[17] This constant is related to the constant r_e of Ref. 5 by $a=\frac{2}{3}r_e$.

Thus the first curvature is a constant, and by Eq. (3.2),

$$f_{(1)}=\text{const}. \tag{3.5}$$

The solution (3.4) may be called a circle in Riemann space. It will be noted that this circle is not a closed curve. One might prefer to call it *hyperbola of constant curvature*.[18] One can interpret it as a generalization of the well-known hyperbolic motion in the flat space-time.[19]

4. THE EQUATION OF MOTION AT LOW VELOCITIES

In order to derive the detailed law of motion in the case of motion at nonrelativistic velocity from the DeWitt-Brehme equation, when $F_{\mu\nu}{}^{\text{ext}}=0$, we have simply to expand every term of Eq. (2.4), into a power series in $1/c$. In order to obtain a radiative damping force, we have to continue our expansion to an accuracy up to the seventh order in $1/c$. Since the tail f^μ has already been found up to the seventh order in Ref. 5, we only have to expand the other two terms, $\delta^2\xi^k/\delta s^2$ and R^k.

Let us begin with R^k. Since e^2 is of order four,[20] it remains to expand $\delta^3\xi^\mu/\delta s^3$ up to the third order. The other term in R^k, namely,

$$\frac{d\xi^k}{ds}\frac{\delta^3\xi^\sigma}{\delta s^3}\frac{d\xi_\sigma}{ds},$$

need not be expanded since it is of order 5.

A direct calculation gives

$$\frac{\delta^3\xi^k}{\delta s^3}=\frac{d^3\xi^k}{ds^3}+\Gamma^k{}_{\alpha\beta,\gamma}\frac{d\xi^\alpha}{ds}\frac{d\xi^\beta}{ds}\frac{d\xi^\gamma}{ds}+3\Gamma^k{}_{\alpha\beta}\frac{d^2\xi^\alpha}{ds^2}\frac{d\xi^\beta}{ds}+\Gamma^k{}_{\alpha\beta}\Gamma^\beta{}_{\rho\sigma}\frac{d\xi^\alpha}{ds}\frac{d\xi^\rho}{ds}\frac{d\xi^\sigma}{ds}. \tag{4.1}$$

The last two terms on the right of Eq. (4.1) will not contribute up to the third order since both the Christoffel symbols and $d^2\xi^\alpha/ds^2$ are of order two. The first term on the right side of Eq. (4.1), however, has the third-order term $\partial^3\xi^k/\partial t^3$. Writing the second term explicitly

$$\Gamma^k{}_{\alpha\beta,\gamma}\frac{d\xi^\alpha}{ds}\frac{d\xi^\beta}{ds}\frac{d\xi^\gamma}{ds}=\Gamma^k{}_{00,\gamma}\frac{d\xi^\gamma}{ds}\left(\frac{d\xi^0}{ds}\right)^2+2\Gamma^k{}_{0s,\gamma}\frac{d\xi^s}{ds}\frac{d\xi^\gamma}{ds}\frac{d\xi^0}{ds}+\Gamma^k{}_{rs,\gamma}\frac{d\xi^\gamma}{ds}\frac{d\xi^s}{ds}\frac{d\xi^r}{ds}, \tag{4.2}$$

we see that its contribution to the third order is given by $\Gamma^k{}_{00,\gamma}\dot{\xi}^\gamma$, where $\Gamma^k{}_{00}$ has to be calculated at its second order. Now

$$\Gamma^k{}_{00}\simeq\tfrac{1}{2}\eta^{k\lambda}(2g_{\lambda0,0}-g_{00,\lambda})\simeq\tfrac{1}{2}g_{00,k}. \tag{4.3}$$

Thus we get, for the case of the Schwarzschild field

$$\Gamma^k{}_{00}\simeq-(M/R)_{,k}, \tag{4.4}$$

where R is defined by $R^2=x^sx^s$. Hence we get for the seventh order of R^k

$${}_7R^k=\tfrac{2}{3}e^2\left[\frac{\partial^3\xi^k}{\partial t^3}-\left(\frac{M}{r}\right)_{,\xi^k\xi^s}\dot{\xi}^s\right], \tag{4.5}$$

where the index written as a subscript on the left of R^k denotes its order. But the second term on the right-hand side of Eq. (4.5) is (except for a sign reversal) just the nonconservative part (seventh-order term) of the nonlocal force term, $f_{\text{NC}}{}^k$ [see Eq. (1.7)] found in Ref. 5. Thus the contribution of the right-hand side of Eq. (2.4) to the sixth-order term is

$${}_6R^k+{}_6f^k=-(e^2M/r)(1/r)_{,\xi^k}, \tag{4.6}$$

and

$${}_7R^k+{}_7f^k=\tfrac{2}{3}e^2\partial^3\xi^k/\partial t^3 \tag{4.7}$$

for the seventh-order term, which is the traditional radiative damping force term.

Expanding the left-hand side of Eq. (2.4), and taking into account the fact that m is a test particle (i.e., $m\ll M$, in addition to $m^2\ll e^2$), we get[21]

$$m(\delta^2\xi^k/\delta s^2)\simeq m\{\ddot{\xi}^k-(M/r)_{,\xi^k}-M[(\dot{\xi}^s\dot{\xi}^s-4(M/r))(1/r)_{,\xi^k}-(4/r)_{,\xi^s}\dot{\xi}^s\dot{\xi}^k]\}. \tag{4.8}$$

We can finally write the equation of motion of the charge in the form

$$m\ddot{\xi}^k-m(M/r)_{,\xi^k}={}_6F^k+{}_7F^k, \tag{4.9}$$

where

$${}_6F^k=mM\{(\dot{\xi}^s\dot{\xi}^s-4(M/r))(1/r)_{,\xi^k}-4\dot{\xi}^s(1/r)_{,\xi^s}\dot{\xi}^k\}-e^2M(1/r)(1/r)_{,\xi^k}, \tag{4.10}$$

and

$${}_7F^k=\tfrac{2}{3}e^2\partial^3\xi^k/\partial t^3. \tag{4.11}$$

Equations (4.9)–(4.11) are identical with that obtained by Bażański.[22]

We thus come to the conclusion that *both the EIH*

[18] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), p. 12.

[19] T. Fulton and F. Rohrlich, Ann. Phys. (N. Y.) **9**, 499 (1960).

[20] See, for example, S. Bażański, in *Recent Developments in General Relativity* (Pergamon Press, Warsaw, 1962), p. 137.

[21] See, for example, J. N. Goldberg, in *Gravitation, an Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 3, Eq. (3-3.20).

[22] In fact, Bażański derived the equation of motion for two charged particles up to the seventh order in $1/c$. In order to obtain Eq. (4.9) from Bażański's equation one has, of course, to set e_2 equal to zero as well as letting m_1 tend to zero.

and the DeWitt-Brehme method give the same results for slow motion, provided one does not neglect any term of the DeWitt-Brehme equation of motion when expanding it in a power series in $1/c$.

Still another point should be clarified: the meaning of neglecting R^k in the DeWitt-Brehme equation within the framework of accuracy of the seventh order in $1/c$.

We have already shown that the seventh-order term of R^k is given by Eq. (4.5); lower order terms do not exist. Equation (4.5) can also be written in the form

$$ {}_7R^k=\tfrac{2}{3}e^2(d/dt)[\ddot{\xi}^k-(M/r)_{,\xi^k}], \qquad (4.12)$$

the right-hand side of which is (except for a coefficient) just the time-derivative of Newton's law of motion. Thus neglecting R^k in this approximation implies the use of Newton's law of motion, which is the lowest order of the undamped equation of motion. This alone enables us to write the result obtained, which is the traditional radiative damping force, in the form given by Eq. (1.7), rather than conversely, as has been stated in reference 5.[23,24]

ACKNOWLEDGMENTS

The author gratefully acknowledges the hospitality extended to him at Lehigh University by Professor P. Havas, and wishes to thank him and Dr. S. Bażański for helpful discussions.

Part of this work was done while the author was at the Technion, Haifa. Discussions with Professors N. Rosen and A. Peres are gratefully acknowledged.

I would like to thank also Professor B. S. DeWitt for his valuable criticism.

[23] The same procedure of using Newton's law of motion in order to write the traditional radiative damping force term in terms of the Riemann tensor was used by Peres (Ref. 15). It can easily be shown, however, that the form suggested by Peres [Eq. (12) in Ref. 15] for this term is *not* fully covariant.

[24] It will be noted that we conclude nothing about the phenomenon of preacceleration since we feel that the slow-motion approximation is not the proper tool for it.

PHYSICAL REVIEW VOLUME 138, NUMBER 4B 24 MAY 1965

Master Equations and Markov Processes

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The processes described by generalized master equations (GME), derived from the Liouville equation on the basis of various physical and dynamic arguments, have been termed Markovian or non-Markovian depending upon whether the GME did not or did involve an explicit time integration. We show that these designations are not in accord with the (very specific) mathematical definitions of Markovian and non-Markovian processes. We demonstrate that the GME does not contain sufficient information to determine whether or not the stochastic process described by it is Markovian or non-Markovian.

I. INTRODUCTION

A NUMBER of generalized master equations have been derived recently.[1] Zwanzig[2] has shown that all of these equations are equivalent if not identical. These generalized master equations differ from the Pauli equation[3] in that they involve an explicit time integration. On the basis of this time integration, these equations have been termed "*non-Markovian*" by the various authors. We shall demonstrate that this nomenclature is quite misleading and may, in fact, lead to erroneous conclusions as to the nature of the stochastic processes since it does not agree with the well-defined *mathematical* usage of the terms *Markovian* and *non-Markovian.* We suggest, in view of the development to be presented below, that the physical generalized master equations (and the stochastic processes described by them) be termed non-Paulian to distinguish them from the Pauli equation (and processes) which do not involve explicit time integrations.

In Sec. II we present definitions of joint and conditional probabilities, we give the mathematical definition of Markov processes and we derive equations for the temporal development of the joint and con-

* This work was supported in part by the National Science Foundation.

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[2] R. Zwanzig, Physica **30**, 1109 (1964).

[3] W. Pauli, *Probleme der Modernen Physik* (S. Hirzel, Leipzig, 1928), p. 30.