High-Frequency Conductivity of a Plasma in Quasiequilibrium. I. Formulation of the General Theory*

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A general expression for the high-frequency conductivity is derived from the Bogolyubov-Born-Green-Kirkwood-Yvon hierarchy for a fully ionized plasma whose unperturbed state is stable. The result includes all the effects due to the high-frequency field and the collective interactions up to first order in the plasma parameter.

I. INTRODUCTION

THE recent studies of incoherent scattering of electromagnetic waves[1,2] and other wave problems in plasma[3–5] have revealed the importance of the collisional correction to the dielectric constant, discussed in the past in the self-consistent field approximation (or the Landau-Vlasov theory). For instance, the effect of collisions is shown to give rise to significant broadening of the resonance line in the scattering problem and to result in a constant damping of the electron oscillations in the long-wavelength limit (the usual Landau damping vanishes as the wave number k approaches zero). For these and other reasons, discussion of the dielectric constant or conductivity of a plasma in the presence of frequency and spatial dispersions has attracted considerable interest[6–13] in recent years.

The existing theoretical discussions may be divided into two groups because of their distinct approaches; one has made use of the diagrammatic method[3,9,11] originally developed in quantum many-body theory, and the other has applied the kinetic-theoretical technique[4,6,7] formulated by Bogolyubov, Born, Green, Kirkwood, and Yvon (the BBGKY hierarchy[14]). Although the two approaches are difficult to compare according to the formalism, the available results are in agreement. Nevertheless, to the best of the author's knowledge, all these discussions have been presented under one common assumption, that is, that the unperturbed plasma is in thermodynamic equilibrium (electrons and ions have Maxwellian distributions with equal temperature). Because in reality the condition of thermodynamic equilibrium is usually not met, a more general theory is desirable.

This paper presents the first part of a series of studies on the high-frequency conductivity of a plasma in quasiequilibrium. As a first step we attempt to derive a general expression for the conductivity based on the truncated BBGKY hierarchy. We are able to express it in a general but reasonably simple form in which the velocity distribution functions of electrons and ions in the unperturbed plasma are considered arbitrary. We are interested in the case in which the unperturbed plasma is stable under small perturbations according to the linearized Vlasov theory, and we assume that the time of relaxation of the plasma toward a final-equilibrium Maxwellian distribution is very long compared to the period of the applied field.

A general expression for the conductivity will be determined in terms of an integral operator discussed in a previous publication.[15] In principle, this result includes all the effects of high-frequency dispersion and collective interactions up to first order in the plasma parameter, $\epsilon=1/n\lambda_D{}^3$ (where n is particle density and λ_D is the Debye distance). From this general expression, the result for the equilibrium case previously derived by Oberman, Ron, and Dawson[7] can be recovered immediately.

In a subsequent paper (II, of this series) we shall apply the result obtained here to the case of a nonisothermal plasma ($T_i \neq T_e$).

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13 In order to shorten our reference list we have excluded those works which include the effect of the external magnetic field or analyses based on Boltzmann and Fokker-Planck equations. However, some of these works will be referenced in our later publications whenever such reference seems desirable.
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II. MATHEMATICAL FORMULATION

The Governing Equations

We assume that both the unperturbed plasma and the applied high-frequency field are homogeneous. Consequently, we can list as follows the first two members of the truncated BBGKY hierarchy,[14] which describes a fully ionized plasma with Coulomb interactions only:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\nabla_1+\frac{e_s}{m_s}\mathbf{E}\cdot\nabla_{\mathbf{v}_1}\right)\mathfrak{F}_s(1,t)=\frac{1}{m_s}\sum_r n_r\int\frac{\partial\phi_{sr}(1,2)}{\partial\mathbf{r}_1}\cdot\frac{\partial}{\partial\mathbf{v}_1}\mathcal{G}_{sr}(1,2,t)d^3r_2d^3v_2,\tag{1}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\nabla_1+\mathbf{v}_2\cdot\nabla_2+\frac{e_s}{m_s}\mathbf{E}\cdot\nabla_{\mathbf{v}_1}+\frac{e_r}{m_r}\mathbf{E}\cdot\nabla_{\mathbf{v}_2}\right)\mathcal{G}_{sr}(1,2,t)-\frac{1}{m_s}\frac{\partial\mathfrak{F}_s(1,t)}{\partial\mathbf{v}_1}\\&\quad\times\sum_q n_q\int\frac{\partial\phi_{sq}(1,3)}{\partial\mathbf{r}_1}\mathcal{G}_{rq}(2,3,t)d^3r_3d^3v_3-\frac{1}{m_r}\frac{\partial\mathfrak{F}_r(2,t)}{\partial\mathbf{v}_2}\cdot\sum_q n_q\int\frac{\partial\phi_{rq}(2,3)}{\partial\mathbf{r}_2}\mathcal{G}_{sq}(1,3,t)d^3r_3d^3v_3\\&\qquad=\frac{\partial\phi_{sr}}{\partial\mathbf{r}_1}\cdot\left(\frac{\mathfrak{F}_r(2,t)}{m_s}\frac{\partial\mathfrak{F}_s(1,t)}{\partial\mathbf{v}_1}-\frac{\mathfrak{F}_s(1,t)}{m_r}\frac{\partial\mathfrak{F}_r(2,t)}{\partial\mathbf{v}_2}\right).\end{aligned}\tag{2}$$

Here, $\mathfrak{F}(1,t)$ is the one-particle distribution function; $\mathcal{G}(1,2,t)$ is the pair-correlation function; the subscripts $s, r, \cdots$ designate the species of particles; e_s and m_s are the charge and mass of the s type of particles; 1, 2, $\cdots$ denote the state variable in phase space; for instance, $(\mathbf{r}_1,\mathbf{v}_1)$, $(\mathbf{r}_2,\mathbf{v}_2)$, $\cdots$; $\phi_{sr}(1,2)=(e_se_r)/|\mathbf{r}_1-\mathbf{r}_2|$ is the interparticle Coulomb potential; $\nabla_1\equiv\partial/\partial\mathbf{r}_1$; $\nabla_{\mathbf{v}_1}\equiv\partial/\partial\mathbf{v}_1$; and $\mathbf{E}$ is the applied field which can be also written as

$$\mathbf{E}=\mathbf{E}_0e^{i\omega t}.\tag{3}$$

Let us split $\mathfrak{F}_s(1,t)$ and $\mathcal{G}_{sr}(1,2,t)$ into two parts:

$$\mathfrak{F}_s(1,t)=F_s(\mathbf{v}_1,t)+f_s(\mathbf{v}_1,t),$$
$$\mathcal{G}_{sr}(1,2,t)=G_{sr}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r}_1-\mathbf{r}_2,t)+g_{sr}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{r}_1-\mathbf{r}_2,t),$$

where F_s and G_{sr} designate, respectively, the distribution function and correlation function of the unperturbed plasma, and f_s and g_{sr} apply in a similar manner to the perturbed part. We assume,

$$F_s\gg f_s,\quad G_{sr}\gg g_{sr},$$

since the applied field $\mathbf{E}$ is considered to be weak. Furthermore, since we have postulated that the period of the oscillating field, $1/\omega$, is very short compared to the time of relaxation of the unperturbed plasma toward equilibrium, we may remark that $\mathfrak{F}_s$ and $\mathcal{G}_{sr}$ belong to a long time scale (slow process) but f_s and g_{sr} vary according to a much finer time scale (fast process).

Linearizing Eqs. (1) and (2) we obtain two sets of equations. The first set describes the relaxation of the unperturbed (nonequilibrium) plasma and takes the form

$$\frac{\partial F_s}{\partial t}=\frac{1}{m_s}\sum_r n_r\int\frac{\partial\phi_{sr}(1,2)}{\partial\mathbf{r}_1}\cdot\frac{\partial}{\partial\mathbf{v}_1}G_{sr}(1,2,t)d^3r_2d^3v_2,\tag{4}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\nabla_1+\mathbf{v}_2\cdot\nabla_2\right)G_{sr}(1,2,t)-\frac{1}{m_s}\frac{\partial F_s(1)}{\partial\mathbf{v}_1}\cdot\sum_q n_q\int\frac{\partial\phi_{sq}(1,3)}{\partial\mathbf{r}_1}G_{rq}(2,3)d^3r_3d^3v_3\\&\quad-\frac{1}{m_r}\frac{\partial F_r(2)}{\partial\mathbf{v}_2}\cdot\sum_q n_q\int\frac{\partial\phi_{rq}(2,3)}{\partial\mathbf{r}_2}G_{sq}(1,3)d^3r_3d^3v_3=\frac{\partial\phi_{sr}}{\partial\mathbf{r}_1}\cdot\left(\frac{F_r(2)}{m_s}\frac{\partial F_s(1)}{\partial\mathbf{v}_1}-\frac{F_s(1)}{m_r}\frac{\partial F_r(2)}{\partial\mathbf{v}_2}\right).\end{aligned}\tag{5}$$

The second set, which describes the small perturbations due to the high-frequency field, can be written as

$$\frac{\partial f_s}{\partial t}+\mathbf{v}_1\cdot\nabla f_s+\frac{e_s}{m_s}\mathbf{E}\cdot\frac{\partial F_s}{\partial\mathbf{v}_1}=\frac{1}{m_s}\sum_r n_r\int\frac{\partial\phi_{sr}(1,2)}{\partial\mathbf{r}_1}\cdot\frac{\partial}{\partial\mathbf{v}_1}g_{sr}(1,2,t)d^3r_2d^3v_2,\tag{6}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_1\cdot\nabla_1+\mathbf{v}_2\cdot\nabla_2\right)g_{sr}-\frac{1}{m_s}\frac{\partial F_s(1)}{\partial \mathbf{v}_1}\cdot\sum_q n_q\int\frac{\partial\phi_{sq}(1,3)}{\partial \mathbf{r}_1}g_{rq}(2,3)d^3r_3d^3v_3-\frac{1}{m_r}\frac{\partial F_r(2)}{\partial \mathbf{v}_2}\cdot\sum_q n_q\int\frac{\partial\phi_{rq}(2,3)}{\partial \mathbf{r}_2}g_{sq}(1,3)d^3r_3d^3v_3$$
$$=\frac{1}{m_s}\frac{\partial f_s(1)}{\partial \mathbf{v}_1}\cdot\sum_q n_q\int\frac{\partial\phi_{sq}(1,3)}{\partial \mathbf{r}_1}G_{rq}(2,3)d^3r_3d^3v_3+\frac{1}{m_r}\frac{\partial f_r(2)}{\partial \mathbf{v}_2}\cdot\sum_q n_q\int\frac{\partial\phi_{rq}(2,3)}{\partial \mathbf{r}_2}G_{sq}(1,3)d^3r_3d^3v_3$$
$$+\frac{\partial\phi_{sr}}{\partial \mathbf{r}_1}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial \mathbf{v}_2}\right)[f_s(1)F_r(2)+F_s(1)f_r(2)]+\frac{e_s}{m_s}\mathbf{E}\cdot\frac{\partial G_{sr}}{\partial \mathbf{v}_1}+\frac{e_r}{m_r}\mathbf{E}\cdot\frac{\partial G_{sr}}{\partial \mathbf{v}_2}. \quad (7)$$

Obviously, if one makes use of Bogolyubov's adiabatic approximation and synchronization assumption,[16] Eqs. (4) and (5) constitute essentially the usual plasma kinetic equation derived independently by Balescu,[17] Lenard,[18] Guernsey[19] and others[20,21] provided F_s is stable, subject to small perturbation.

Preliminary Derivation of the Conductivity

Our next task is to determine a general expression for the conductivity. It is convenient to rewrite Eq. (6) in terms of the Fourier transform of g_{sr}:

$$\frac{\partial f_s}{\partial t}+\frac{e_s}{m_s}\mathbf{E}_0e^{i\omega t}\cdot\frac{\partial F_s}{\partial \mathbf{v}_1}=-\frac{\partial}{\partial \mathbf{v}_1}\frac{i}{(2\pi)^3}\int d^3k\frac{4\pi\mathbf{k}e_s}{m_sk^2}\sum_r n_re_r\int d^3v_2g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty), \quad (8)$$

where $g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t\to\infty)$ represents the asymptotic solution of $g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)$ at large time and

$$g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=\int d^3r_1e^{-i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)}g_{sr}(\mathbf{r}_1-\mathbf{r}_2,\mathbf{v}_1,\mathbf{v}_2,t). \quad (9)$$

On the other hand, the equation governing $g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)$ takes the following form:

$$\left(\frac{\partial}{\partial t}+i\mathbf{k}_1\cdot\mathbf{v}_1+i\mathbf{k}_2\cdot\mathbf{v}_2-\frac{4\pi e_s}{m_sk_1^2}i\mathbf{k}_1\cdot\frac{\partial F_s}{\partial \mathbf{v}_1}\sum_s n_se_s\int d^3v_1-\frac{4\pi e_r}{m_rk_2^2}i\mathbf{k}_2\cdot\frac{\partial F_r}{\partial \mathbf{v}_2}\sum_r n_re_r\int d^3v_2\right)g_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=R_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t), \quad (10)$$

where R_{sr} is the forcing function, which can be written as follows:

$$R_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2,t)=\frac{4\pi e_se_ri\mathbf{k}}{k^2}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial \mathbf{v}_2}\right)[f_r(\mathbf{v}_2,t)F_s(\mathbf{v}_1)+f_s(\mathbf{v}_1,t)F_r(\mathbf{v}_2)]$$
$$+\frac{1}{m_s}\frac{\partial f_s}{\partial \mathbf{v}_1}\cdot 4\pi e_s\sum_q n_qe_q\frac{i\mathbf{k}}{k^2}\int d^3v_3G_{rq}(-\mathbf{k},\mathbf{v}_2,\mathbf{v}_3)-\frac{1}{m_r}\frac{\partial f_r}{\partial \mathbf{v}_2}\cdot 4\pi e_r\sum_q n_qe_q\frac{i\mathbf{k}}{k^2}$$
$$\times\int d^3v_3G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3)-\mathbf{E}_0e^{i\omega t}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}_1}+\frac{1}{m_r}\frac{\partial}{\partial \mathbf{v}_2}\right)G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2) \quad (11)$$

and

$$G_{sr}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2,t)=\int d^3r_1e^{-i\mathbf{k}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)}G_{sr}(\mathbf{r}_1-\mathbf{r}_2,\mathbf{v}_1,\mathbf{v}_2,t).$$

Since we are interested only in the high-frequency conductivity (by the term "high frequency" we mean that the frequency is assumed to be much higher than any collision frequency for particles of arbitrary species[22]), the

[16] N. H. Bogolyubov, *Problems of a Dynamic Theory in Statistical Physics* (Moscow, 1946); translated by E. K. Gora, AFCRC-TR-59-235; or *Studies in Statistical Mechanics*, edited by J. de Boer and G. E. Uhlenbeck (North-Holland Publishing Company, Amsterdam, 1962), Vol. 1.
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[18] A. Lenard, Ann. Phys. (N. Y.) **10**, 390 (1960).
[19] R. L. Guernsey, dissertation, University of Michigan, 1960 (unpublished).
[20] N. Rostoker and M. Rosenbluth, Phys. Fluids **3**, 1 (1960).
[21] J. Hubbard, Proc. Roy. Soc. (London) **A260**, 114 (1961).
[22] For an isothermal plasma, the electron-electron collision frequency is the highest collision frequency. However, in a nonisothermal plasma this is not always true.

reactive solution of Eq. (8) is certainly dominant. That is,

$$f_s(\mathbf{v}_1,t)=-\frac{e_s}{m_s}\frac{\mathbf{E}_0e^{i\omega t}}{i\omega}\cdot\frac{\partial F_s}{\partial\mathbf{v}_1}. \tag{12}$$

This situation suggests that if we require the solution of g_{sr} to be accurate only to first order in $1/\lambda_D{}^3n$, we replace f_s in Eq. (11) by the reactive solution given by Eq. (12). Substituting Eq. (12) into Eq. (11), we obtain

$$\begin{aligned}R_{sr}(t)=-E_0e^{i\omega t}\Big\{&\frac{4\pi e_se_r}{\omega k^2}\mathbf{k}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial\mathbf{v}_2}\right)\hat{k}_0\cdot\left[\frac{e_s}{m_s}\frac{\partial F_s}{\partial\mathbf{v}_1}F_r(v_2)+\frac{e_r}{m_r}\frac{\partial F_r}{\partial\mathbf{v}_2}F_s(v_1)\right]\\&+\frac{1}{m_s\omega}\frac{\partial}{\partial\mathbf{v}_1}\left(\frac{e_s}{m_s}\hat{k}_0\cdot\frac{\partial F_s}{\partial\mathbf{v}_1}\right)\cdot4\pi e_s\sum_q n_qe_q\frac{\mathbf{k}}{k^2}\int d^3v_3G_{rq}(-\mathbf{k},\mathbf{v}_2,\mathbf{v}_3)\\&-\frac{1}{m_r\omega}\frac{\partial}{\partial\mathbf{v}_2}\left(\frac{e_r}{m_r}\hat{k}_0\cdot\frac{\partial F_r}{\partial\mathbf{v}_2}\right)\cdot4\pi e_r\sum_q n_qe_q\frac{\mathbf{k}}{k^2}\int d^3v_3G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3)+\hat{k}_0\cdot\left(\frac{e_s}{m_s}\frac{\partial}{\partial\mathbf{v}_1}+\frac{e_r}{m_r}\frac{\partial}{\partial\mathbf{v}_2}\right)G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)\Big\}\\\equiv&-E_0e^{i\omega t}R_{sr}{}^0,\end{aligned} \tag{13}$$

where $\hat{k}_0$ is a unit vector parallel to E.

Since according to Eq. (20) in Ref. 15,

$$\sum_r n_re_r\int d^3v_2g_{sr}(t\to\infty)=\int_0^t d\tau Q_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\tau,\mathbf{k},-\mathbf{k})R_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2',t-\tau), \tag{14}$$

where $Q_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\tau,\mathbf{k},-\mathbf{k})$ is an operator which in the present case takes the following form:

$$\begin{aligned}Q_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';t,\mathbf{k},-\mathbf{k})=&\frac{1}{(2\pi i)^2}\int_{-\infty-i\gamma_1}^{+\infty-i\gamma_1}d\tilde{u}_1\int_{-\infty-i\gamma_2}^{+\infty-i\gamma_2}d\tilde{u}_2e^{i(k\tilde{u}_1-k\tilde{u}_2)t}\frac{1}{(u_1+\tilde{u}_1)}\\&\times\left[\int d^3v_1'\delta(\mathbf{v}_1-\mathbf{v}_1')-\frac{D_s(\mathbf{v}_1,\mathbf{k})}{\epsilon(\tilde{u}_1,\mathbf{k})}\int d^3v_1'\frac{\sum_s n_se_s}{u_1+\tilde{u}_1}\right]\int d^3v_2'\frac{\sum_r n_re_r}{\epsilon(\tilde{u}_2,\mathbf{k})(u_2'+\tilde{u}_2)},\end{aligned}$$

where

$$D_s(\mathbf{v}_1,\mathbf{k})=-\frac{4\pi e_s\mathbf{k}}{m_sk^3}\cdot\frac{\partial F_s}{\partial\mathbf{v}_1},\quad u_1=\frac{\mathbf{k}\cdot\mathbf{v}_1}{\mathbf{k}},$$

$$\epsilon(\tilde{u}_1,\mathbf{k})=1+\sum n_se_s\int d^3v_1\frac{D_s(\mathbf{v}_1,\mathbf{k})}{u_1+\tilde{u}_1}.$$

We may rewrite Eq. (8) in the following form:

$$\frac{\partial f_s}{\partial t}+\frac{e_s}{m_s}\mathbf{E}_0e^{i\omega t}\cdot\frac{\partial F_s}{\partial\mathbf{v}_1}=\frac{\partial}{\partial\mathbf{v}_1}\frac{i}{(2\pi)^3}\int d^3k\frac{4\pi e_s\mathbf{k}}{m_sk^2}E_0e^{i\omega t}\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k},-\mathbf{k})R_{sr}{}^0(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2), \tag{15}$$

where we have extended the upper limit of the τ integral in Eq. (14) to infinity and used the definition

$$\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k},-\mathbf{k})=\int_0^\infty dte^{-i\omega t}Q_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';t,\mathbf{k},-\mathbf{k}).$$

In the present case,[15]

$$\begin{aligned}\tilde{Q}_{sr}=&-\frac{1}{ik}\int d^3v_1'\frac{\delta(\mathbf{v}_1'-\mathbf{v}_1)}{\epsilon(-\omega/k-u_1,\mathbf{k})}\int d^3v_2'\frac{\sum_r n_re_r}{(u_2'-\omega/k-u_1)}+\frac{1}{2\pi ik}\int_{-\infty-i\gamma_1}^{+\infty-i\gamma_1}d\tilde{u}_1\frac{D_s(\mathbf{v}_1)}{i(u_1+\tilde{u}_1)\epsilon(\tilde{u}_1,\mathbf{k})\epsilon(-\omega/k+\tilde{u}_1,\mathbf{k})}\\&\times\int\frac{d^3v_1'\sum_s n_se_s}{(u_1'+\tilde{u}_1)}\int\frac{d^3v_2'\sum_r n_re_r}{(u_2'-\omega/k+\tilde{u}_1)}.\end{aligned}$$

Now, let us multiply Eq. (15) by $n_s e_s \mathbf{v}_1$, integrate with respect to $\mathbf{v}_1$, and then sum up all components. If we introduce the definition of the current density $\mathbf{J}$ as

$$\mathbf{J}=\sum_s n_s e_s \int d^3v_1 \mathbf{v}_1 f_s(\mathbf{v}_1,t)\,,$$

then we have

$$\frac{\partial \mathbf{J}}{\partial t}-\sum_s \frac{n_s e_s^2}{m_s}\mathbf{E}_0 e^{i\omega t}=-\frac{i}{(2\pi)^3}\sum_s \int d^3k \frac{4\pi n_s e_s^2 \mathbf{k}}{m_s k^2}E_0 e^{i\omega t}\int d^3v_1 \tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega)R_{sr}{}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')\,, \tag{16}$$

or

$$\mathbf{J}=-\sum_s \frac{i\omega_s^2}{4\pi\omega}\mathbf{E}_0 e^{i\omega t}-\frac{E_0 e^{i\omega t}}{(2\pi)^3}\sum_s \frac{\omega_s^2}{\omega}\int d^3k \frac{\mathbf{k}}{k^2}\int d^3v_1 \tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega)R_{sr}{}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')\,, \tag{17}$$

where $\omega_s^2=4\pi n_s e_s^2/m_s$. If the conductivity σ is defined in the usual way,

$$\mathbf{J}=\sigma \mathbf{E}_0 e^{i\omega t}\,, \tag{18}$$

then we have

$$\sigma=-\sum_s \frac{i\omega_s^2}{4\pi\omega}-\frac{1}{(2\pi)^3}\sum_s \frac{\omega_s^2}{\omega}\int d^3k \frac{\mathbf{k}\cdot\hat{k}_0}{k^2}\int d^3v_1 \tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega)R_{sr}{}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')\,. \tag{19}$$

The first part of Eq. (19) yields the usual dominant reactive contribution which is designated as

$$\sigma_0=-\sum_s (i\omega_s^2/4\pi\omega) \tag{20}$$

and the second term is the higher order correlation contribution σ_1:

$$\sigma_1=-\frac{1}{(2\pi)^3}\sum_s \frac{\omega_s^2}{\omega}\int d^3k \frac{\mathbf{k}\cdot\hat{k}_0}{k^2}\int d^3v_1 \tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega)R_{sr}{}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')\,. \tag{21}$$

To complete the derivation, let us return to Eq. (13) and discuss possible simplification of the expression for σ_1. First, we should remark that the solution of $G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$ at large time for the stable case has the following form[23]:

$$\begin{aligned}G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)=&\frac{1}{\mathbf{k}\cdot(\mathbf{v}_2-\mathbf{v}_1)+i\lambda}\Bigg\{-kD_r(\mathbf{k},\mathbf{v}_2)\left[\frac{e_sF_s(\mathbf{v}_1)}{\epsilon^-(\mathbf{k},-u_1)}+kD_s(\mathbf{k},\mathbf{v}_1)\int_{-\infty}^{+\infty}d^3v'\frac{\sum n_s e_s^2 F_s(\mathbf{v}')}{(\mathbf{k}\cdot\mathbf{v}'-\mathbf{k}\cdot\mathbf{v}_1+i\lambda)|\epsilon^-(\mathbf{k},-u')|^2}\right]\\&+kD_s(\mathbf{k},\mathbf{v}_1)\left[\frac{e_rF_r(\mathbf{v}_2)}{\epsilon^+(\mathbf{k},-u_2)}+kD_r(\mathbf{k},\mathbf{v}_2)\int_{-\infty}^{+\infty}d^3v'\frac{\sum n_r e_r^2 F_r(\mathbf{v}')}{(\mathbf{k}\cdot\mathbf{v}'-\mathbf{k}\cdot\mathbf{v}_2-i\lambda)|\epsilon^-(\mathbf{k},-u')|^2}\right]\Bigg\}\,.\end{aligned} \tag{22}$$

If we substituted (22) into (13), the result should obviously be woefully complicated. Fortunately, the manipulation can be handled in a much better way, as we shall discuss in Sec. III.

III. REDUCTION OF THE GENERAL EXPRESSION

Obviously, the result given by Eq. (21) with $R_{sr}{}^0$ defined by Eq. (13) is by no means satisfactory, since it is too lengthy and complicated for application. Any possible reduction or simplification is undoubtedly desirable. The principal task of this section is to show how this simplification can be made.

Let us take a closer look at the definition of $R_{sr}{}^0(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$ from Eq. (13). It is observed that we can rewrite $R_{sr}{}^0(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$ in the following form by interchanging the order of differentiation:

$$\begin{aligned}R_{sr}{}^0(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)=&\frac{e_s\hat{k}_0}{m_s\omega}\cdot\frac{\partial}{\partial\mathbf{v}_1}\Bigg[\omega G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)+\frac{e_s}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1}\cdot 4\pi\frac{\mathbf{k}}{\mathbf{k}^2}\sum_q n_q e_q\int d^3v_3 G_{rq}(-\mathbf{k},\,\mathbf{v}_2,\,\mathbf{v}_3)\\&+\frac{4\pi e_s e_r}{k^2}\mathbf{k}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial\mathbf{v}_2}\right)F_s(\mathbf{v}_1)F_r(\mathbf{v}_2)\Bigg]+\frac{e_r\hat{k}_0}{m_r\omega}\cdot\frac{\partial}{\partial\mathbf{v}_2}\\&\times\Bigg[\omega G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)-\frac{e_r}{m_r}\frac{\partial F_r}{\partial\mathbf{v}_2}\cdot 4\pi\frac{\mathbf{k}}{k^2}\sum_q n_q e_q\int d^3v_3 G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3)+\frac{4\pi e_s e_r}{k^2}\mathbf{k}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial\mathbf{v}_2}\right)F_s(\mathbf{v}_1)F_r(\mathbf{v}_2)\Bigg]\,.\end{aligned} \tag{23}$$

[23] We omit the derivation of such a solution here since it is straightforward. One may first make use of the operator discussed in Ref. 15 to obtain $\sum_r n_r e_r \int d^3v_2 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$ and $\sum_s n_s e_s \int d^3v_1 G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$. Then by substituting these results into the Fourier transforms of Eq. (15), one can obtain the solution of $G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$ after some simple reduction.

However, from the Fourier transform of Eq. (5), we see that the asymptotic solution $G_{sr}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, t\to\infty)$ should satisfy the following equation:

$$(-i\delta+\mathbf{k}\cdot\mathbf{v}_1-\mathbf{k}\cdot\mathbf{v}_2)G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)-\frac{\mathbf{k}}{k^2}\cdot\frac{1}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1}4\pi e_s\sum_q n_q e_q\int d^3v_3 G_{rq}(-\mathbf{k},\mathbf{v}_2,\mathbf{v}_3)$$
$$+\frac{\mathbf{k}}{k^2}\cdot\frac{1}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2}4\pi e_s\sum_q n_q e_q\int d^3v_3 G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3)=\frac{4\pi e_s e_r}{k^2}\mathbf{k}\cdot\left(\frac{1}{m_s}\frac{\partial}{\partial \mathbf{v}_1}-\frac{1}{m_r}\frac{\partial}{\partial \mathbf{v}_2}\right)F_s(\mathbf{v}_1)F_r(\mathbf{v}_2), \quad (24)$$

where δ is a positive and arbitrarily small number ($\delta\to 0$).

Making use of Eq. (24), we are justified in writing

$$R_{sr}^0(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)=\frac{e_s\hat{k}_0}{m_s\omega}\cdot\frac{\partial}{\partial \mathbf{v}_1}\left[(\omega+\mathbf{k}\cdot\mathbf{v}_1-\mathbf{k}\cdot\mathbf{v}_2)G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)+\frac{\mathbf{k}}{k^2}\cdot\frac{e_r}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2}4\pi\sum_q n_q e_q\int d^3v_3 G_{sq}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_3)\right]$$
$$+\frac{e_r}{m_r}\frac{\hat{k}_0}{\omega}\cdot\frac{\partial}{\partial \mathbf{v}_2}\left[(\omega+\mathbf{k}\cdot\mathbf{v}_1-\mathbf{k}\cdot\mathbf{v}_2)G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)-\frac{\mathbf{k}}{k^2}\frac{e_s}{m_s}\cdot\frac{\partial F_s}{\partial \mathbf{v}_1}4\pi\sum_q n_q e_q\times\int d^3v_3 G_{rq}(-\mathbf{k},\mathbf{v}_2,\mathbf{v}_3)\right]. \quad (25)$$

Moreover,

$$R_{sr}^0(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)=\frac{e_s}{m_s}\frac{\hat{k}_0\cdot\hat{k}}{\omega}G_{sr}(k,v_1,v_2)-\frac{e_r}{m_r}\frac{\hat{k}_0\cdot\hat{k}}{\omega}G_{sr}(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2)$$
$$+\frac{e_s\hat{k}_0}{m_s\omega}\cdot\left[(\omega+\mathbf{k}\cdot\mathbf{v}_1-\mathbf{k}\cdot\mathbf{v}_2)-\frac{\mathbf{k}}{k^2}\cdot\frac{e_s}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1}4\pi\sum_s n_s e_s\int d^3v_1+\frac{\mathbf{k}}{k^2}\cdot\frac{e_r}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2}4\pi\sum_r n_r e_r\int d^3v_2\right]$$
$$\times\frac{\partial G_{sr}}{\partial \mathbf{v}_1}+\frac{e_r\hat{k}_0}{m_r\omega}\cdot\left[(\omega+\mathbf{k}\cdot\mathbf{v}_1-\mathbf{k}\cdot\mathbf{v}_2)-\frac{\mathbf{k}}{k^2}\cdot\frac{e_s}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1}4\pi\sum_s n_s e_s\int d^3v_1+\frac{\mathbf{k}}{k^2}\cdot\frac{e_r}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2}4\pi\sum_r n_r e_r\int d^3v_2\right]\frac{\partial G_{sr}}{\partial \mathbf{v}_2}, \quad (26)$$

where we have added two *null* terms, namely,

$$\frac{\mathbf{k}}{k^2}\cdot\frac{e_s}{m_s}\frac{\partial F_s}{\partial \mathbf{v}_1}4\pi e_s\sum_s n_s e_s\int d^3v_1\frac{\partial G_{sr}}{\partial \mathbf{v}_1},$$

and

$$\frac{\mathbf{k}}{k^2}\cdot\frac{e_r}{m_r}\frac{\partial F_r}{\partial \mathbf{v}_2}4\pi e_s\sum_r n_r e_r\int d^3v_2\frac{\partial G_{sr}}{\partial \mathbf{v}_2}.$$

The advantage of writing R_{sr}^0 in the above form will be immediately apparent. Since in order to determine σ_1 we only need the result of $\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})R_{sr}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')$, we shall pay special attention to this quantity. As we can see from the discussion of Q_{sr}[15] that

$$\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})R_{sr}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')=\sum_r n_r e_r\int d^3v_1\int d^3v_2\tilde{P}_{sr}(\mathbf{v}_1,\mathbf{v}_2|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})R_{sr}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2').$$

Here $\tilde{P}_{sr}(\mathbf{v}_1,\mathbf{v}_2|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})$ is an operator which can be defined as follows:

$$\tilde{P}_{sr}(\mathbf{v}_1,\mathbf{v}_2|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})R_{sr}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')$$
$$\equiv -i\left[\omega+\mathbf{k}\cdot\mathbf{v}_1-\mathbf{k}\cdot\mathbf{v}_2-\frac{4\pi e_s}{m_s}\frac{\mathbf{k}}{k^2}\cdot\frac{\partial F_s}{\partial \mathbf{v}_1}\sum_s n_s e_s\int d^3v_1+\frac{4\pi e_r}{m_r}\frac{\mathbf{k}}{k^2}\cdot\frac{\partial F_r}{\partial \mathbf{v}_2}\sum_r n_r e_r\int d^3v_2\right]^{-1}R_{sr}^0(\mathbf{k},\mathbf{v}_1,\mathbf{v}_2). \quad (27)$$

Therefore,

$$\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})R_{sr}^0(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2')=\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})\left[\frac{e_s}{m_s}-\frac{e_r}{m_r}\right]\frac{\hat{k}_0\cdot\hat{k}}{\omega}G_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2') \quad (28)$$

since

$$\int d^3v_1\int d^3v_2\left[\frac{e_s}{m_s}\frac{\hat{k}_0}{\omega}\cdot\frac{\partial G_{sr}}{\partial \mathbf{v}_1}+\frac{e_r}{m_r}\frac{\hat{k}_0}{\omega}\cdot\frac{\partial G_{sr}}{\partial \mathbf{v}_2}\right]=0.$$

Substituting (28) into (26), we now obtain

$$\sigma_1=\frac{-\sum_s\omega_s^2}{3(2\pi)^3\omega^2}\int d^3k\frac{(\mathbf{k}\cdot\hat{k}_0)^2}{k^2}\int d^3v_1\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})\left(\frac{e_s}{m_s}-\frac{e_r}{m_r}\right)G_{sr}(\mathbf{k},\mathbf{v}_1',\mathbf{v}_2'). \tag{29}$$

As discussed in Ref. 15, for the stable case, $\tilde{Q}_{sr}(\mathbf{v}_1|\mathbf{v}_1',\mathbf{v}_2';\omega,\mathbf{k})$, may be written as

$$\tilde{Q}_{sr}(\omega,\mathbf{k})=\frac{2\pi}{k}\int d^3v_1'\frac{\delta(\mathbf{v}_1-\mathbf{v}_1')}{\epsilon^-(-\omega/k-u_1,\,u\mathbf{k})}\int d^3v_2'\sum_r n_re_r\delta_-\left(\frac{\omega}{k}+u_1-u_2'\right)+\frac{2\pi}{k}$$

$$\times\int_{-\infty+i0+}^{+\infty+i0+}d\tilde{u}_1\frac{D_s(\mathbf{v}_1)}{(\tilde{u}_1-u_1)\epsilon^+(-\tilde{u}_1\,\mathbf{k})\epsilon^-(-\omega/k-\tilde{u}_1,\,\mathbf{k})}\int d^3v_1'\sum_s n_se_s\delta_+(\tilde{u}_1-u_1')\int d^3v_2'\sum_r n_re_r\delta_-\left(\frac{\omega}{k}+\tilde{u}_1-u_2'\right),$$

where

$$\delta_\pm(\alpha)=\tfrac{1}{2}\delta(\alpha)\pm\frac{i}{2\pi}P\frac{1}{\alpha}=\mp\frac{1}{2\pi i}\lim_{\gamma\to0+}\frac{1}{\alpha\pm i\gamma},$$

and other notations are defined similary as in Ref. 15.

IV. CONCLUDING REMARKS

A general expression for the high-frequency conductivity has been derived for a plasma whose unperturbed state is assumed to be stable. Besides the above assumption, our result is quite general and reasonably simple, allowing an arbitrary number of species and arbitrary distribution functions for the electrons and ions.[24,25] The theory can be extended to the weakly unstable case by using an expansion scheme similar to that recently proposed by Frieman and Rutherford.[25]

The correctness of our result may be examined by considering the equilibrium case

$$G_{sr}(k,v_1,v_2)=\frac{-4\pi e_se_r}{\kappa T(k^2+k_D^2)}F_s(v_1)F_r(v_2), \tag{30}$$

where κ is the Boltzmann constant, k_D is the Debye wave number, and F_s and F_r are Maxwellian distributions. From (30) and (29), we readily obtain

$$\sigma_1=\frac{-4\pi}{3(2\pi)^3\kappa T\omega^2}\int_0^\infty dk\frac{k}{k^2+k_D^2}\sum_s\omega_s^2\int d^3v_1\tilde{Q}_s(\mathbf{v}_1,|\mathbf{v}_1',\mathbf{v}_2';\omega_s,\mathbf{k})\left[\frac{e_r\omega_s^2}{n_s}-\frac{e_s\omega_r^2}{n_r}\right]F_s(v_1')F_r(v_2'),$$

which [see Eq. (53) in Ref. 15] leads to the result that Oberman, Ron, and Dawson[7] derived by using the singular-integral-equation technique.

In the next paper we will discuss the application of Eq. (29) to the nonisothermal case.

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[24] A formal expression for the hf conductivity has been given by Dupree recently [T. H. Dupree, Phys. Fluids 6, 1714 (1963)]. Although his formalism is completely different from ours, his result in the long-wavelength limit is believed to be equivalent to Eq. (19) of this paper.

[25] E. A. Frieman and R. H. Rutherford, Princeton University Plasma Physics Laboratory Report MATT 179, 1963 (unpublished).