Magnetoacoustic Effect in Impure Metals

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(Received 16 November 1964)

A simple but approximate theory is given for the acoustic attenuation in impure metals or semimetals in a strong magnetic field. In the extreme quantum case the acoustic attenuation is an oscillatory function of the magnetic field. It is shown that the qualitative nature of the oscillation depends on two parameters $\omega_q\tau$ and $q_z l$, where ω_q is the phonon frequency, τ the mean collision time, $\mathbf{q}$ the phonon wave vector, and l the electron mean free path, and the z axis is chosen as the direction of the field. For $\omega_q\tau \gg 1$ the quantum oscillation is gigantic, as pointed out by Gurevich *et al.* The large absorption occurs when the electrons near the Fermi level drift in phase with the sound. In the intermediate region where $\omega_q\tau \lesssim 1$ but $q_z l \gg 1$, the oscillation is still giant in the sense that only one quantum level contributes to the absorption, but the oscillation is purely a density-of-states effect. When $q_z l \lesssim 1$, all levels contribute to the absorption, and the oscillation is of the de Haas–van Alphen type. The effect of varying the direction of sound propagation relative to the field is also discussed in all three regions.

I. INTRODUCTION

THE acoustic-attenuation measurement is a powerful tool for studying the electronic structure of metals. Our classic example of its usefulness is that it gives a direct measurement of the dimensions of the Fermi surface. Measurements have also been made on metals under high magnetic fields and low temperatures.[1] In this case, as a consequence of the Landau level structure, the acoustic attenuation exhibits quantum oscillations as a function of the field.[2] Gurevich, Skobov, and Firsov[3] predicted that, if the specimen can be made pure enough, the quantum oscillations can become "giant oscillations." Several experimental observations of such giant oscillations have been reported in the last few years. Recently, one of the authors (A. M. T.) reviewed the present knowledge on the giant oscillations and noted some important quantitative discrepancies between the theory of Gurevich *et al.* and the experiments.[4] It seems that the main difficulty of the theory is that the finite mean-free-path effect is not properly taken into account. Gurevich *et al.* first derived a result based on perturbation theory on free electrons. Then they relaxed the energy-conservation condition by replacing the δ function by a Lorentzian function. This, of course, is not the proper way to do the problem. Later, Skobov[5] made a more sophisticated study by introducing a set of randomly distributed scattering centers into the model and calculating the acoustic attenuation in the presence of scattering. The paper itself is a brilliant work of analysis, but for a person whose interest is in the physical effects due to collisions, it is of little value. Moreover, Skobov neglected the phonon energy in his final formula. This is justified in practice, but tends to obscure the physical effect because, for instance, one cannot recover the perturbation-theory result by letting the mean-free-path approach infinity. The spin splitting is also ignored in this work.

In the present paper we give a more general formula for the acoustic attenuation by including the spin splitting as well as the phonon energy. A part of this paper duplicates the work of Skobov. However, by exhibiting more intermediate mathematical steps in the derivation, we hope to achieve greater clarity. We also point out that the qualitative nature of the oscillations depends on two parameters $\omega_q\tau$ and $q_z l$, where ω_q is the phonon frequency, q_z the component of phonon wave vector in the direction of the field, τ the mean collision time, and l the mean free path of the electrons. Three different regions may be distinguished, namely, the giant oscillation region ($\omega_q\tau \gg 1$), the intermediate region ($\omega_q\tau \lesssim 1$, $q_z l \gg 1$), and the de Haas–van Alphen region ($q_z l \lesssim 1$). The amplitude of oscillation, the line shape, and the tilt-angle effect are discussed in detail for the three cases.

We use the same method for the acoustic attenuation as Gurevich *et al.*[3] and Skobov.[5] This method is a direct generalization of the golden-rule calculation for the phonon life time. We treat the finite-free-path effect of the electrons by the Green's function technique. This calculation is not rigorous because it treats the electrons as damped plane waves between successive interactions with phonons. This simplification is valid only when the electron mean free path is long compared with the phonon wavelength. In the opposite case this treatment ignores a diffusion effect. The rigorous treatment of this effect was first given by Pippard[6] for normal metals. Tsuneto[7] reformulated the theory in the standard framework of many-body problems. It will be shown by

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[1] D. H. Reneker, Phys. Rev. **115**, 303 (1959); J. G. Mavroides, B. Lax, K. J. Button, and Y. Shapira, Phys. Rev. Letters **9**, 541 (1962); Y. Shapira, *ibid.* **13**, 162 (1964).

[2] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. **117**, 937 (1960); J. J. Quinn and S. Rodriguez, *ibid.* **128**, 2494 (1962).

[3] V. L. Gurevich, V. G. Skobov, and Yu. A. Firsov, Zh. Eksperim. i Teor. Fiz. **40**, 786 (1961) [English transl.: Soviet Phys.—JETP **13**, 552 (1961)].

[4] A. M. Toxen and S. Tansal, Phys. Rev. **137**, A211 (1965).

[5] V. G. Skobov, Zh. Eksperim. i Teor. Fiz. **40**, 1446 (1961) [English transl.: Soviet Phys.—JETP **13**, 1014 (1961)].

[6] A. B. Pippard, Phil. Mag. **46**, 1104 (1955). See also C. Kittel, Acta Met. **3**, 295 (1955), and T. Holstein, Phys. Rev. **113**, 479 (1959).

[7] T. Tsuneto, Phys. Rev. **121**, 402 (1961).

direct comparison that, in the case of no magnetic field, our result is rather close to the results of Pippard for both longitudinal and transverse phonons. Even in the short mean-free-path region, where the error should be the largest, our result has the correct frequency and free-path dependence and is only about 20% larger in numerical value. In the presence of a magnetic field we can see no reason why the diffusion effect should be more important. But our method has the advantage of being much easier to carry out. So, it is hoped that we can at least obtain a first understanding of the physical effect from this calculation. A better calculation of the Pippard or Tsuneto type is reserved for the future.

II. SCATTERING OF LANDAU STATES BY IMPURITIES

In the effective-mass approximation the unperturbed Hamiltonian for an electron gas in a field is

$$\mathcal{H}_0=(1/2m)\textstyle\sum_i[\mathbf{p}_i-e\mathbf{A}(\mathbf{r}_i)]^2, \tag{2.1}$$

where m is the effective mass, $\mathbf{p}_i$ the momentum, and e the algebraic charge of the ith electron; $\mathbf{A}(\mathbf{r})$ is the vector potential. We work in the unit system with $c=\hbar=1$. Choosing the external dc field $\mathbf{H}$ to lie in the z direction and the gauge

$$\mathbf{A}=(0,-Hx,0), \tag{2.2}$$

we can write down the one-particle wave function

$$\psi(n,k_y,k_z|\,\mathbf{r})=L^{-1}e^{i(k_y y+k_z z)}\chi_n[x+(k_y/m\omega_c)], \tag{2.3}$$

where n, k_y, k_z are the quantum numbers of the state, χ_n is the one-dimensional harmonic oscillator wave function, L the linear dimension of the sample, and ω_c the cyclotron frequency

$$\omega_c=eH/m. \tag{2.4}$$

The energy of the state is

$$\epsilon(n,k_y,k_z)=\frac{k_z^2}{2m}+(n+\tfrac{1}{2})\omega_c. \tag{2.5}$$

If the spin splitting is included, the wave function is simply multiplied by a spinor and the energy becomes

$$\epsilon(n,k_y,k_z,s)=(k_z^2/2m)+(n+\tfrac{1}{2})\omega_c+s\omega_0, \tag{2.6}$$

where $s=\pm\frac{1}{2}$ and $\omega_0=g\mu_B H$ is the Zeeman energy. For bismuth ω_0 is very nearly equal to ω_c. We shall simplify the notation by using Greek letters $\{\alpha\}$, $\{\beta\}$, etc., to denote the set of quantum numbers n, k_y, k_z. The wave functions and energies will be denoted by $\psi(\alpha s|\,\mathbf{r})$ and $\epsilon(\alpha,s)$, respectively. For the many-body system, we write the wave function

$$\Psi(\mathbf{r})=\textstyle\sum_{\alpha s} c_{\alpha s}\psi(\alpha,s|\,\mathbf{r}), \tag{2.7}$$

and the Hamiltonian

$$\mathcal{H}_0=\textstyle\sum_{\alpha s}\epsilon(\alpha,s)c_{\alpha s}^{\dagger}c_{\alpha s}, \tag{2.8}$$

where $c_{\alpha s}$ is the fermion destruction operator associated with the single-particle state $\psi(\alpha,s)$.

We now introduce a set of randomly distributed scattering centers. To make the mathematics tractable, we assume that the scattering potentials are δ functions in space

$$V(\mathbf{r})=u\Omega_0\textstyle\sum_i\delta(\mathbf{r}-\mathbf{r}_i), \tag{2.9}$$

where u is the strength of the potential and Ω_0 is the volume of the unit cell. The sum is over the positions of all the scatterers. The use of this potential corresponds to the assumption that s-wave scattering predominates and the scattering length concept is applicable. Using the wave functions (2.7), we may write $V(\mathbf{r})$ in the second quantized form

$$V=\sum_{\alpha\alpha' s}\langle\alpha' s|\,V\,|\alpha s\rangle c_{\alpha' s}^{\dagger}c_{\alpha s}. \tag{2.10}$$

The matrix elements are

$$\begin{aligned}&\langle\alpha' s|\,V\,|\alpha s\rangle\\&\quad=uLN^{-1}\textstyle\sum_i e^{i(k_y-k_y')y_i+i(k_z-k_z')z_i}\\&\qquad\times\chi_{n'}{}^{*}(x_i+k_y'/m\omega_c)\chi_n[x_i+(k_y/m\omega_c)],\end{aligned} \tag{2.11}$$

where N is the total number of ions in the sample.

The effect of scattering on the Landau states is studied by the Green's function technique.[8] We define a Green's function

$$G(\mathbf{r},\mathbf{r}';\tau)=\langle T\Psi(\mathbf{r},\tau)\Psi^{\dagger}(\mathbf{r}',0)\rangle, \tag{2.12}$$

where

$$\begin{aligned}&\Psi(\mathbf{r},\tau)=\exp[-\tau(\mu\mathfrak{N}-\mathcal{H})]\Psi(\mathbf{r})\exp[\tau(\mu\mathfrak{N}-\mathcal{H})],\\&\mathcal{H}=\mathcal{H}_0+V,\\&\mathfrak{N}=\textstyle\sum_{\alpha s}c_{\alpha s}^{\dagger}c_{\alpha s},\end{aligned} \tag{2.13}$$

and μ is the Fermi energy. The operator T is the τ-ordering operator such that

$$\begin{aligned}G(\mathbf{r},\mathbf{r}';\tau)&=\langle\Psi(\mathbf{r},\tau)\Psi^{\dagger}(\mathbf{r}',0)\rangle, &&\tau>0\\&=-\langle\Psi^{\dagger}(\mathbf{r}',0)\Psi(\mathbf{r},\tau)\rangle, &&\tau<0.\end{aligned}$$

Using Eq. (2.7) we may write

$$G(\mathbf{r},\mathbf{r}';\tau)=\sum_{\alpha\alpha' ss'}\psi^{*}(\alpha' s'|\,\mathbf{r}')\psi(\alpha s|\,\mathbf{r})G(\alpha s,\alpha' s';\tau), \tag{2.14}$$

with

$$G(\alpha s,\alpha' s';\tau)=\langle Tc_{\alpha s}(\tau)c_{\alpha' s'}{}^{\dagger}(0)\rangle. \tag{2.15}$$

Furthermore, both $G(\mathbf{r},\mathbf{r}';\tau)$ and $G(\alpha s,\alpha' s';\tau)$ can be Fourier expanded in τ

$$G(\alpha s,\alpha' s';\tau)=(1/\beta)\textstyle\sum_\nu G(\alpha s,\alpha' s';\omega_\nu)e^{-i\omega_\nu\tau}, \tag{2.16}$$

where $\beta=(kT)^{-1}$, $\omega_\nu=(2\nu+1)\pi/\beta$, $\nu=$integer.

For the unperturbed system it is straightforward to find that

$$G^{(0)}(\alpha s,\alpha' s';\omega_\nu)=\frac{\delta_{\alpha\alpha'}\delta_{ss'}}{\epsilon(\alpha,s)-\mu-i\omega_\nu}. \tag{2.17}$$

[8] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskii, Zh. Eksperim. i Teor. Fiz. **36**, 900 (1959) [English transl.: Soviet

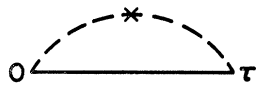

FIG. 1. The diagram for calculating the electron-impurity scattering lifetime.

The perturbed Green's function is related to the unperturbed Green's function by the Dyson equation

$$G(\alpha s,\alpha' s';\omega_\nu)=G^{(0)}(\alpha s;\omega_\nu)\delta_{\alpha\alpha'}\delta_{ss'}+\sum_{\alpha''s''}G^{(0)}(\alpha s;\omega_\nu)\times\Sigma(\alpha s,\alpha''s'';\omega_\nu)G(\alpha''s'',\alpha's';\omega_\nu), \quad (2.18)$$

where Σ is the self-energy whose meaning will be apparent shortly. In the lowest order perturbation theory it can be shown that G depends on only one set of quantum numbers and the self-energy is independent of the quantum numbers $\{\alpha\}$. Then the Dyson equation gives

$$G(\alpha s;\omega_\nu)=[\epsilon(\alpha,s)-\mu-i\omega_\nu-\Sigma(s,\omega_\nu)]^{-1}. \quad (2.19)$$

It is now clear that $\Sigma(s,\omega_\nu)$ is the self-energy due to impurity scattering. In the renormalized theory the self-energy is given by the diagram in Fig. 1 where the solid line represents the true Green's function. Explicitly,

$$\Sigma(\alpha s;\omega_\nu)=\sum_{\alpha's'}\langle\alpha s|V|\alpha's'\rangle\langle\alpha's'|V|\alpha s\rangle G(\alpha s;\omega_\nu). \quad (2.20)$$

Since the scattering preserves the spin direction, we have

$$s=s'.$$

After averaging over the random position of the impurities the matrix element product becomes

$$\langle\langle V\rangle\langle V\rangle\rangle_{\rm av}=cu^2LN^{-1}\int\left|\chi_n\left(x+\frac{k_y}{m\omega_c}\right)\right|^2\times\left|\chi_{n'}\left(x+\frac{k_y'}{m\omega_c}\right)\right|^2dx,$$

where c is the density of the impurities. If we assume that the renormalized self-energy is also independent of the quantum numbers $\{\alpha\}$, we can easily carry out the sum over k_y' in Eq. (2.20).

$$\sum_{k_y'}\left|\chi_{n'}\left(x+\frac{k_y'}{m\omega_c}\right)\right|^2=\frac{m\omega_cL}{2\pi}, \quad (2.21)$$

for L much larger than the cyclotron radius. The final x integration is trivial, so we obtain

$$\sum_{k_y'}\langle\langle V\rangle\langle V\rangle\rangle_{\rm av}=(2\pi N)^{-1}cu^2m\omega_cL^2. \quad (2.22)$$

Putting this result into Eq. (2.20) and making use of Eq. (2.19) we find the following equation for the self-energy:

$$\Sigma(s;\omega_\nu)=\frac{cu^2m\omega_cL^2}{2\pi N}\times\sum_{k_z'n'}\frac{1}{\epsilon(\alpha',s)-\mu-i\omega_\nu-\Sigma(s,\omega_\nu)}. \quad (2.23)$$

From the expression for Σ we may evaluate the energy shift Δ and level width Γ of the state with energy ω from the Fermi level.

$$\Delta(s;\omega)=\tfrac{1}{2}\{\Sigma(s;(1/i)[\omega+i\delta])+\Sigma(s;(1/i)[\omega-i\delta])\},$$
$$\Gamma(s;\omega)=(1/i)\{\Sigma(s;(1/i)[\omega+i\delta])-\Sigma(s;(1/i)[\omega-i\delta])\}. \quad (2.24)$$

One usually ignores the level shift because it does not lead to any important consequences. Then we find, after some simple manipulations, the following equation for the level width

$$\Gamma(s;\omega)=\frac{\omega_c\Gamma_0}{2\sqrt{\mu}}\times\sum_{n'}\frac{\sin\frac{1}{2}\theta_{n'}}{\{[\omega_c(n'+\frac{1}{2})+s\omega_0-\mu-\omega]^2+[\frac{1}{2}\Gamma(s;\omega)]^2\}^{1/4}}, \quad (2.25)$$

where

$$\tan\theta_{n'}=\frac{\frac{1}{2}\Gamma(s;\omega)}{\omega_c(n'+\frac{1}{2})+s\omega_0-\mu-\omega},\quad 0\le\theta_{n'}\le\pi, \quad (2.26)$$

and

$$\Gamma_0=cu^2m\Omega_0(2m\mu)^{1/2}/\pi.$$

The quantity Γ_0 may be recognized to be the width of an energy level near the Fermi level in the absence of the magnetic field. A similar expression for $\Gamma(s;\omega)$, not including spin splitting, was derived by Skobov.[9]

It is clear from Eq. (2.25) that, when the scattering is weak, the level width becomes large when the energy corresponds to a cyclotron level. Furthermore, when ω crosses a cyclotron level the quantity $\sin\frac{1}{2}\theta_{n'}$ varies from nearly 1 to nearly 0. This means that $\Gamma(s;\omega)$ must vary rapidly here. Since the Landau levels are equally spaced, the level width is a periodic function of the energy. Similarly when the field varies, the level width goes through an oscillation every time a cyclotron level passes through ω. So $\Gamma(s;\omega)$ makes de Haas–van Alphen type of oscillations. A sketch of $\Gamma(s;\omega)$ as a function of ω is shown in Fig. 2.

Finally, it is well known that the Green's function can be expressed in the following spectral representation:

$$G(\alpha s;\omega_\nu)=\int\frac{d\omega}{2\pi}\frac{A(\alpha s;\omega)}{\omega-i\omega_\nu}, \quad (2.27)$$

Phys.—JETP **9**, 636 (1959)]. See also L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962).

[9] V. G. Skobov, Zh. Eksperim. i Teor. Fiz. **37**, 1467 (1959) [English transl.: Soviet Phys.—JETP **10**, 1039 (1960)].

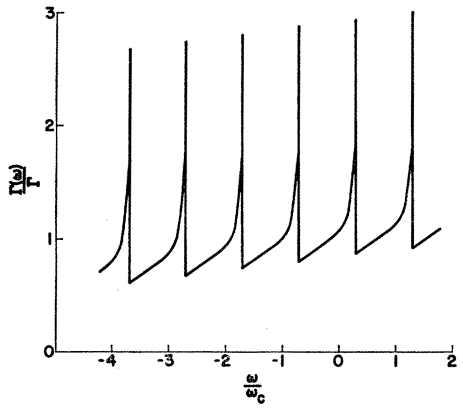

FIG. 2. The electron level width (inverse lifetime) as a function of the energy measured from the Fermi level.

where the spectral density $A(\alpha s;\omega)$ is given by

$$A(\alpha s;\omega)=\Gamma(s;\omega)/([\epsilon(\alpha,s)-\mu-\omega]^2+[\tfrac{1}{2}\Gamma(s;\omega)]^2). \quad (2.28)$$

III. CALCULATION OF ACOUSTIC ATTENUATION

The acoustic attenuation coefficient is calculated from the phonon level width as a result of electron-phonon interaction. The diagram for this process is just the self energy diagram for phonons as shown in Fig. 3. Explicitly, the self-energy $\Pi(\mathbf{q};\tau)$ is

$$\Pi(\mathbf{q};\tau)=-\sum_{\alpha\alpha' s}|U(\alpha',\alpha)|^2 G(\alpha s;\tau)G(\alpha' s;-\tau), \quad (3.1)$$

where $U(\alpha,\alpha')$ is the matrix element for electron-phonon interaction. The expression for U will be derived from the deformation potential model. We then take the Fourier component $\Pi(\mathbf{q};\omega_m)$ of Eq. (3.1). Since phonons obey Bose statistics, the frequency $\omega_m=2m\pi/\beta$, where m is an integer. The phonon level width is evaluated by the relation

$$\Gamma_p(\mathbf{q})=(1/i)\{\Pi[\mathbf{q},(1/i)(\omega_q+i\delta)]-\Pi[\mathbf{q},(1/i)(\omega_q-i\delta)]\}. \quad (3.2)$$

The acoustic attenuation coefficient is related to Γ_p by

$$\alpha(\mathbf{q})=\Gamma_p(\mathbf{q})/v_s, \quad (3.3)$$

where v_s is the sound velocity.

The deformation potential as given by Bardeen and Shockley[10] is

$$\delta U=E_1\Delta(\mathbf{r}), \quad (3.4)$$

where $\Delta(\mathbf{r})$ is the strain induced by the phonons

$$\Delta(\mathbf{r})=(2\rho\omega_q L^3)^{-1/2}qe^{i\mathbf{q}\cdot\mathbf{r}}, \quad q=|\mathbf{q}|, \quad (3.5)$$

and ρ is the density of the material. The quantity E_1 is the energy of the interaction whose value depends on the band structure of the material. For a degenerate band, E_1 is of the order of the Fermi energy. Hence, the matrix element $U(\alpha,\alpha')$ of the electron-phonon interaction is

$$U(\alpha,\alpha')=\langle\alpha' s|\delta U|\alpha s\rangle=(E_1 q/(2\rho\omega_q L^3)^{1/2})\delta_{k_y',k_y+q_y}\delta_{k_z',k_z+q_z}J_{n'n}, \quad (3.6)$$

$$J_{n'n}=\int e^{iq_x x}\chi_{n'}{}^*\left(x'+\frac{q_y}{2m\omega_c}\right)\chi_n\left(x-\frac{q_y}{2m\omega_c}\right)dx.$$

For a material with a spherical Fermi surface, the deformation-potential model applies only to the longitudinal phonons. But for bismuth the Fermi surface is highly anisotropic. As a result there should be no clear distinction between longitudinal and transverse phonons. Based on this reasoning, we use the same model of interaction for both kinds of phonons as a first approximation.

In the case of no magnetic field a simple calculation gives the following result for the acoustic attenuation:

$$\alpha(q)=(m^2E_1{}^2q/\pi^2\rho v_s)\tan^{-1}ql, \quad (3.7)$$

where l is the electron mean free path. This relation holds when $\omega_q\tau\ll 1$, and τ is the mean collision time of the electrons. In Figs. 4 and 5 the quantity $\alpha(\mathbf{q})/q$ as a function of ql is compared with the rigorous results of Pippard for longitudinal and transverse phonons. The value of E_1 is chosen to fit the curves in the short-wave limit because this is the region where the golden rule applies and the discrepancy between our method and the rigorous method vanishes. It is apparent that the approximate formula is in rather good over-all agreement with the rigorous ones. In the long-wavelength limit where the discrepancy is large, our result gives the correct wavelength and mean-free-path dependence. This gives us confidence that our result for the magnetic-field-dependent acoustic attenuation should be qualitatively correct. The quantity E_1 comes out to be about 0.6–0.7 times the Fermi energy for both cases.

Going back to the main problem, we substitute into Eq. (3.1) the spectral representation for the Green's functions, take the Fourier components of both sides,

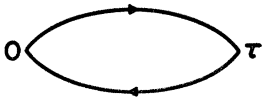

FIG. 3. The diagram for calculating the acoustic attenuation.

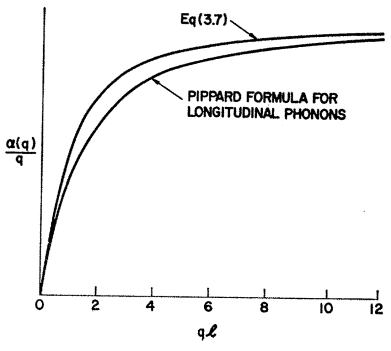

FIG. 4. Comparison of the acoustic attenuation in zero field obtained from the approximate method, Eq. (3.7), with the more rigorous result of Pippard for longitudinal phonons. The curves are matched at the very high-frequency limit.

[10] J. Bardeen and W. Shockley, Phys. Rev. **80**, 72 (1950); A. Sommerfeld and H. Bethe, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1953), pp. 499–579.

and calculate the phonon level width. The result is

$$\Gamma_p(\mathbf{q})=\sum_{\alpha\alpha' s}|U(\alpha',\alpha)|^2\int\frac{d\omega}{2\pi}A(\alpha s,\omega)A(\alpha' s,\omega-\omega_q)\times[n(\omega-\omega_q)-n(\omega)],\quad(3.8)$$

where $n(\omega)=1/(e^{\beta\omega}+1)$. The procedures for evaluating the various sums and integrals in the above equation are given in the Appendix. The final result for the acoustic attenuation is

$$\alpha(q)=\frac{E_1^2q^2(2m)^{3/2}\omega_c\beta}{128\pi^2\rho v_s}\sum_s\sum_{n=0}^{N_1}|J_{nn}|^2\int d\omega[\mathrm{sech}^2(\tfrac{1}{2}\beta\omega)]\times\frac{1}{\zeta_0\sqrt{\epsilon}}\left\{\frac{\delta/\zeta_0}{(\sqrt{\epsilon}-\sqrt{\epsilon_0})^2+(\delta/\zeta_0)^2}+\frac{\delta/\zeta_0}{(\sqrt{\epsilon}+\sqrt{\epsilon_0})^2+(\delta/\zeta_0)^2}\right\},\quad(3.9)$$

where the quantities δ, ζ_0, ϵ, ϵ_0, N_1 are defined as follows:

$$\begin{aligned}&\delta=\tfrac{1}{2}\Gamma(\omega),\\&\zeta_0=q_z/(2m)^{1/2},\\&\epsilon=\omega+\mu-\omega_c(n+\tfrac{1}{2})-s\omega_0,\\&\epsilon_0=(\omega_q/2\zeta_0)^2,\\&N_1=\text{the index of the last cyclotron level for which }\epsilon>0.\end{aligned}\quad(3.10)$$

This result is quite different from that of Gurevich *et al.* because their intuitive way of introducing the mean-free-path effect is not justified. Equation (3.9) is a more general result than Skobov's because both the phonon energy and the spin splitting are included.

IV. DISCUSSION OF THE RESULT

We give a qualitative discussion of the result obtained in the last section. From Fig. 2 we see that δ is almost a constant except when ω is near a cyclotron level. The periodic variation is caused by the change in density of states of the electrons. For most values of ω we may still use the notions of relaxation time

$$\tau=1/\delta,\quad(4.1)$$

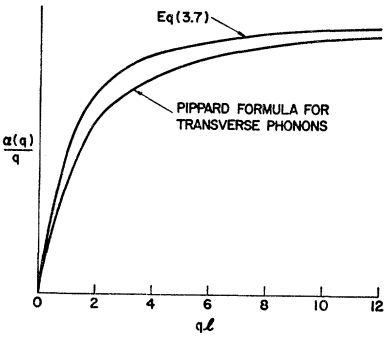

FIG. 5. Comparison of the acoustic attenuation in zero field obtained from the approximate method, Eq. (3.7), with the more rigorous result of Pippard for transverse phonons. The curves are matched at the very high-frequency limit.

and mean free path

$$l=v_F/\delta,\quad(4.2)$$

where $v_F=(2\mu/m)^{1/2}$ is the Fermi velocity. Furthermore, to a good approximation τ and l may be treated as constants. It can be seen from Eq. (3.9) that the acoustic attenuation depends on the product of three peaked functions, the thermal broadening factor $\mathrm{sech}^2(\frac{1}{2}\beta\omega)$ which is peaked at $\omega=0$, the density of states $(\epsilon)^{-1/2}$ which is peaked at $\epsilon=0$, and the resonance factor in the parenthesis which is peaked at $\epsilon=\epsilon_0$. The properties of the last two peaked functions, and hence the nature of the oscillations, depend on two dimensionless quantities $\omega_q\tau$ and q_zl. For bismuth

$$(\omega_q\tau/q_zl)=(v_s/v_F)\cong10^{-3}.$$

Hence, $q_zl\gg\omega_q\tau$ unless the direction of propagation is very close to being perpendicular to the external field. We discuss separately the three regions, $\omega_q\tau\gg1$, $\omega_q\tau\lesssim1$, but $q_zl\gg1$, and $q_zl\lesssim1$.

A. The Giant Oscillation Region

This region is characterized by $\omega_q\tau\gg1$, or equivalently $\epsilon_0\gg(\delta/\zeta_0)^2$. The density of states function is only peaked at $\epsilon=0$, so the above condition implies that the resonance function is sharper than the density-of-states function as long as the direction of phonon propagation is not nearly perpendicular to the external field. Since usually $\epsilon_0\ll\omega_c$, one can see that $\alpha(\mathbf{q})$ can be large if $\epsilon\cong\epsilon_0$ and $\omega\cong0$. This implies that the N_1th cyclotron level must lie below the Fermi level by the amount ϵ_0. When the dc field is varied so that the cyclotron level varies around this value, the variation of $\alpha(\mathbf{q})$ is extremely large due to the peaked nature of the resonance factor. The peak is due to the N_1th level only. It is easy to show that the condition $\epsilon=\epsilon_0$ is equivalent to demanding energy momentum conservation for the electron phonon interaction. When it is satisfied the z component of the electron velocity coincides with the z component of the sound velocity. This is the condition for giant oscillations as described by Gurevich *et al.* In

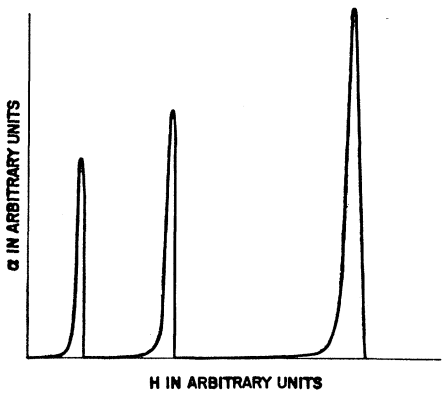

FIG. 6. A set of quantum oscillations in the intermediate region. The asymmetry in line shape is barely noticeable near the bottom of the peaks. The Fermi energy and the spin splitting are chosen to represent bismuth; the temperature is about 1.5°K.

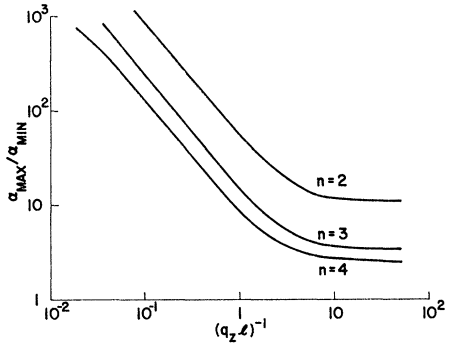

FIG. 7. The ratio $\alpha_{\max}/\alpha_{\min}$ plotted as a function of $q_z l$. The number n represents the ratio μ/ω_c. A peak occurs whenever this quantity has roughly an integral value. The quantity $\alpha_{\min}$ is defined as the trough on the high-field side of the peak.

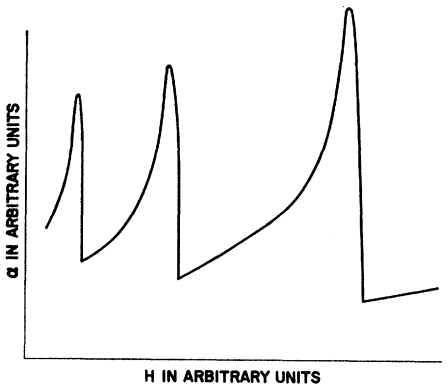

FIG. 8. A set of quantum oscillations in the de Haas–van Alphen region.

fact, if we let $\delta \to 0$, we recover the formula they obtained by perturbation theory. For finite δ the line shape is a thermally broadened Lorentzian function. The variation in $\alpha(\mathbf{q})$ is typically of the order of 10^3 or more at $\sim 1°$K. The giant oscillation is also a sensitive function of the direction of phonon propagation when it is nearly perpendicular to the field. In this case the width of the resonance becomes very large and the only variation in $\alpha(\mathbf{q})$ will be due to the density of states effect. The amplitude of oscillation is thereby reduced. This effect is somewhat analogous to the tilt effect discussed by Spector[11] and Eckstein.[12]

In practice the condition $\omega_q \tau \gg 1$ is extremely difficult to achieve. For phonons at the frequency ~ 10 Mc, we need $\tau \gg 2\times 10^{-8}$ sec. Taking the Fermi velocity to be $\sim 10^8$ cm/sec, we find that the mean free path should be of the order of a few centimeters, roughly the size of the sample.

B. Intermediate Oscillation Region

In this region $\omega_q \tau \lesssim 1$ but $q_z l \gg 1$. The resonance factor is no longer as sharply peaked as the density of states. We may put $u \cong u'$ in Eq. (3.14) and obtain the following formula for $\alpha(\mathbf{q})$:

$$\alpha(\mathbf{q}) \cong \frac{E_1^2 m^2 q^2 l}{\pi^2 \rho v_s} \frac{\beta\omega_c}{16\sqrt{\mu}} \sum_s \sum_{n=1}^{N_1} |J_{nn}|^2 \times \int d\omega \, \mathrm{sech}^2(\tfrac{1}{2}\beta\omega) \frac{\delta^2}{\sqrt{\epsilon[\epsilon\zeta_0^2+\delta^2]}}. \quad (4.3)$$

This formula is the final answer in Skobov's paper. Again N_1 is the last cyclotron level for which $\epsilon > 0$. The density of states become very large when ω is close to but above a cyclotron level, so $\alpha(\mathbf{q})$ is large when the N_1th level is near the Fermi energy. When the field varies, $\alpha(\mathbf{q})$ also oscillates. The quantity

$$(\delta/\zeta_0)^2 = \omega_c(4\mu/\omega_c)(1/q_z l)^2 \ll \omega_c,$$

because μ/ω_c is of the order unity. Thus, again, only one level contributes to the sum because the factor $\delta/[\epsilon\zeta_0^2+\delta^2]$ is small for $\epsilon \gg \omega_c$. This oscillation differs from the giant oscillation in that the peak is not due to the resonance factor but rather to the large variation of the density of states.

It is quite interesting to study the line shape in this region. We have seen that $\alpha(\mathbf{q})$ is large when a cyclotron level is near the Fermi level. If the field is decreased, $\alpha(\mathbf{q})$ drops smoothly as described by Eq. (4.3). However, if the field is increased so that this cyclotron level is raised above the Fermi level, the corresponding term no longer appears in the sum. As a result $\alpha(\mathbf{q})$ drops abruptly to a very low value within a small region of field corresponding to $\Delta\omega_c \sim kT$. Therefore, the line shape is asymmetrical, being sharper on the high-field side and more gradual on the low-field side. When the temperature is sufficiently low the line shape on the low-field side reflects the variation in density of states while on the high-field side it reflects the thermal broadening. For this reason the line shape contains valuable information about the band structure of the material. At higher temperatures the large thermal smearing may render the asymmetry less noticeable.

The total variation of $\alpha(\mathbf{q})$ is still quite large in this region. Typical ratio of $\alpha_{\max}$ to $\alpha_{\min}$ is of the order of $10 \sim 1000$ depending on the mean free path and the temperature. A set of such oscillations is displayed in Fig. 6. One can still get a tilt effect by varying the angle between the magnetic field and the phonon propagation. As the angle approaches 90°, ζ_0 becomes very small. Equation (4.3) shows that all the levels with $n < N_1$ contribute to the sum. This increases the minimum value of $\alpha(\mathbf{q})$ and reduces the net variation $\alpha_{\max} - \alpha_{\min}$. In Fig. 7 the dependence of the ratio $\alpha_{\max}/\alpha_{\min}$ is plotted as a function of $q_z l$.

C. de Haas–van Alphen Region

This is the region where $q_z l \lesssim 1$ or $(\delta/\zeta_0)^2 \lesssim \omega_c$. This may either be due to short mean free path or small q_z when q is nearly perpendicular to the field direction. All cyclotron levels contribute to the sum in Eq. (4.3), but

[11] H. N. Spector, Phys. Rev. **120**, 1261 (1960).
[12] S. G. Eckstein, Phys. Rev. **131**, 1087 (1963).

the level closest and below the Fermi energy gives rise to the peak. The oscillation is no longer large, $\alpha_{\max}/\alpha_{\min}\sim 1$. The asymmetry in line shape is more pronounced. Once in this region, further reduction in $q_z l$ has no effect on $\alpha(\mathbf{q})$. Thus, if the mean free path is short, varying the angle between the field and the phonon propagation will not affect the absorption. A set of oscillations of this kind is depicted in Fig. 8.

V. ACKNOWLEDGMENTS

The authors wish to thank Dr. P. B. Miller, Dr. D. C. Mattis, Dr. T. Tsuneto, and Professor L. P. Kadanoff for valuable discussions.

APPENDIX

Evaluation of the Integrals in Eq. (3.8)

We give here in detail the steps leading from Eq. (3.8) to Eq. (3.9). Inserting the electron-phonon matrix element Eq. (3.6) into Eq. (3.8), we find

$$\Gamma_p(\mathbf{q})=\frac{E_1^2q^2}{2\rho\omega_q L^3}\sum_s\sum_{nn'}|J_{n'n}|^2\int\frac{d\omega}{2\pi}\{\sum_{k_yk_y'}\delta_{k_y',k_y+q_y}\}\times\{\sum_{k_zk_z'}\delta_{k_z',k_z+q_z}A(\alpha s;\omega)A(\alpha' s;\omega-\omega_q)\}\times\{n(\omega-\omega_q)-n(\omega)\}. \quad (A1)$$

From Eq. (2.31) we see that $A(\alpha s;\omega)$ does not depend on k_y, so the summation over k_y, k_y' is easy to carry out:

$$\sum_{k_yk_y'}\delta_{k_y',k_y+q_y}=m\omega_c L^2/2\pi.$$

The limits on k_y are such that the center of the cyclotron orbit should be contained in L. For the summation over k_z and k_z', it is convenient to define the following shorthand symbols:

$$\begin{aligned}&\zeta=k_z/(2m)^{1/2},\quad \zeta'=k_z'/(2m)^{1/2},\quad \zeta_0=q_z/(2m)^{1/2},\\&\epsilon=\omega+\mu-\omega_c(n+\tfrac{1}{2})-s\omega_0,\\&\epsilon'=\omega-\omega_q+\mu-\omega_c(n'+\tfrac{1}{2})-s\omega_0,\\&\delta=\tfrac{1}{2}\Gamma(\omega),\quad \delta'=\tfrac{1}{2}\Gamma(\omega-\omega_q).\end{aligned} \quad (A2)$$

Then we can write

$$\sum_{k_z,k_z'}\delta_{k_z',k_z+q_z}A(\alpha s;\omega)A(\alpha' s;\omega-\omega_q)=\frac{(2mL)^{1/2}}{(2\pi)^2}\int_{-\infty}^{\infty}e^{i(\xi'-\xi-\zeta_0)\xi}d\xi\times\int\frac{2\delta d\zeta}{[\zeta^2-\epsilon]^2+\delta^2}\int\frac{2\delta' d\zeta'}{[\zeta'^2-\epsilon']^2+\delta'^2} \quad (A3)$$

after a simple transformation. The ζ, ζ' integrations are done by the contour method. We define

$$\begin{aligned}&\tan\theta=\delta/\epsilon,\\&\tan\theta=\delta'/\epsilon',\quad 0\le\theta,\quad \theta'\le\pi.\end{aligned} \quad (A4)$$

Then the poles of the integrands are at

$$\pm(\epsilon\pm i\delta)^{1/2}=\pm(u\pm iv),$$

and

$$\pm(\epsilon'\pm i\delta')^{1/2}=\pm(u'+iv'),$$

where

$$u=(\epsilon^2+\delta^2)^{1/4}\cos\tfrac{1}{2}\theta,\quad v=(\epsilon^2+\delta^2)^{1/4}\sin\tfrac{1}{2}\theta, \quad (A5)$$

and similar expressions for u', v'. The choice of θ, θ' makes u, v, u', $v'\ge 0$. We close the contours of the ζ, ζ' integrations in such a way that the final integration over ξ should converge. For instance, for $\xi>0$, we must close the ζ contour by an infinite semicircle below the real axis, and the ζ' contour above the real axis. For $\xi<0$, the reversed choice is made. The contour integrals and the ξ integral are all easy to do, and the result for the quantity in Eq. (A3) is

$$\begin{aligned}&(2m)^{1/2}L/2\,\mathrm{Re}\{((u+iv)(u'+iv')[v+v'-i(u+u'-\zeta_0)])^{-1}\\&+((u+iv)(u'-iv')[v+v'+i(u'-u+\zeta_0)])^{-1}\\&+((u-iv)(u'+iv')[v+v'-i(u'-u-\zeta_0)])^{-1}\\&+((u-iv)(u'-iv')[v+v'+i(u+u'+\zeta_0)])^{-1}\}.\end{aligned} \quad (A6)$$

Finally, the thermal factor, the third bracket in Eq. (A1), may be written as

$$n(\omega-\omega_q)-n(\omega)\cong\tfrac{1}{4}\beta\omega_q\,\mathrm{sech}^2(\tfrac{1}{2}\beta\omega). \quad (A7)$$

Thus the final expression for the attenuation constant contains sums over n and n' and an integration over ω.

The result we obtain so far is still highly complex. To facilitate further discussion, we must make a number of approximations. Since the phonon frequency is usually extremely small compared with the cyclotron frequency, transitions from one cyclotron level to another are forbidden by energy conservation. This eliminates all terms with $n\neq n'$ from the sum. The first and last terms in the parentheses of Eq. (A6) are not of the Lorentzian form because $u+u'$ appears. Consequently they are not important because they never resonate. For the other terms we observe that the quantity ϵ may be positive or negative depending on the value of the index n. As long as $|\epsilon|\gg\delta$ and ω_q,

$$\begin{aligned}&u+iv=\sqrt{\epsilon}+i(\delta/2\sqrt{\epsilon}),\\&u'+iv'\cong\sqrt{\epsilon}-(\omega_q/2\sqrt{\epsilon})+i(\delta/2\sqrt{\epsilon}),\end{aligned} \quad (A8)$$

for $\epsilon>0$, and

$$u+iv=u'+iv'=i\sqrt{\epsilon}, \quad (A9)$$

for $\epsilon<0$. It is clear that if δ is small enough ($\delta\ll\omega_c$) the contribution of those levels with $\epsilon<0$ are highly damped compared with those with $\epsilon>0$. This allows us to carry the sum up to N_1 where N_1 designates the last level for which $\epsilon>0$. With these simplifications it is easy to show that the acoustic-attenuation constant reduces to the expression in Eq. (3.9).

The approximations in Eqs. (A8) and (A9) are only valid for $|\epsilon|\gg\delta$ but the ω integration requires that all ϵ values must be considered. This is not a serious objection to the approximation procedure because the integral about $\epsilon=0$ is convergent. Hence, we make only a small error in replacing $\epsilon+i\delta$ by ϵ even for $|\epsilon|<\delta$.