Nonlinear Dynamics of Longitudinal Oscillations in a Homogeneous Collisionless Plasma with an External Magnetic Field*

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The effect of wave-wave scattering of longitudinal oscillations in a homogeneous collisionless plasma with an external magnetic field is derived. The nonlinear dynamics of the modes existing near multiples of the electron cyclotron frequency are discussed, and it is shown that resonant wave-wave scattering of unstable low harmonic modes is an important mechanism for transferring energy to the higher harmonics.

I. INTRODUCTION

In the linearized theory of collisionless plasma dynamics, a large number of collective oscillations are known to exist which have the property that initially small disturbances grow exponentially in time. However, as these unstable excitations increase in amplitude, the nonlinear terms in the equations of motion become important in determining the dynamics and ultimate state of the system. A large class of these linearly unstable excitations, called micro-instabilities, have the property that nonlinear effects limit the amplitudes so that the total perturbed energy remains small compared to the unperturbed energy, and the state of the excitation remains well defined. In homogeneous field free systems, for example, these characteristics for high-frequency oscillations are equivalent to the requirements that

$$\frac{1}{8\pi}\int d\mathbf{x}\frac{\mathbf{E}^2+\mathbf{B}^2}{nKT}\ll 1\,,$$

and $\omega_k \gg \bar{\gamma}_k$, where $\mathbf{E}$ and $\mathbf{B}$ are the perturbed electromagnetic fields, nKT the total kinetic energy of the system, and $\omega_\mathbf{k}$ and $\gamma_\mathbf{k}$ are the characteristic frequency and growth rate of the excitations with wave vector $\mathbf{k}$. Because of the small amount of energy available to such modes, their excitation cannot produce a gross distortion of the plasma. But, phenomena such as turbulence, enhanced resistivity, and anomalous diffusion of particles across magnetic field lines are to be anticipated.

The determination of the turbulent state of the system brought about by the excitation of linearly unstable modes requires analysis of the nonlinear terms in the equations of motion. For modes in the micro-instability class the dominant nonlinear effects can be analyzed into the "elementary processes" of wave-induced particle diffusion in phase space and wave-wave scattering.

In this paper the effect of wave-wave scattering of longitudinal oscillations in a homogeneous collisionless plasma with an external magnetic field is analyzed. Many of these modes have instabilities of the "velocity-space" type which have linearized growth rates that depend on detail characteristic of the electron and ion distribution functions. The amplitudes of these modes are then limited by wave-induced particle diffusion in velocity space, but do attain amplitudes so that three and four wave-wave scattering processes are as important in the dynamics as the linear processes. Particular emphasis is placed on describing the wave-wave scattering of modes which exist at multiples of the electron cyclotron frequency. It will be shown that because these modes satisfy the three-wave resonance condition

$$\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}=\omega_{\mathbf{k}_3}\,,\qquad \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3\,, \tag{1.1}$$

wave-wave scattering is an important mechanism for transferring energy between the various harmonics.

II. THE NONLINEAR EQUATIONS

The time evolution of the system's dynamics is governed by the Maxwell-Vlasov equations,

$$\left(\frac{\partial}{\partial t}-i\mathbf{k}\cdot\mathbf{v}+\frac{e}{m}\left(\frac{\mathbf{v}\times\mathbf{B}_0}{c}\right)\cdot\frac{\partial}{\partial\mathbf{v}}\right)f_\mathbf{k}+\frac{e}{m}E_\mathbf{k}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}} =-\frac{e}{m}\sum_{\mathbf{k}'\neq 0}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial f_{\mathbf{k}'}}{\partial\mathbf{v}} \tag{2.1a}$$

and

$$E_\mathbf{k}=\frac{4\pi n}{-ik}\int ef_\mathbf{k}\,, \tag{2.1b}$$

where

$$\int\{\quad\}\equiv\sum_{\text{species}}\int d\mathbf{v}\{\quad\}.$$

In Eq. (2.1) all quantities have been expanded in Fourier series, e.g., $\mathbf{E}(x,t)=\sum_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{E}_\mathbf{k}(t)$, and transverse wave effects have been neglected. $f_\mathbf{k}$ is either the ion or the electron distribution function and superscripts denoting species have been dropped, but are implicit. The function, $g(\mathbf{v},t)=f_{\mathbf{k}=0}(\mathbf{v},t)$, is the usual background velocity distribution function which is assumed to be

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slowly varying in time and determines the usual quasi-linear dispersion relation.[1–3]

For the high-frequency modes to be considered, the linear coupling of transverse to longitudinal waves is proportional to

$$\beta=\frac{nKT}{B_0^2/8\pi}\ll 1,$$

and hence is negligible. However, the transverse waves can couple to longitudinal waves nonlinearly, and if the three wave resonance condition (1.1) is satisfied, provide an important source or sink of energy for the longitudinal modes. For simplicity these effects will be ignored, but the generalization necessary to include them is obvious.

Introducing the Green's function $G_{\mathbf{k}}$, defined for $t-t'>0$, and which satisfies the equation

$$\left\{\frac{\partial}{\partial t}-i\mathbf{k}\cdot\mathbf{v}+\frac{e}{m}\left(\frac{\mathbf{v}\times\mathbf{B}_0}{c}\right)\cdot\frac{\partial}{\partial\mathbf{v}}\right\}G_{\mathbf{k}}=\delta(t-t')\delta(\mathbf{v}-\mathbf{v}'),\quad(2.2)$$

Eq. (2.1) can be written

$$[G_{\mathbf{k}}]^{-1}f_{\mathbf{k}}=-\frac{e}{m}E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}-\frac{e}{m}\sum_{\mathbf{k}'\neq 0}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial f_{\mathbf{k}'}}{\partial\mathbf{v}}.\quad(2.3)$$

Neglecting terms proportional to initial values of $f_{\mathbf{k}}$, which if nicely behaved phase-mix away for long times in the equation for $E_{\mathbf{k}}(t)$ and $g(\mathbf{v},t)$, Eq. (2.3) is simply

$$f_{\mathbf{k}}(\mathbf{v},t)=-\frac{e}{m}G_{\mathbf{k}}E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}-\frac{e}{m}\sum_{\mathbf{k}'\neq 0}G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial f_{\mathbf{k}'}}{\partial\mathbf{v}}.\quad(2.4)$$

$G_{\mathbf{k}}$ is now the operator

$$G_{\mathbf{k}}\{\;\}=\int_0^\infty dt'\int d\mathbf{v}'G_{\mathbf{k}}(\mathbf{v},\mathbf{v}';t,t')\{\;\}\quad(2.5)$$

and operates on all terms to the right of it. Iterating the Eq. (2.4) for $f_{\mathbf{k}}$ in powers of $E_{\mathbf{k}}$, up to order $E_{\mathbf{k}}^4$, one finds

$$f_{\mathbf{k}}(\mathbf{v},t)=-\frac{e}{m}G_kE_k\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}-\frac{e}{m}\sum_{\mathbf{k}'\neq 0}G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}\left\{-\frac{e}{m}G_{\mathbf{k}'}E_{\mathbf{k}'}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial g}{\partial\mathbf{v}}\right.$$
$$\left.+\left(\frac{e}{m}\right)^2\sum_{\mathbf{k}''}G_{\mathbf{k}'}E_{\mathbf{k}'-\mathbf{k}''}\frac{(\mathbf{k}'-\mathbf{k}'')}{|\mathbf{k}'-\mathbf{k}''|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}''}E_{\mathbf{k}''}\frac{\mathbf{k}''}{k''}\cdot\frac{\partial g}{\partial\mathbf{v}}\right\}.\quad(2.6)$$

Using Poisson's equation, the equation for $E_{\mathbf{k}}(t)$ is

$$E_{\mathbf{k}}(t)-\frac{1}{ik}\int\omega_p^2G_{\mathbf{k}}E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}=-\sum_{\mathbf{k}'}\int\frac{\omega_p^2e}{ikm}G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}E_{\mathbf{k}'}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial g}{\partial\mathbf{v}}$$
$$+\sum_{\mathbf{k}',\mathbf{k}''}\int\frac{\omega_p^2}{ik}\left(\frac{e}{m}\right)^2G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}E_{\mathbf{k}'-\mathbf{k}''}\frac{\mathbf{k}'-\mathbf{k}''}{|\mathbf{k}'-\mathbf{k}''|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}''}E_{\mathbf{k}''}\frac{\mathbf{k}''}{k''}\cdot\frac{\partial g}{\partial\mathbf{v}}.\quad(2.7)$$

The last term on the right-hand side of Eq. (2.7) is simplified when use is made of the condition that initially, the phases of the waves are random. To the order that is being considered here, then only the terms $\mathbf{k}''=\mathbf{k}$, and $\mathbf{k}''=\mathbf{k}'-\mathbf{k}$ contribute to the dynamics. Therefore, Eq. (2.7) becomes

$$\epsilon(\mathbf{k})E_{\mathbf{k}}(t)=-\sum_{\mathbf{k}'}\int\frac{\omega_p^2}{ik}\left(\frac{e}{m}\right)G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}E_{\mathbf{k}'}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial g}{\partial\mathbf{v}}$$
$$-\sum_{\mathbf{k}'}\int\frac{\omega_p^2}{ik}\left(\frac{e}{m}\right)^2G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}\left\{E_{\mathbf{k}-\mathbf{k}'}{}^*\frac{\mathbf{k}'-\mathbf{k}}{|\mathbf{k}'-\mathbf{k}|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}}E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}\right.$$
$$\left.+E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'-\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}{}^*\frac{(\mathbf{k}'-\mathbf{k})}{|\mathbf{k}'-\mathbf{k}|}\cdot\frac{\partial g}{\partial\mathbf{v}}\right\},\quad(2.8)$$

[1] W. E. Drummond and D. Pines, Suppl. Nucl. Fusion Part 3, 1049 (1962).
[2] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Suppl. Nucl. Fusion Part 2, 465 (1962).
[3] W. E. Drummond and M. N. Rosenbluth, Phys. Fluids 5, 1507 (1962).

where the operator $\epsilon(\mathbf{k})$ is defined as

$$\epsilon(\mathbf{k})=1-\frac{1}{ik}\int\omega_p{}^2G_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}\{\quad\}. \tag{2.9}$$

Neglecting the nonlinear terms in Eq. (2.8) and assuming

$$E_{\mathbf{k}}(t)\simeq\exp\left(i\int_0^t\omega_k(t')dt'\right), \tag{2.10}$$

where ω_k satisfies $(1/\omega_{\mathbf{k}}{}^2)(d\omega_{\mathbf{k}}/dt)\ll1$, Eq. (2.8) reduces to

$$\epsilon(k,\omega_k)E_k(t)=0, \tag{2.11}$$

where $\epsilon(k,\omega_k)=0$ is the usual quasilinear dispersion relation. Making use of the assumption that the linearized modes, whose frequencies are defined by $\epsilon(k,\omega_k)=0$, remain well defined, the sum on $\mathbf{k}'$ in the first term on the right-hand side of Eq. (2.8) is separated into two regions. The resonant region $R^1(\mathbf{k},\mathbf{k}')$ is defined by

$$\omega_{\mathbf{k}'}{}^0+\omega_{\mathbf{k}-\mathbf{k}'}{}^0\simeq\omega_{\mathbf{k}}{}^0\quad\omega_{\mathbf{k}}{}^0=\mathrm{Re}\,\omega_{\mathbf{k}}$$

and $R^2(\mathbf{k},\mathbf{k}')$ the remainder of $\mathbf{k}'$ space. Using the assumption (2.10), the fast time dependence of the first nonlinear term on the right-hand side of Eq. (2.8) is seen to be $\sim e^{i\omega_k t}$ for $\mathbf{k}'$ in $R^1(\mathbf{k},\mathbf{k}')$ and all possible beat frequencies of $\omega_{\mathbf{k}}$ for $\mathbf{k}'$ in R^2. Fields of the latter type are nonsecular and denoted by $E_{\mathbf{k}}{}^{(2)}$, the former by $E_{\mathbf{k}}{}^{(1)}$. To the order of relevance,

$$E_{\mathbf{k}}{}^{(2)}(t)=-\sum_{k'\in R^2}\int\frac{\omega_p{}^2}{ik}\left(\frac{e}{m}\right)\frac{1}{\epsilon(\mathbf{k})}G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}{}^{(1)}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}E_{\mathbf{k}'}{}^{(1)}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial g}{\partial\mathbf{v}}. \tag{2.12}$$

$E_{\mathbf{k}}{}^{(2)}$ then enters the dynamics for the main part of $E_{\mathbf{k}}$, $E_{\mathbf{k}}{}^{(1)}$, only through the four-wave term obtained by letting $E_{\mathbf{k}'}=E_{\mathbf{k}'}{}^{(2)}$, and $E_{\mathbf{k}-\mathbf{k}'}=E_{\mathbf{k}-\mathbf{k}'}{}^{(2)}$ in the first nonlinear term in Eq. (2.8). In terms of $E_{\mathbf{k}}{}^{(1)}$ only [dropping the superscript (1), all fields now have the fast time dependence of $e^{i\omega_k t}$] the equation for the modes becomes

$$\begin{aligned}
\epsilon(\mathbf{k})E_{\mathbf{k}}(t)=&-\sum_{k'\in R'}\int\frac{\omega_p{}^2}{ik}\left(\frac{e}{m}\right)G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}E_{\mathbf{k}'}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial g}{\partial\mathbf{v}}\\
&-\sum_{\mathbf{k}'}\int\frac{\omega_p{}^2}{ik}\left(\frac{e}{m}\right)^2G_{\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}\left\{E_{\mathbf{k}-\mathbf{k}'}{}^*\frac{(\mathbf{k}'-\mathbf{k})}{|\mathbf{k}'-\mathbf{k}|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}}E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}\right.\\
&\left.+E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'-\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}{}^*\frac{(\mathbf{k}'-\mathbf{k})}{|\mathbf{k}'-\mathbf{k}|}\cdot\frac{\partial g}{\partial\mathbf{v}}\right\}+\sum_{\mathbf{k}'}\int\frac{\omega_p{}^2}{ik}\left(\frac{e}{m}\right)G_{\mathbf{k}}\\
&\times\left\{\left[\frac{\omega_p{}^2}{i|\mathbf{k}-\mathbf{k}'|}\frac{e}{m}\frac{1}{\epsilon(\mathbf{k}-\mathbf{k}')}G_{\mathbf{k}-\mathbf{k}'}\left(E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}G_{-\mathbf{k}'}E_{\mathbf{k}'}{}^*\frac{\mathbf{k}'}{k'}\cdot\frac{\partial g}{\partial\mathbf{v}}\right.\right.\right.\\
&\left.\left.+E_{\mathbf{k}'}{}^*\frac{\mathbf{k}'}{k'}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}}E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}\right)\right]\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}E_{\mathbf{k}'}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial g}{\partial\mathbf{v}}\\
&+E_{\mathbf{k}-\mathbf{k}'}\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'}\frac{\mathbf{k}'}{k'}\cdot\frac{\partial g}{\partial\mathbf{v}}\left[\int\frac{\omega_p{}^2}{ik'}\left(\frac{e}{m}\right)\frac{1}{\epsilon(\mathbf{k}')}G_{\mathbf{k}'}\left(E_{\mathbf{k}-\mathbf{k}'}{}^*\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}\right.\right.\\
&\left.\left.\left.\times G_{\mathbf{k}}E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial g}{\partial\mathbf{v}}+E_{\mathbf{k}}\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial\mathbf{v}}G_{\mathbf{k}'-\mathbf{k}}E_{\mathbf{k}-\mathbf{k}'}{}^*\frac{(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}g\right)\right]\right\}.
\end{aligned} \tag{2.13}$$

For any particular wavelength in this system, many modes may exist. To conveniently account for this degeneracy, the notation is generalized so that "$\mathbf{k}$" now refers to both the wave vector and a particular mode at that wave vector. Again, using the assumption that the linearized modes remain well defined, or explicitly,

$$E_{\mathbf{k}}(t)=\mathcal{E}_{\mathbf{k}}(t)\exp\left[i\int_0^t\omega_{\mathbf{k}}(t')dt'\right], \tag{2.14}$$

where $\omega_{\mathbf{k}}(t)$ is defined by (2.11) and

$$(1/\mathcal{E}_k\omega_k)(\partial\mathcal{E}_k/\partial t)\ll1, \tag{2.15}$$

the time integrals in Eq. (2.13) can be done. Typically, the time integrals are of the form

$$I=\int_0^{t_1}f(t_1,\tau)d\tau E_{\mathbf{k}}(t-\tau). \tag{2.16a}$$

With the assumption (2.15)

$$E_k(t-\tau)\simeq e^{-i\omega_k\tau}\left[E_k(t)-\tau\left(\frac{\partial E_k}{\partial t}-i\omega_k E_k\right)\right], \quad (2.16b)$$

and hence

$$I=E_k(t)\int_0^{t_1} f(t_1,\tau)d\tau e^{-i\omega_k\tau}+\left[\frac{\partial E_k(t)}{\partial t}-i\omega_k E_k(t)\right] \times\frac{\partial}{\partial i\omega_k}\int_0^{t_1} f(t_1,\tau)d\tau e^{-i\omega_k\tau}. \quad (2.16c)$$

Using this simplification the left-hand side of Eq. (2.13) becomes

$$\frac{\partial\epsilon(k,\omega_k)}{\partial i\omega_k}\left[\frac{\partial E_k}{\partial t}-i\omega_k E_k\right]. \quad (2.17)$$

As $(\partial E_k/\partial t)-i\omega_k E_k$ is of order E_k^2, this may be neglected in all terms but the first when the time integrals are completed on the right-hand side of (2.13). In the first term this expression occurs and is evaluated in lowest approximation. This gives a term of $O(E_k^3)$ to be included. One finally arrives at the nonlinear equation, correct up to order E_k^4

$$\frac{\partial E_k}{\partial t}=i\omega_k E_k+\sum_{k'\in R_1} M_{k,k'}E_{k'}(t)E_{k-k'}(t) +\sum_{k'} L_{k,k'}|E_{k'}(t)|^2E_k(t). \quad (2.18)$$

Using the Green's function appropriate to the system in cylindrical coordinates, $\mathbf{v}=(v_\parallel,v_\perp,\theta)$, $\mathbf{k}=(k_\parallel,k_\perp,\varphi)$, and Ω is the cyclotron frequency for the species

$$G_k(\mathbf{v},\mathbf{v}';t,t')=\frac{e^{ik_\parallel v_\parallel'\tau}}{v_\perp'}e^{i(k_\perp v_\perp'/\Omega)[\sin(\theta'-\varphi)-\sin(\theta'-\Omega\tau-\varphi)]}\delta(v_\parallel-v_\parallel')\delta(v_\perp-v_\perp')\delta(\theta-\theta'+\Omega\tau), \quad \text{for} \quad \tau=t-t'>0, \quad (2.19)$$

the velocity integrals may be completed and explicit expressions found for $\epsilon(\mathbf{k},\omega_k)$, $M_{\mathbf{k},\mathbf{k}'}$, and $L_{\mathbf{k},\mathbf{k}'}$. For $\epsilon(\mathbf{k},\omega_k)$, one obtains the longitudinal dielectric function

$$\epsilon(\mathbf{k},\omega_k)=1-\frac{1}{ik^2}\sum_{\text{species}}\omega_p^2\int d\mathbf{v}\mathbf{k}\cdot\frac{\partial g}{\partial\mathbf{v}}\int_0^\infty d\tau e^{i(k_\parallel v_\parallel-\omega_k)\tau}e^{i(k_\perp v_\perp/\Omega)[\sin(\theta-\varphi)-\sin(\theta-\Omega\tau-\varphi)]}, \quad (2.20a)$$

or

$$\epsilon(\mathbf{k},\omega_k)=1-\frac{1}{k^2}\sum_{\substack{\text{species}\\ -\infty\leq n,n_1\leq\infty}}\omega_p^2\int d\mathbf{v}\mathbf{k}\cdot\frac{\partial g}{\partial\mathbf{v}}\frac{J_n(k_\perp v_\perp/\Omega)J_{n_1}(k_\perp v_\perp/\Omega)}{(k_\parallel v_\parallel-\omega_k+n_1\Omega)}e^{i(n-n_1)(\theta-\varphi)}. \quad (2.20b)$$

In the absence of waves, g is just the equilibrium distribution function which must be of the form, $g=g(v_\parallel,v_\perp)$. Neglecting the small anisotropy generated by the fluctuations, the usual dielectric function

$$\epsilon(\mathbf{k},\omega_k)=1-\sum_{\substack{\text{species}\\ -\infty\leq n\leq\infty}}\frac{\omega_p^2}{k^2}\int\frac{d\mathbf{v}J_n^2(k_\perp v_\perp/\Omega)}{(k_\parallel v_\parallel-\omega_k+n\Omega)}\left[k_\parallel\frac{\partial g}{\partial v_\parallel}+\frac{n\Omega}{v_\perp}\frac{\partial g}{\partial v_\perp}\right] \quad (2.20c)$$

is obtained.

Similarly, one obtains for $M_{\mathbf{k},\mathbf{k}'}$:

$$M_{\mathbf{k},\mathbf{k}'}=-\frac{1}{\partial\epsilon(\mathbf{k},\omega_k)/\partial\omega_k}\sum_{\text{species}}\frac{\omega_p^2}{k}\left(\frac{e}{m}\right)\sum_{n_1,n}\int\frac{d\mathbf{v}J_{n_1}(k_\perp' v_\perp/\Omega)e^{in_1(\theta-\varphi')}}{(k_\parallel' v_\parallel-\omega_{k'}+n_1\Omega)}e^{-ik_\perp' v_\perp/\Omega\sin(\theta-\varphi')}\left[\frac{k_\parallel'}{k'}\frac{\partial g}{\partial v_\parallel}+\frac{n_1\Omega}{k' v_\perp}\frac{\partial g}{\partial v_\perp}\right] \times\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|}\cdot\frac{\partial}{\partial\mathbf{v}}\left[\frac{e^{-in(\theta-\varphi)}J_n(k_\perp v_\perp/\Omega)}{(k_\parallel v_\parallel-\omega_{k-k'}-\omega_{k'}+n\Omega)}e^{ik_\perp v_\perp/\Omega\sin(\theta-\varphi)}\right], \quad (2.21a)$$

or

$$M_{\mathbf{k},\mathbf{k}'}=\frac{1}{\partial\epsilon(\mathbf{k},\omega_k)/\partial\omega_k}\sum_{\text{species}}\frac{\omega_p^2}{k}\left(\frac{e}{m}\right)\frac{1}{k'|\mathbf{k}-\mathbf{k}'|}\sum_{n,n_1,n_2}\int\frac{d\mathbf{v}J_{n_1}(k_\perp' v_\perp/\Omega)J_{n_2}(k_\perp' v_\perp/\Omega)}{(k_\parallel' v_\parallel-\omega_{k'}+n_1\Omega)}$$

$$\times J_n(k_\perp v_\perp/\Omega)J_{n+n_2-n_1}(k_\perp v_\perp/\Omega)e^{-i(n_2-n_1)(\varphi-\varphi')}\left[k_\parallel'\frac{\partial g}{\partial v_\parallel}+\frac{n_1\Omega}{v_\perp}\frac{\partial g}{\partial v_\perp}\right]$$

$$\times\left\{\frac{k_\parallel(k_\parallel-k_\parallel')}{(k_\parallel v_\parallel-\omega_k+n\Omega)^2}+\frac{k_\perp^2}{[k_\parallel v_\parallel-\omega_k+(n+1)\Omega][k_\parallel v_\parallel-\omega_k+(n-1)\Omega]}\right.$$

$$\left.-\frac{k_\perp k_\perp'}{2}\left[\frac{e^{i(\varphi-\varphi')}}{[k_\parallel v_\parallel-\omega_k+n\Omega][k_\parallel v_\parallel-\omega_k+(n+1)\Omega]}+\frac{e^{-i(\varphi-\varphi')}}{[k_\parallel v_\parallel-\omega_k+n\Omega][k_\parallel v_\parallel-\omega_k+(n-1)\Omega]}\right]\right\}. \quad (2.21b)$$

The coefficient, $L_{\mathbf{k},\mathbf{k}'}$, in Eq. (2.18) can also be reduced to velocity integral form, but except in the cases of zero[4] or "infinite"[5] magnetic field is extremely complicated. For example, it is evident from Eq. (2.13) that $L_{\mathbf{k},\mathbf{k}'}$ involves velocity integrals of a product of six Bessel functions and as such makes even zero-temperature approximations extremely tedious. If one is interested in the energy being scattered into a particular mode which has a low amplitude so that the mode of interest, $E_{\mathbf{k}}$ satisfies $|E_{\mathbf{k}}|^2 \ll |E_{\mathbf{k}-\mathbf{k}'}|^2 \simeq |E_{\mathbf{k}'}|^2$, for $\mathbf{k}'$ in $R^1(\mathbf{k},\mathbf{k}')$, then only $M_{\mathbf{k},\mathbf{k}'}$ has to be calculated.

The main quantities of interest for characterizing the turbulent state of the plasma system are the squares of the amplitudes of the excited modes, $|E_{\mathbf{k}}(t)|^2$. A closed equation for the amplitudes can be obtained from Eq. (2.18) if use is made of our lack of knowledge of initial conditions. In particular, it is assumed that the initial conditions are just those encountered by the excitation of the modes by thermal noise, that is, the modes have phases which are initially completely random. One then generates an infinite chain of coupled equations by multiplying Eq. (2.18) by all powers of $E(t)$ and averaging the equations over the initial phases. For example,

$$\frac{\partial |E_{\mathbf{k}}(t)|^2}{\partial t} = 2\gamma_k |E_{\mathbf{k}}|^2 + \{\sum_{\mathbf{k}'} M_{\mathbf{k},\mathbf{k}'} \langle E_{\mathbf{k}}^* E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle + \sum_{\mathbf{k}'} L_{\mathbf{k},\mathbf{k}'} |E_{\mathbf{k}'}|^2 |E_{\mathbf{k}}|^2 + \text{complex conjugate}\} \tag{2.22a}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\langle E_{\mathbf{k}}^* E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle = i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}-\mathbf{k}'} - \omega_{\mathbf{k}}^*)\langle E_{\mathbf{k}}^* E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle + \sum_{\mathbf{k}_1} M_{\mathbf{k},\mathbf{k}_1}^* \langle E_{\mathbf{k}_1}^* E_{\mathbf{k}-\mathbf{k}_1}^* E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle \\ + \sum_{\mathbf{k}_2} M_{\mathbf{k}',\mathbf{k}_2} \langle E_{\mathbf{k}}^* E_{\mathbf{k}'-\mathbf{k}_2} E_{\mathbf{k}_2} E_{\mathbf{k}-\mathbf{k}'}\rangle + \sum_{\mathbf{k}_3} M_{\mathbf{k}-\mathbf{k}',\mathbf{k}_3} \langle E_{\mathbf{k}}^* E_{\mathbf{k}'} E_{\mathbf{k}_3} E_{\mathbf{k}-\mathbf{k}'-\mathbf{k}_3}\rangle + O(E_{\mathbf{k}}^6),\end{aligned} \tag{2.22b}$$

where in Eq. (2.22) the brackets $\langle\ \rangle$ denote an average over initial phases and have been dropped from $|E_{\mathbf{k}}|^2$ for convenience. The chain of equations continues of course, but if use is made of the assumption that the linearized modes remain well-defined Eqs. (2.22a) and (2.22b) actually close to the approximation considered.

In the absence of nonlinearities Eq. (2.18) shows that the phase of $E_{\mathbf{k}}$ remains constant in time. This equation of course neglects the small effects due to details of the initial particle distribution function. Hence, without nonlinearities and with random initial phases the phase average of all odd products of $E(t)$ vanish, and only those terms in the phase average of even products remain which have identically cancelling phases. In the presence of the nonlinearities the phase of the wave changes, but slowly compared to $\omega_{\mathbf{k}}$. The Eqs. (2.22) are complete when the average four product is expressed in terms of the $|E_{\mathbf{k}}|^2$. For example,

$$\langle E_{\mathbf{k}_1}^* E_{\mathbf{k}-\mathbf{k}_1} E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle = (\delta_{\mathbf{k}',\mathbf{k}_1} + \delta_{\mathbf{k}',\mathbf{k}-\mathbf{k}_1}) |E_{\mathbf{k}'}|^2 |E_{\mathbf{k}-\mathbf{k}'}|^2 + O(E^6). \tag{2.23}$$

This decomposition is suggested by the linearized theory and in the presence of nonlinearities can be shown to be correct self-consistently. The self-consistency of (2.23) is found by deriving the equation for $\langle E_{\mathbf{k}_1}^* E_{\mathbf{k}-\mathbf{k}_1} E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle$ which is coupled to $\langle E_{\mathbf{k}_1}^* E_{\mathbf{k}_1-\mathbf{k}_2}^* E_{\mathbf{k}-\mathbf{k}_1} E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle$. The average five product is then expressed in terms of products of two and three products, as would be the case in the linearized theory, and Eq. (2.23) is shown to be correct. Using Eq. (2.23), Eq. (2.22b) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\langle E_{\mathbf{k}}^* E_{\mathbf{k}'} E_{\mathbf{k}+\mathbf{k}'}\rangle = i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}-\mathbf{k}'} - \omega_{\mathbf{k}}^*)\langle E_{\mathbf{k}}^* E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle + 2H_{\mathbf{k},\mathbf{k}'}^* |E_{\mathbf{k}'}|^2 |E_{\mathbf{k}-\mathbf{k}'}|^2 \\ - 2H_{\mathbf{k}',\mathbf{k}} |E_{\mathbf{k}}|^2 |E_{\mathbf{k}-\mathbf{k}'}|^2 - 2H_{\mathbf{k}-\mathbf{k}',\mathbf{k}} |E_{\mathbf{k}}|^2 |E_{\mathbf{k}'}|^2, \\ H_{\mathbf{k},\mathbf{k}'} = (M_{\mathbf{k},\mathbf{k}'} + M_{\mathbf{k},\mathbf{k}-\mathbf{k}'})/2.\end{aligned} \tag{2.24}$$

Equation (2.24) can now be integrated using the fact that

$$\omega_k^0 \gg \frac{1}{|E_{\mathbf{k}}|^2} \frac{\partial |E_{\mathbf{k}}|^2}{\partial t},$$

and asymptotically one finds

$$\langle E_{\mathbf{k}}^* E_{\mathbf{k}'} E_{\mathbf{k}-\mathbf{k}'}\rangle = 2\pi\delta(\omega_{\mathbf{k}'}^0 + \omega_{\mathbf{k}-\mathbf{k}'}^0 - \omega_{\mathbf{k}}^0)\{H_{\mathbf{k},\mathbf{k}'}^* |E_{\mathbf{k}'}|^2 |E_{\mathbf{k}-\mathbf{k}'}|^2 - H_{\mathbf{k}',\mathbf{k}} |E_{\mathbf{k}}|^2 |E_{\mathbf{k}-\mathbf{k}'}|^2 - H_{\mathbf{k}-\mathbf{k}',\mathbf{k}} |E_{\mathbf{k}}|^2 |E_{\mathbf{k}'}|^2\}, \tag{2.25}$$

where $\delta(x)$ is the Dirac delta function, and Eq. (2.25) is valid in the limit of infinite volume. The equation for the

[4] W. E. Drummond and D. Pines, Ann. Phys. (N. Y.) (to be published).
[5] R. E. Aamodt and W. E. Drummond, Phys. Fluids 8, 171 (1965).

energy in the **k**th mode is then

$$\frac{\partial}{\partial t}|E_{\mathbf{k}}|^2=2\gamma_k|E_{\mathbf{k}}|^2+4\pi\sum_{\mathbf{k}'}|H_{\mathbf{k},\mathbf{k}'}|^2\delta(\bar{\omega})|E_{\mathbf{k}'}|^2|E_{\mathbf{k}-\mathbf{k}'}|^2-2\pi\sum_{\mathbf{k}'}[H_{\mathbf{k},\mathbf{k}'}H_{\mathbf{k}',\mathbf{k}}+H_{\mathbf{k},\mathbf{k}'}{}^*H_{\mathbf{k}',\mathbf{k}}{}^*]\delta(\bar{\omega})|E_{\mathbf{k}-\mathbf{k}'}|^2|E_{\mathbf{k}}|^2$$
$$-2\pi\sum_{\mathbf{k}'}[H_{\mathbf{k},\mathbf{k}'}H_{\mathbf{k}-\mathbf{k}',\mathbf{k}}+H_{\mathbf{k},\mathbf{k}'}{}^*H_{\mathbf{k}-\mathbf{k}',\mathbf{k}}{}^*]\delta(\bar{\omega})|E_{\mathbf{k}'}|^2|E_{\mathbf{k}}|^2+\sum_{\mathbf{k}'}[L_{\mathbf{k},\mathbf{k}'}+L_{\mathbf{k},\mathbf{k}'}{}^*]|E_{\mathbf{k}'}|^2|E_{\mathbf{k}}|^2, \quad (2.26)$$

$$\bar{\omega}=\omega_{\mathbf{k}}{}^0-\omega_{\mathbf{k}'}{}^0-\omega_{\mathbf{k}-\mathbf{k}'}{}^0.$$

Equation (2.26) is the basic equation for the energy in the kth mode for micro-instabilities in homogeneous plasma systems. If the quasilinear equations[1,2] for the particles and fields are used to describe the initial buildup of the amplitudes it is evident that when $|E_{\mathbf{k}}|^2$ becomes of order γ_k that the nonlinear terms are as important as the linear term in Eq. (2.26). For the mild bump velocity space instabilities in a homogeneous plasma, with[5] and without[1,2] a magnetic field, and the ion cyclotron instability[3] the quasilinear theory does predict equilibrium amplitudes of the order of γ_k.

III. DISCUSSION

A particularly interesting set of modes for study in a finite magnetic field are the modes which can exist at integral multiples of the electron cyclotron frequency. In a homogeneous plasma many of the harmonics can be unstable if the electron-velocity distribution function is sufficiently anisotropic,[6] or if groups of electrons have drift velocities along the field relative to one another. The latter type of instability is brought about by the usual particle-wave resonance phenomena where the phase velocity of the wave along the magnetic field is such as to pick up energy from particles.

For these modes, $\{\mathbf{k}\}$ conveniently denotes $(\mathbf{k},n)$ where $\omega_{\mathbf{k},n}{}^0=n\Omega_e+\bar{\omega}(\mathbf{k},n)$, $n=\pm1, \pm2, \cdots$, $\Omega_e=|(e/mc)B_0|$, and $\bar{\omega}(\mathbf{k},n)$ are wavelength-dependent factors accounting for slight shifts of the modes from exact cyclotron resonance. To illustrate the effect of wave-wave scattering it is assumed that the plasma parameters are such that only the $n=1$ mode is linearly unstable.

The quasilinear theory of the $n=1$ mode predicts that the mode grows exponentially in time causing a concomitant diffusion of particles in velocity space.[1–3] For the case of the instability induced by the wave resonanting with a drifting gentle beam of electrons, the mode becomes "quasilinearly-stationary" when the beam velocity distribution function has flattened in the direction of the external magnetic field, at which time $|E_{\mathbf{k},n=1}{}^{(1)}|$ is of order $\gamma_{\mathbf{k},n=1}(t=0)$. As $|E_{\mathbf{k}}{}^{(1)}|^2$ approaches the quasilinear amplitude, the nonlinear terms in Eq. (2.26) become of order $(\gamma_k)^2$ which is the same order as the linear term. If the three wave resonance condition can be satisfied for $\{\mathbf{k}'\}=(\mathbf{k}', n=1)$, $\{\mathbf{k}-\mathbf{k}'\}=(\mathbf{k}-\mathbf{k}', n=1)$ and $\{\mathbf{k}\}=(\mathbf{k}, n=2)$, that is

$$\Omega_e+\bar{\omega}(\mathbf{k}',1)+\Omega_e+\bar{\omega}(\mathbf{k}-\mathbf{k}', n=1)=2\Omega_e+\bar{\omega}(\mathbf{k}, n=2), \quad (3.1)$$

the $n=1$ modes start supplying energy to the $n=2$ modes. As the $n=2$ mode was assumed purely oscillating or damped, $|E_{\mathbf{k},n=2}|^2$ is much less than order γ_k and for small times $|E_{\mathbf{k},n=2}|^2$ satisfies the equation

$$\frac{\partial}{\partial t}|E_{\mathbf{k},n=2}|^2=4\pi\sum_{\mathbf{k}'}|H_{\mathbf{k},\mathbf{k}'}|^2\delta(\bar{\omega})|E_{\mathbf{k}',n=1}|^2\times|E_{\mathbf{k}-\mathbf{k}',n=1}|^2. \quad (3.2)$$

Hence, $|E_{\mathbf{k},n=2}|^2$ grows linearly in time until it reaches an amplitude of order (γ) when the scattering out of $(\mathbf{k}, n=2)$ becomes important and the terms proportional to $|E_{\mathbf{k},n=2}|^2$ in Eq. (2.26) are as large as those kept in Eq. (3.2). If the $n=2$ mode is damped, the quasilinear term transfers energy from the $n=2$ mode back into the particles. Also, the $n=2$ mode can resonantly scatter with the $n=1$ mode and the $n=2$ mode to supply energy to the $n=3$ and $n=4$ modes. If the transfer of energy to the particles is small, it is evident that all of the cyclotron modes which directly or indirectly couple resonantly through three wave scatterings to the unstable modes will reach an amplitude of the order of $\gamma_k(t=0)$. Of course, details of the spectrum depend on the structure of $H_{\mathbf{k},\mathbf{k}'}$, $L_{\mathbf{k},\mathbf{k}'}$, and $\gamma_{\mathbf{k}}$.

In any event, if the three wave resonance condition is satisfied, nonlinear wave-wave scattering is an important mechanism for supplying energy to the higher cyclotron harmonics. These nonlinear processes may furnish a partial explanation of the radiation experimentally observed in various plasma devices at high cyclotron harmonics.

[6] E. G. Harris (to be published).