Vortices in an Imperfect Bose Gas. I. The Condensate*

ALEXANDER L. FETTER†‡
Department of Physics, University of California, Berkeley, California
(Received 7 October 1964)

A theoretical study of rectilinear vortices in an imperfect Bose gas shows a close correspondence with classical hydrodynamics. The energy and momentum of a vortex pair in an unbounded fluid are calculated. The similarity between a vortex pair and a vortex ring leads to an estimate of the critical velocity v_c of liquid He II in a tube of radius R that includes the effect of the walls $v_c = C\hbar/2mR$, where C is a constant of order unity. A variational treatment of a system of many identical vortices in a container shows that the energy is lowest for a uniform distribution, and that the number of vortices per unit area ν agrees with Feynman's result $\nu = 2m\omega/h$. In the classical limit ($h \to 0$), the angular momentum and energy approach the values for solid-body rotation.

INTRODUCTION

THE suggested existence of vortices with quantized circulation[1,2] in liquid He II has stimulated both experimentalists and theorists. Measurements of the circulation have confirmed the predicted values of h/m.[3,4] The initial theory of quantized circulation was based on general arguments[2] and for this reason it was unable to predict the detailed structure of the vortices. The investigation of such structure in liquid He II is difficult because the comparatively high density makes perturbation theory unsuitable. A dilute Bose gas with short-range repulsive interactions provides a convenient model, one that is known to have states with most of the properties of a single classical vortex.[5,6]

This paper investigates systems of rectilinear vortices, using an imperfect Bose gas as a model for liquid He II. In Sec. I, a brief review is given of a single vortex in an unbounded fluid. The properties of two configurations are studied in detail: the vortex pair[7] (Sec. II), and a system of identical rectilinear vortices[8] (Sec. III).

I. SINGLE VORTEX IN AN UNBOUNDED FLUID

The condensate of an imperfect Bose gas can be characterized by a wave function ψ, whose normalization yields the total number of condensed particles.[9,10] In the low-density limit, ψ obeys a nonlinear field equation[11]

$$[(2m)^{-1}\hbar^2\nabla^2+\mu]\psi(\mathbf{r}) = \int d^3r' v(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r})|\psi(\mathbf{r}')|^2, \quad (1)$$

where v is the interparticle potential, μ is the chemical potential, and m is the mass of one particle. Several derivations of Eq. (1) have been given, leading to essentially equivalent forms.[5,6,11] The particular choice here is appropriate in the context of many-particle Green's functions[12] for a Bose system.[11]

The nonlinearity of (1) leads to many difficulties, and exact solutions[13] are possible only for a short-range repulsive potential (or pseudopotential)

$$v(\mathbf{r}-\mathbf{r}') = V_0\delta(\mathbf{r}-\mathbf{r}'). \quad (2)$$

With this simplifying assumption, the wave function is constant for a uniform system and is expressible in terms of elliptic functions for a one-dimensional finite channel.[14,15] More general situations cannot be integrated in terms of known functions.

It is possible, however, to show that for a large system with a fixed number of particles, the state of lowest energy is a uniform distribution. In this calculation, the effect of the boundaries may be neglected, for the following reason. The wave function vanishes at the walls of the container and reaches a finite value within a thin boundary layer, whose structure is independent of the remaining fluid. Hence the boundary layer contributes equally to the energy in all configurations and cancels when computing the difference between the energy of two states. The expectation value of the energy is

$$E = (2m)^{-1}\hbar^2\int d^3r|\nabla\psi(\mathbf{r})|^2 + \tfrac{1}{2}V_0\int d^3r|\psi(\mathbf{r})|^4, \quad (3)$$

* Research supported in part by the U. S. Air Force Office of Scientific Research, Grant No. AF-AFOSR-130-63.
† Fellow of the Miller Institute for Basic Research in Science.
‡ A portion of this work is based on a Ph.D. thesis submitted to Harvard University, 1963.
[1] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949).
[2] R. P. Feynman, *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. I, p. 17.
[3] W. F. Vinen, Proc. Roy. Soc. (London) **A260**, 218 (1961).
[4] G. W. Rayfield and F. Reif, Phys. Rev. Letters **11**, 305 (1963).
[5] L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. **40**, 646 (1961) [English transl.: Soviet Phys.—JETP **13**, 451 (1961)].
[6] E. P. Gross, Nuovo Cimento **20**, 454 (1961).
[7] A preliminary account has been given in A. L. Fetter, Phys. Rev. Letters **10**, 507 (1963).
[8] A preliminary account has been given in A. L. Fetter, Bull. Am. Phys. Soc. **8**, 33 (1963).
[9] N. Bogoliubov, J. Phys. (USSR) **11**, 23 (1947).
[10] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).
[11] P. C. Hohenberg, Ph.D. thesis, Harvard University, 1962 (unpublished); P. C. Martin, J. Math. Phys. **4**, 208 (1963); P. C. Hohenberg and P. C. Martin, Phys. Rev. Letters **12**, 69 (1964) and to be published.
[12] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).
[13] A general review is found in E. P. Gross, J. Math. Phys. **4**, 195 (1963).
[14] V. L. Ginzburg and L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. **34**, 1240 (1958) [English transl.: Soviet Phys.—JETP **7**, 858 (1958)].
[15] T. T. Wu, J. Math. Phys. **2**, 105 (1961).

where the delta-function interaction has been used. The wave function for the uniform system is

$$\psi = n_0^{1/2}, \tag{4}$$

in which case the energy is

$$E_0 = \tfrac{1}{2} V_0 n_0^2 \int d^3r. \tag{5}$$

The normalization of ψ is fixed by the constant number of particles

$$\int d^3r\, |\psi(\mathbf{r})|^2 = n_0 \int d^3r, \tag{6}$$

and it follows that

$$\int d^3r [|\psi(\mathbf{r})|^2 - n_0]^2 = \int d^3r [|\psi(\mathbf{r})|^4 - n_0^2] \geqq 0. \tag{7}$$

From (3), (5), and (7), it is easy to find the essentially positive quantity

$$E - E_0 = (2m)^{-1}\hbar^2 \int d^3r\, |\nabla\psi(\mathbf{r})|^2 + \tfrac{1}{2} V_0 \int d^3r [|\psi(\mathbf{r})|^4 - n_0^2] \geqq 0, \tag{8}$$

where the equality holds only for constant ψ. Therefore, any nonuniform configuration represents an excited state of the medium.

With the approximation of a short-range potential, Eq. (1) becomes

$$[(2m)^{-1}\hbar^2\nabla^2 + \mu]\psi = V_0 \psi |\psi|^2. \tag{9}$$

In cylindrical coordinates, there are solutions of the form[5,6]

$$\psi = A e^{il\theta} f_l(r), \tag{10}$$

where l is an integer, f_l is real, and $f_l(r) \to 1$ for large r. The density at infinity is A^2, so that $A = n_0^{1/2}$. The nonlinearity of (9) determines the chemical potential as

$$\mu = n_0 V_0. \tag{11}$$

Equation (10) has been interpreted as the wave function for a vortex in the condensate.[5,6] In contrast to a classical vortex, the structure of the core is fully determined because the radial function f vanishes for $r=0$. The natural core dimension is the de Broglie wavelength a, given by

$$a^2 = \hbar^2/2m\mu = \hbar^2/2mn_0V_0, \tag{12}$$

and for the energetically favorable case of $l^2=1$, f behaves linearly for small r. The exact solution for f has been found numerically,[14] and the energy E_v per unit length of the vortex has been calculated as[5]

$$E_v = \pi\hbar^2 n_0 m^{-1} \ln(1.46R/a). \tag{13}$$

The logarithmic dependence on the outer radius R of the cylinder is typical of the energy of a single vortex.

A simple analytic approximation for f can be found,

$$f(r) = r(r^2+a^2)^{-1/2}. \tag{14}$$

It is shown for comparison with the exact solution[14] in Fig. 1. With Eq. (14), the total number of particles N, the kinetic energy T, and the total potential energy V_T (all per unit length) are

$$N = n_0\pi R^2 - 2\pi a^2 n_0 \ln(R/a), \tag{15}$$

$$T = \pi\hbar^2 n_0 m^{-1}[\ln(R/a) + \tfrac{1}{4}], \tag{16}$$

$$V_T = \tfrac{1}{2}\pi n_0^2 V_0 R^2 - 2\pi n_0^2 V_0 a^2[\ln(R/a) - \tfrac{1}{4}]. \tag{17}$$

The major portion of the total potential energy represents the interaction energy of a uniform medium. The energy necessary to form a vortex is found by subtracting from (17) the energy $\bar{V}$ of the same total number of particles N distributed uniformly over the same area πR^2

$$\bar{V} = \tfrac{1}{2} V_0 N^2 (\pi R^2)^{-1}. \tag{18}$$

Thus the energy per unit length of a vortex is

$$E_v = T + V_T - \bar{V} = \pi\hbar^2 n_0 m^{-1} \ln(1.65R/a), \tag{19}$$

where we have neglected terms that vanish as (R/a) becomes very large. This is to be compared with (13); the two expressions differ only by a small additive logarithmic constant. The approximation (14) for f will be used in all further calculations reported here.

The momentum space wave function is interesting, and, with the approximation for $f(r)$, the integral is straightforward.[16] We find

$$\psi(\mathbf{k}) = \int d^2r e^{-i\mathbf{k}\cdot\mathbf{r}} \psi(\mathbf{r}) = -2\pi i n_0^{1/2} e^{i\chi} k^{-2} e^{-ka}(1+ka), \tag{20}$$

where (k,χ) are the polar coordinates of the vector $\mathbf{k}$. The distribution in momentum is singular for small k. Nevertheless, particles are found with all components of momentum, although the probability decreases exponentially for $ka \gg 1$. In a uniform medium, for which

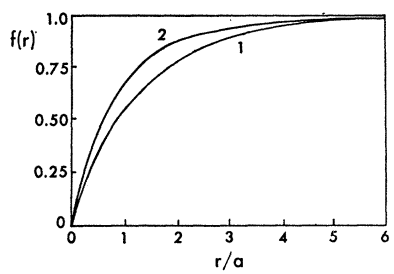

FIG. 1. A comparison of the exact wave function for the vortex (labeled 1) with the approximation, Eq. (14) (labeled 2).

[16] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1962), 2nd ed., p. 434, 13.6(2). Although the integral in Eq. (20) falls outside the range of validity stated by Watson, the result can be verified in several ways, the simplest being an inversion of the Fourier transform using Watson, p. 386, 13.2(7) and (8).

the Fourier transform is $n_0^{1/2}(2\pi)^2\delta(\mathbf{k})$, all the particles are in the zero-momentum state.

II. THE VORTEX PAIR

Infinite Fluid

The simplest configuration of two rectilinear vortices is a vortex pair, for which the circulation about each one is equal in magnitude and opposite in sign. Although bipolar coordinates are the natural choice for the vortex pair, the equation for ψ cannot be separated in this coordinate system. An exact solution requires difficult numerical methods. Instead, we choose an approximate wave function that leads to analytic expressions.

The classical theory of vortices is a linear one, and the effects due to different vortices can be found with the superposition principle. The nonlinearity of the field equation (9) is only apparent near the positions of the vortices, where $\mu - V_0|\psi|^2$ fails to vanish. Thus it is reasonable to approximate the exact solution by the solution for two independent vortices. If the pair of

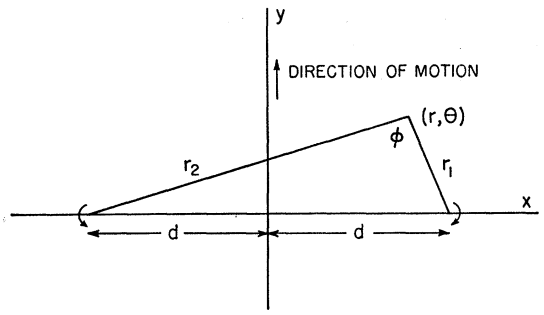

FIG. 2. The geometry of the vortex pair, showing the direction of motion and the sense of rotation.

vortices is situated on the x axis at a distance $\pm d$ from the origin, the wave function will be taken as a product of the wave functions for the separate vortices

$$\psi = n_0^{1/2} e^{-i\varphi} f(r_1) f(r_2), \tag{21}$$

where the geometry is shown in Fig. 2, and the functions f are those given in (14). The wave function (21) should be a good approximation when the distance between the vortices is large compared to the core size a, and the interaction between the cores is negligible.

The velocity of the fluid can be calculated from the definition

$$\mathbf{v} = (2mi|\psi|^2)^{-1}\hbar[\psi^*\nabla\psi - (\nabla\psi^*)\psi], \tag{22}$$

and for the wave function (21), this yields

$$\mathbf{v} = -(\hbar/m)\nabla\varphi. \tag{23}$$

The solution of the differential equation

$$dx/v_x = dy/v_y \tag{24}$$

gives the stream lines, but it is simpler to observe that

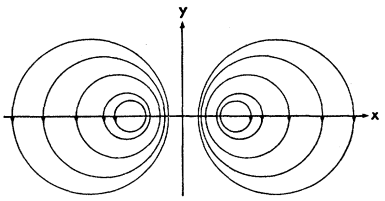

FIG. 3. The stream lines of the vortex pair.

$\hbar\varphi/m$ is the classical velocity potential Φ. A short calculation shows that Φ is the real part of the complex potential

$$F(z) = \Phi + i\Psi = \hbar i m^{-1} \ln[(z+d)(z-d)^{-1}]. \tag{25}$$

The stream lines are then the curves of constant Ψ. The complex potential (25) is identical with that for a classical vortex pair situated at the points $\pm d$ with circulation $\mp h/m$.[17] The stream lines (Fig. 3) form a system of coaxal circles

$$(x - d\coth\lambda)^2 + y^2 = d^2\,\mathrm{csch}^2\lambda, \tag{26}$$

where λ is a constant. In particular, the x component of velocity vanishes along the line $x=0$.

There is a close analogy between rectilinear vortices and current filaments.[18] The velocity field (23) and stream lines (26) correspond exactly to the lines of magnetic field between a pair of long parallel conductors carrying opposite currents. The magnetostatic energy of a single current filament diverges, as in (13). For a pair of oppositely directed current filaments, however, the total energy is finite[18] because of cancellation of the magnetic field at large distances. The same result will be found for a vortex pair, since the velocity fields cancel at infinity.

The analogy between vortices and current filaments is not perfect, however, because the forces are quite different in the two cases. Oppositely directed current filaments repel each other and move apart. In contrast, the energy of a vortex pair decreases when the vortices move toward each other, so that there is an attractive force. Furthermore, a vortex behaves gyroscopically, in that it moves perpendicular to an applied force. It is not difficult to show that the vortex pair moves with constant separation along the perpendicular bisector of the line joining the centers. Thus for the system in Fig. 2, the pair moves in the positive y direction.

There is a simpler explanation of the motion of a system of vortices. A classical rectilinear vortex does not contribute to its own motion; it moves with a velocity due to the superposition of all the other vortices. For example, each member of a vortex pair moves in the same direction as the fluid midway between them, and

[17] A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press Inc., New York, 1950), p. 142.

[18] A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press Inc., New York, 1950), p. 153, and *Electrodynamics* (Academic Press Inc., New York, 1952), pp. 178 and 198.

the configuration shows no tendency to collapse.[19] More generally, a system of rectilinear vortices behaves in many respects like a system of point masses, in which the circulation corresponds to the mass (but can take either sign). The "center of mass" and the "angular momentum" of the system of vortices are constants of the motion.[20] In particular, the "center of mass" of the vortex pair is at infinity, so that any tendency to decrease the separation is incompatible with the constants of the motion.

An extension of the above discussion to quantum vortices is extremely difficult. The center of the vortex cannot be localized with arbitrary precision, and the translational velocity v_t must be defined in terms of the wave function. It is plausible to identify v_t with the group velocity, $v_t=\partial E/\partial P$. The velocity so calculated from classical values of E and P agrees with the classical translational velocity of a vortex pair and (with a small logarithmic discrepancy) a vortex ring. We shall show below that there is a close correspondence between the quantum and classical values of momentum and energy, and it is a plausible assumption that the quantum vortices obey essentially the same dynamics as the classical ones (at least if the vortices are widely separated compared to the dimensions of the core).

Momentum is not a well-defined quantity in classical hydrodynamics and is commonly replaced by the impulse necessary to generate the actual motion from rest.[21] The impluse per unit length of a classical vortex pair with circulation $\pm\kappa$ is[22]

$$P_0=2\rho\kappa d. \tag{27}$$

Here ρ is the density of the fluid, and $2d$ is the separation of the two vortices. It is not difficult to compute the momentum per unit length of the quantum vortex pair. Using

$$\mathbf{P}=-\tfrac{1}{2}\hbar i\int d^2r[\psi^*\nabla\psi-(\nabla\psi^*)\psi], \tag{28}$$

we find from (21) that P_x vanishes and that

$$P_y=2m^{-1}h\rho d\{1-\tfrac{1}{2}\delta^{-1}(1+\delta^2)^{-1/2}\times\ln[(1+\delta^2)^{1/2}+\delta]\}, \tag{29}$$

where $\delta=d/a$ and $\rho=n_0m$. The quantity in curly brackets is the ratio of the quantum and classical values; it differs from unity only for $\delta\lesssim 1$.

In a similar way, the total number of particles N, the kinetic energy T, and the potential energy V_T (all per unit length) can be calculated. The results are

$$N=n_0A-2\pi n_0a^2\ln\Xi+a^2\pi n_0\delta^{-1}(1+\delta^2)^{-1/2}\ln[(1+\delta^2)^{1/2}+\delta], \tag{30}$$

$$T=\frac{\hbar^2n_0\pi}{2m}\left\{\frac{1-4\delta^4}{16\delta^2(1+\delta^2)^2}-\frac{1+6\delta^2+8\delta^4}{16\delta^3(1+\delta^2)^{5/2}}\ln[(1+\delta^2)^{1/2}+\delta]+1+4\delta(1+\delta^2)^{-1/2}\ln[(1+\delta^2)^{1/2}+\delta]\right\}, \tag{31}$$

$$V_T=\tfrac{1}{2}n_0^2V_0A-2\pi n_0^2V_0a^2\ln\Xi+\pi n_0^2V_0a^2\times\left\{\frac{2}{\delta(1+\delta^2)^{1/2}}\left[1+\frac{1-12\delta^2-16\delta^4}{64\delta^2(1+\delta^2)^2}\right]\ln[(1+\delta^2)^{1/2}+\delta]-[32\delta^2(1+\delta^2)^2]^{-1}[1-18\delta^2-48\delta^4-32\delta^6]\right\}, \tag{32}$$

where

$$\Xi=\{\Lambda+\delta+[(\Lambda+\delta)^2+1]^{1/2}\}\{\Lambda-\delta+[(\Lambda-\delta)^2+1]^{1/2}\}. \tag{33}$$

Here A is the area of the container, and the x integration is cutoff where necessary at $\pm\Lambda a$. The kinetic energy of a vortex pair is independent of the dimensions of the container as previously noted.

The potential energy V associated with the vortex pair is the difference between the total potential energy and the energy of N particles uniformly distributed. The calculation is similar to that for a single vortex, and we find

$$V=V_T-\tfrac{1}{2}V_0N^2A^{-1}=\pi a^2n_0^2V_0\left\{\frac{\ln[(1+\delta^2)^{1/2}+\delta]}{\delta(1+\delta^2)^{1/2}}\left[1+\frac{1-12\delta^2-16\delta^4}{32\delta^2(1+\delta^2)^2}\right]-[32\delta^2(1+\delta^2)^2]^{-1}[1-18\delta^2-48\delta^4-32\delta^6]\right\}. \tag{34}$$

The total energy per unit length of the quantum vortex pair is $T+V$, the sum of (31) and (34). It is independent of the size of the container as the area A of the system becomes infinite.

The energy of a classical vortex pair is wholly kinetic

[19] See H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), 6th ed., p. 222 for a discussion of certain paradoxical aspects of the motion of a vortex pair and vortex ring.

[20] A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press Inc., New York, 1950), pp. 157–160.

[21] C. C. Lin, *Proceedings of the International School of Physics "Enrico Fermi,"* edited by G. Careri (Academic Press Inc., New York, 1963), Course XXI, Liquid Helium, p. 93.

[22] H. Lamb, *Hydrodynamics* (Dover Publications, Inc., New York, 1945), 6th ed., p. 229.

and diverges logarithmically unless special assumptions are made about the core. If each vortex has a hollow core of radius a, the total energy is[23]

$$T_0=(2\pi)^{-1}\rho\kappa^2\ln(2\delta+1). \tag{35}$$

In the limit of large δ, the quantum kinetic energy (31) simplifies greatly:

$$T\rightarrow(2\pi)^{-1}(h/m)^2\rho[\ln(2\delta)+\tfrac{1}{4}]. \tag{36}$$

For circulation $\kappa=h/m$, (35) and (36) are very similar. In this limit, the quantum potential energy (34) reduces to a constant

$$V\rightarrow\pi a^2n_0^2V_0=(8\pi)^{-1}(h/m)^2\rho. \tag{37}$$

The equality of (37) and the constant term in (36) is suggestive of a zero-point energy, in analogy with the harmonic oscillator.

Figure 4 shows the quantum kinetic energy (31), the quantum potential energy (34), and the classical kinetic energy (35) as functions of the separation δ of the vortices. The potential energy V is small, and the kinetic energy T is close to the classical value T_0 for all δ. One important difference is that both T and V are finite for for small δ, in contrast to T_0. This reflects the quantum effect of the core: The curvature of the wave function leads to additional kinetic energy, and the decrease in density requires work against the repulsive interaction potential.

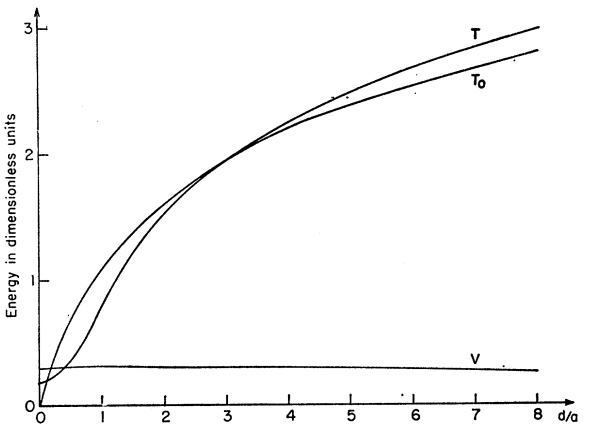

FIG. 4. The energy of the vortex pair, measured in units of $(\rho\kappa^2/2\pi)$, as a function of the separation of the vortices. T and V are the quantum kinetic and potential energies, and T_0 is the classical kinetic energy.

Finite Channels

In classical hydrodynamics, the image of a single vortex in a plane is a second vortex with opposite circulation. The image guarantees the correct boundary condition of vanishing normal component of velocity. A single vortex parallel to a wall is therefore equivalent to a vortex pair. An alternative point of view is that an infinite plane may be inserted along the perpendicular bisector of the line joining the centers of the pair with no change in the flow pattern.

The method of images fails in the quantum fluid for two reasons: (1) The equation is nonlinear so that the superposition principle fails. (2) The boundary condition requires that the wave function vanish at a rigid wall.[24] In principle, for a vortex near a wall, the field equation (9) must be solved for $x\geqq0$, subject to the condition that ψ vanish at $x=0$ and that ψ behave like Eq. (10) near the position of the vortex. In the absence of a vortex, the solution of (9) for an infinite half-space is[14,15]

$$\psi(x)=n_0^{1/2}\tanh(x/a\sqrt{2}). \tag{38}$$

The effect of a plane boundary vanishes exponentially in the interior of the fluid, in contrast to the effect of a vortex, where $n_0-|\psi|^2$ vanishes only quadratically far from the axis.

An obvious approximation to the exact wave function for a vortex near a wall is a product of (21) and (38). For this wave function, however, the energy cannot be found in terms of tabulated functions. (If a computer were used at all, an exact solution would be preferable to an evaluation with approximate wave functions.) Hence the results for a vortex pair (31) and (34) will be used in a qualitative description of a vortex near a plane boundary.

The more interesting problem is a large vortex ring of radius r_0, which is essentially equivalent to a rectilinear vortex of length $2\pi r_0$. When the ring is enclosed in a coaxial cylinder of radius R $(R>r_0)$, the corresponding two-dimensional configuration is a vortex at a distance $R-r_0$ from the wall. In several theories,[25,26] such a vortex ring accounts for the critical velocity of liquid He II. According to Landau's criterion for superfluid flow,[27] the critical velocity v_c is the minimum value of the ratio E/P for all the elementary excitations in the fluid. If only phonons and rotons are included, the predicted value of v_c is far too large, but with the inclusion of large vortex rings, it has been possible to get rather good agreement with experiment.[25,26]

This approach has been criticized for failing to include the effect of the boundary.[23] A proper classical treatment of the images shows that v_c is sensitive to the assumed core structure. For a hollow core, the theoretical v_c vanishes,[23] and for a core with uniform vorticity, the theoretical v_c is an order of magnitude smaller than the experimental values.[26] The dependence on the core is an unsatisfactory feature of the classical calculation. A

[23] J. C. Fineman and C. E. Chase, Phys. Rev. **129**, 1 (1963).
[24] The importance of the boundary condition has been emphasized by Gross, Ref. 13.

[25] B.T. Geilikman, Zh. Eksperim. i Teor. Fiz. **37**, 891 (1959) [English transl.: Soviet Phys.—JETP **10**, 635 (1960)], and W. F. Vinen, *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1961), Vol. III, p. 38.
[26] E. S. Raja Gopal, Ann. Phys. (N. Y.) **25**, 196 (1963).
[27] L. Landau, J. Phys. (USSR) **5**, 71 (1941).

quantum vortex has a definite structure, and no special assumptions are needed.

The critical velocity will now be estimated with the wave function for the vortex pair. Such a calculation is rather crude, but it does predict a qualitatively correct dependence on the radius of the tube. The energy per unit length of a single vortex near a wall is given by one-half the sum of (31) and (34), the kinetic and potential energy of the vortex pair. When the vortex approaches the wall ($r_0 \to R$), the energy becomes

$$\tfrac{1}{2}(T+V) \to \tfrac{1}{2}C(\hbar^2 n_0 \pi/m), \tag{39}$$

where C is a constant of order unity. The detailed expressions (31) and (34) for T and V yield the result $C=11/12$. The circumference of the ring is $2\pi R$, so that the total energy E_R of the ring is

$$E_R = C\hbar^2 n_0 \pi^2 R/m. \tag{40}$$

In the classical treatment, the impulse of a vortex ring is unchanged by the presence of the containter.[21] This is assumed to remain true for the quantum vortex ring, so that[23]

$$P_R = \pi\rho\kappa R^2 = 2\pi^2 \hbar n_0 R^2, \tag{41}$$

apart from quantum corrections of order (a/R). The ratio of (40) to (41), which is in fact the minimum value of E/P, yields

$$v_c = C(\hbar/2mR). \tag{42}$$

For a tube of radius $R \approx 10^{-3}$ cm, the critical velocity calculated from (42) (for $C=1$) is ≈ 0.08 cm/sec, roughly an order of magnitude smaller than the experimental value of ≈ 1 cm/sec.[23] The observed critical velocity is approximately proportional to R^{-1}, in agreement with (42).

The calculation of the critical velocity assumes that the vortices act independently. Although this is valid when the vortices are far apart, it certainly fails for a vortex near a wall, particularly since ψ must vanish at the boundary. A numerical solution of the field equation should be possible with relaxation methods,[28] which are especially useful in two-dimensional problems. The nonlinearity poses no essential difficulty, but the complex nature of ψ requires the solution of coupled equations for the real and imaginary parts. Two configurations are particularly interesting: a rectilinear vortex near a plane boundary, and a vortex ring of small radius. Both these structures are two-dimensional and involve the quantum corrections to the simple product wave function. The rectilinear vortex near a plane is important in a calculation of v_c; the boundary condition on ψ is expected to keep the vortex a finite distance from the wall. The energy of a vortex ring almost certainly has a minimum at a finite radius. Such a small vortex ring forms a model of a roton. The ring differs from the vortex pair because of the curvature of the axis of the vortex. A good initial trial function can shorten the relaxation procedure considerably. For both of the above cases, an approximate product wave function would be a suitable choice.

In the calculation of v_c, it should be remembered that, out of necessity, all but the first term in the low-density expansion for ψ has been neglected. Any attempt to equate this model with liquid He II underestimates the interaction energy, and therefore the critical velocity. A more realistic calculation, taking into account the finite compressibility of the fluid, should yield the correct order of magnitude for v_c.

III. SYSTEM OF MANY VORTICES

Rotating liquid He II is generally assumed to form an array of rectilinear vortices.[2,29,30] With an imperfect Bose gas as a model fluid, we calculate the energy of a system of vortices as a function of their positions. The actual distribution is that which minimizes the energy, subject to the constraint of fixed angular momentum. The force between vortices of like circulation is repulsive, so that the distribution is uniform, but random, in contrast to the lattice structure of flux vortices in type-II superconductors.

In the limit of many vortices, the fluid becomes a continuum with uniformly distributed vorticity, and the interaction energy of the vortices is equivalent to the rotational energy of the fluid. In this calculation it is necessary to include the effects of the image vortices due to the presence of the boundaries. Without the images, the energy of a circular cluster of vortices is proportional to $R^4 \ln R$, where R is the radius of the cluster. At large distances from the cluster, this configuration is indistinguishable from a single vortex, and the factor $\ln R$ is the same as that in the energy of a single vortex, Eq. (13). The images cancel the $R^4 \ln R$ term, leaving an energy proportional to R^4, which is, of course, the correct dependence of the moment of inertia of a circular cylinder about its axis.

The system considered here consists of $\mathfrak{N}$ vortices with circulation $\kappa = h/m$ at the positions $\mathbf{r}_i \equiv (r_i, \theta_i)$. Figure 5 illustrates the geometry, where

$$\boldsymbol{\varrho}_i = \mathbf{r} - \mathbf{r}_i \tag{43}$$

and φ_i is the angle measured about the point $\mathbf{r}_i$. It is consistent to assume that the vortices are far apart, because the energy increases whenever two vortices approach within a distance a (12) of each other. Hence the quantum corrections are negligible and a simple product wave function is a reasonable choice for ψ

$$\psi(\mathbf{r}) = n_0^{1/2} \textstyle\prod_j g_j, \tag{44}$$

where

$$g_j = \exp(i\varphi_j) f(\rho_j) \equiv \exp(i\varphi_j)\rho_j(\rho_j^2 + a^2)^{-1/2}. \tag{45}$$

[28] R. V. Southwell, *Relaxation Methods in Theoretical Physics* (Oxford University Press, New York, 1946).

[29] H. E. Hall and W. F. Vinen, Proc. Roy. Soc. (London) **A238**, 204 and 215 (1956), and G. Careri, W. D. McCormick and F. Scaramuzzi, Phys. Letters **1**, 61 (1962).

[30] See Lin, Ref. 21, for an opposing view.

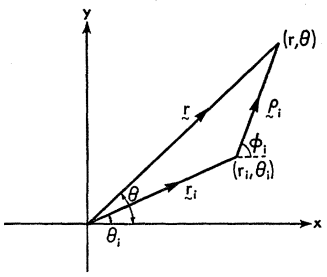

FIG. 5. The geometry of a vortex situated at $\mathbf{r}_i$.

The product in (44) is taken over all vortices from 1 to $\mathfrak{N}$. Far from any vortex, $|\psi| \approx n_0^{1/2}$, and near the jth vortex, $\psi(\mathbf{r}) \approx n_0^{1/2} \exp(i\varphi_j) f(\rho_j)$.

It is straightforward to calculate the observables of the system with the wave function (44). The total number of particles N per unit length is the simplest

$$N = n_0 \int \Pi_j |g_j|^2 = n_0 \int \Pi_j f_j^2 , \tag{46}$$

where the integral is evaluated over a large circle of radius R. The integrand can be expanded as follows:

$$\Pi_j f_j^2 = \Pi_j [1 - a^2(\rho_j^2 + a^2)^{-1}] = 1 - \textstyle\sum_j a^2(\rho_j^2 + a^2)^{-1} + \cdots , \tag{47}$$

and the total number becomes

$$N = n_0 \pi R^2 - \textstyle\sum_j n_0 a^2 \int (\rho_j^2 + a^2)^{-1} + \cdots . \tag{48}$$

The integral in (48) is not difficult; the result is

$$\int (\rho_j^2 + a^2)^{-1} = \pi \ln\{(2a^2)^{-1}[X + (X^2 + 4a^2 r_j^2)^{1/2}]\} , \tag{49}$$

where

$$X = R^2 - r_j^2 + a^2 . \tag{50}$$

The second term of (48) is a correction of order $(a/R)^2$, and it is permissible to set $a=0$ in (49) whenever possible. With this simplification, the total number of particles is

$$N = n_0 \pi R^2 \{1 - (a/R)^2 \textstyle\sum_j \ln[(R^2 - r_j^2)/a^2] + \cdots\} . \tag{51}$$

In the step from (49) to (51), we have assumed that $R^2 - r_j^2 \gg a^2$, which is equivalent to neglecting edge effects. The terms omitted in (51) are at least of order $(a/R)^4 \ln(R/a)$.

The kinetic and potential energy per unit length are given by

$$T = -(4m)^{-1} n_0 \hbar^2 \int [\Pi_i g_i^* \nabla^2 \Pi_j g_j + (\nabla^2 \Pi_i g_i^*) \Pi_j g_j] , \tag{52}$$

$$V_T = \tfrac{1}{2} n_0^2 V_0 \int \Pi_j |g_j|^4 . \tag{53}$$

A calculation similar to that for N (but longer) yields

$$E = T + V_T = \tfrac{1}{2} n_0^2 V_0 \pi R^2 + \tfrac{1}{2} n_0^2 V_0 \pi a^2 \textstyle\sum_{ij}{}' \ln\{|\mathbf{r}_i - \mathbf{r}_j|^{-4} \times [R^4 - 2R^2 r_i r_j \cos(\theta_i - \theta_j) + r_i^2 r_j^2]\} . \tag{54}$$

The summation is over i and j separately, omitting the terms $i=j$. In (54), the limit $a \to 0$ has been taken in the argument of the logarithm. The energy increases whenever $|\mathbf{r}_i - \mathbf{r}_j|$ becomes small; the energy is lowest when the vortices are relatively far apart.

The Hamiltonian remains a constant of the motion in a rotating system. Its value is $E - \omega L$, where L is the total angular momentum and ω is the angular velocity. Hence the quantity $(E - \omega L)$ must be varied with respect to the positions of the separate vortices. An equivalent point of view is that E must be varied, subject to the constraint of fixed angular momentum. Then $E - \lambda L$ is minimized, where the Lagrange multiplier λ is identified with ω in the course of the calculation. In either approach, the total angular momentum must be found from

$$L = \frac{n_0 \hbar}{2i} \int \left[\Pi_i g_i^* \frac{\partial}{\partial \theta} \Pi_j g_j - \left(\frac{\partial}{\partial \theta} \Pi_i g_i^* \right) \Pi_j g_j \right] ; \tag{55}$$

the dominant term is

$$L = \pi n_0 \hbar \textstyle\sum_i (R^2 - r_i^2) , \tag{56}$$

where corrections of order $(a/R)^2$ are neglected.

It is not strictly correct to compute the variation of $E - \omega L$, which does not fix the number of particles; instead the normalized quantity $N^{-1}(E - \omega L)$ should be considered. The total number (51) contains the factor $1 + \mathcal{O}(a^2 \mathfrak{N}/R^2)$, while the energy (54) contains the factor $1 + \mathcal{O}(a^2 \mathfrak{N}^2/R^2)$. The number of vortices $\mathfrak{N}$ is large, even for slow rotations, so that $E - \omega L$ and $N^{-1}(E - \omega L)$ differ only by a correction of order $\mathfrak{N}^{-1}$. (For a rotation of 1 rpm, there are approximately 200 vortices per cm^2.) The neglect of this correction omits self-energy effects that vanish in the limit of a classical continuum. This is discussed in detail below, after the distribution of vortices has been found.

The quantity $J = E - \omega L$ is stationary with respect to the position of the vortices if

$$\partial J / \partial \theta_i = \partial J / \partial r_i = 0 \tag{57}$$

for every $i = 1 \cdots \mathfrak{N}$. The derivative with respect to θ_i is

$$\frac{\partial J}{\partial \theta_i} = \frac{\pi n_0 \hbar^2}{m} \textstyle\sum_j{}' \left\{ \frac{R^2 r_i r_j \sin(\theta_i - \theta_j)}{R^4 - 2R^2 r_i r_j \cos(\theta_i - \theta_j) + r_i^2 r_j^2} - \frac{2 r_i r_j \sin(\theta_i - \theta_j)}{r_i^2 - 2 r_i r_j \cos(\theta_i - \theta_j) + r_j^2} \right\} , \tag{58}$$

where the prime means omit the term $j=i$. The total number of vortices is large, and it is convenient to

replace the summation in (58) by an integration over a vortex density $\nu(r,\theta)$, which is the number of vortices per unit area centered at $\mathbf{r}$. Equation (58) can be rewritten as

$$\frac{\partial J}{\partial\theta}=\frac{\pi n_0\hbar^2}{m}\int_0^R r'dr'\int_{-\pi}^{\pi}d\theta'\nu(r',\theta') \times\sin(\theta-\theta')Y[r,r',\cos(\theta-\theta')]. \quad (59)$$

The function Y is an even function of $(\theta-\theta')$, and (59) can vanish only if $\nu(\theta')$ is also an even function of $(\theta-\theta')$. But θ is arbitrary, so that $\nu(\theta')$ must be independent of θ'.

The problem is thus reduced to finding the radial distribution of vortices that minimizes J. When $\partial J/\partial r_i$ is calculated, and the sum replaced by an integral, we find

$$\frac{\partial J}{\partial r}=2\pi n_0\omega\hbar r-\frac{\pi n_0\hbar^2}{m}\int_0^R r'dr'\nu(r') \times\int_0^{2\pi}d\theta'\left\{\frac{2(r-r'\cos\theta')}{r^2-2rr'\cos\theta'+r'^2} -\frac{r'(rr'-R^2\cos\theta')}{R^4-2R^2rr'\cos\theta'+r^2r'^2}\right\}. \quad (60)$$

Both angular integrals are of the same form,

$$\int_0^{2\pi}d\theta(b-c\cos\theta)(b^2-2bc\cos\theta+c^2)^{-1} =2\pi b^{-1}\eta(b-c), \quad (61)$$

where η is the step-function, $2\eta(x)=1+|x|/x$. The last term of (60) is zero because R^2 is always greater than rr'. The vanishing of (60) yields an integral equation for $\nu(r')$.

$$m\omega r^2=2\pi\hbar\int_0^r r'dr'\nu(r'), \quad (62)$$

whose solution is found by differentiating each side with respect to r. It follows immediately that

$$\nu=2m\omega/h(=\text{const}); \quad (63)$$

this is Feynman's original value.[2]

With the constant vortex density (63), the total angular momentum per unit length can be found from (56),

$$L=\tfrac{1}{2}\pi R^4n_0m\omega=\tfrac{1}{2}MR^2\omega, \quad (64)$$

where $M=\pi R^2n_0m$ is the total mass per unit length. This is just the value associated with the rigid rotation of the fluid. The angular momentum per particle is

$$L/N=\tfrac{1}{2}\mathfrak{N}\hbar, \quad (65)$$

apart from corrections of order $(a/R)^2$. The factor $\frac{1}{2}$ arises because the vortices are uniformly distributed instead of concentrated at the center of the vessel.

A consideration of the interaction energy of this system of vortices will clarify the relation between the above quantum calculation and a classical one. The energy E_v associated with the system of vortices is the difference between the total energy (54) and the energy of N particles uniformly distributed with no angular momentum. The steps are identical with those for a single vortex, and the result is

$$E_v=(2m)^{-1}n_0\pi\hbar^2\{-\textstyle\sum_{ij}{}'\ln|\mathbf{r}_i-\mathbf{r}_j|^2 +\tfrac{1}{2}\sum_{ij}{}'\ln[R^4-2R^2r_ir_j\cos(\theta_i-\theta_j)+r_i^2r_j^2] +\sum_i\ln[(R^2-r_i^2)/a^2]\}, \quad (66)$$

where the relation $a^2=\hbar^2/2mn_0V_0$ has been used. Lin[31] has treated a system of classical rectilinear vortices in the presence of arbitrary boundaries and reduces the problem to that of finding a certain Green's function. In the interior of a simply connected region, the Green's function must vanish at the outer boundary, and for a circular cylinder of radius R, the required solution is

$$2\pi G(\mathbf{r},\mathbf{r}')=\ln|\mathbf{r}-\mathbf{r}'|-\ln|(\mathbf{r}r'/R)-(\mathbf{r}'R/r')|. \quad (67)$$

This is symmetric in $\mathbf{r}$ and $\mathbf{r}'$, and vanishes for $r=R$. It is easy to find the energy of the system of vortices from G;[31] if the vortices are placed at positions $\mathbf{r}_i$ with circulation κ about each one, the interaction energy is

$$E_v=(\rho\kappa^2/8\pi)\{-\textstyle\sum_{ij}{}'\ln|\mathbf{r}_i-\mathbf{r}_j|^2 +\sum_{ij}{}'\ln[R^2-2r_ir_j\cos(\theta_i-\theta_j)+r_i^2r_j^2R^{-2}] +2\sum_i\ln[(R^2-r_i^2)/R]\}. \quad (68)$$

The first term represents the energy of the vortices in the absence of the boundaries,[31,32] while the second and third terms represent the effect of the images. In particular, the last term is a self-energy that is present even for a single vortex in a cylinder.

When the quantized circulation $\kappa=h/m$ is substituted in (68), we see that the energies of the system of quantum and classical vortices differ only in the last two terms. Although no direct provision was made for the images in the wave function (44), the energy (66) includes at least part of this effect automatically.

The limit $(\kappa=h/m\to 0)$ of a classical continuum is particularly interesting. The number of vortices per unit area ν becomes infinite, in such a way that $\kappa\nu=\zeta$ remains constant. Stokes's theorem shows that ζ is the vorticity, defined by $\zeta=|\nabla\times\mathbf{v}|$. For a system in uniform rotation, the vorticity is $\zeta=2\omega$. If the sums in (66) and (68) are replaced by integrals, the self-energy terms vanish linearly with κ. The angular integral can be evaluated by integrating (61) from 0 to b; the radial integrals are then simple. For both (66) and (68), the first term yields an energy proportional to $R^4\ln R$. This

[31] C. C. Lin, Proc. Natl. Acad. Sci. U. S. **27**, 570 and 575 (1941).
[32] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 279 (1949).

behavior is exactly cancelled by the second term, which represents the image vortices. The final result for the interaction energy is

$$E_v=\tfrac{1}{4}\omega^2\rho\pi R^4=\tfrac{1}{2}I\omega^2, \quad (69)$$

where I is the moment of inertia of a cylinder about its axis, $I=\frac{1}{2}MR^2$.

In general, vortices in an imperfect Bose gas exhibit properties similar to those of classical vortices. Here we have investigated the relation in detail for a vortex pair and for a system of many identical vortices. Experimental studies[2–4,29] of liquid He II have been analyzed in terms of classical hydrodynamics; the present work justifies such treatment as long as the distance between vortices is large compared to the dimension of the core.

ACKNOWLEDGMENTS

I should like to thank Professor P. C. Martin for discussion and criticism throughout the development of this work.

PHYSICAL REVIEW VOLUME 138, NUMBER 2A 19 APRIL 1965

Surface Tension of Liquid He⁴†

K. R. Atkins and Y. Narahara*
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania
(Received 30 November 1964)

We have measured the surface tension of liquid He⁴ down to 0.35°, using the capillary-rise method. The results are consistent with the theory that this temperature variation is mainly due to the excitation of surface modes similar to capillary waves. However, the results do not exclude an alternative theory, due to Singh, which considers the effect of the surface on the wave functions of an ideal degenerate Bose-Einstein gas. At the λ point, we observed a discontinuity (or at least a very rapid variation) of the first derivative with respect to temperature.

1. INTRODUCTION

THE surface tension of a liquid is a measure of the free energy associated with unit area of its surface. One of us has suggested[1] that, in the particular case of liquid helium, an important contribution to this surface free energy comes from surface modes of vibration. An exact quantum-mechanical treatment of these surface modes would be difficult, but a simple approach is to assume that they are similar to macroscopic capillary waves which have a frequency-dependent velocity given by the equation

$$c=(2\pi\sigma\nu/\rho)^{1/3}, \quad (1)$$

where σ is the surface tension, ρ the density of the liquid, and ν the frequency of the wave. The assumption that Eq. (1) describes surface modes of all frequencies with a constant value of σ is analogous to the assumption in the Debye theory of the specific heats of solids that the modes of vibration of a solid lattice are sound waves and that there is no dispersion. As in the Debye theory, it is reasonable to assume that the highest frequency mode has a wavelength comparable with the average distance between neighboring atoms. This gives a cutoff frequency $\nu_c\approx 1.5\times10^{11}$ sec^{-1} and a characteristic temperature $\theta_c=h\nu_c/k\approx 7$°K.

The zero-point energy of the surface modes is then found to represent about 60% of the total measured surface tension, emphasizing the importance of these modes. At temperatures small compared with the characteristic temperature ($T\ll\theta_c$), the extra energy per unit area excited in the surface modes is

$$U=2.07(\rho/\sigma)^{2/3}h(k/h)^{7/3}T^{7/3}. \quad (2)$$

The corresponding free energy is

$$U-TS=-\tfrac{3}{4}U \quad (3)$$

$$=-1.55(\rho/\sigma)^{2/3}h(k/h)^{7/3}T^{7/3}. \quad (4)$$

Assuming that this is the only factor influencing the variation of surface tension with temperature, the surface tension σ at a temperature $T(\ll\theta_c)$ is related to the surface tension σ_0 at 0°K by

$$\sigma=\sigma_0-1.55(\rho/\sigma_0)^{2/3}h(k/h)^{7/3}T^{7/3}. \quad (5)$$

Equation (5) is in approximate agreement with previous measurements of the surface tension.[2–4] However, all these measurements were made above 1°K and cannot be unambiguously extrapolated to 0°K to obtain

† Supported by grants from the National Science Foundation and the Research Corporation. This report is based on a thesis submitted to the University of Pennsylvania in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
* Present address: Institute for the Study of Metals, University of Chicago, Chicago, Illinois.
[1] K. R. Atkins, Can. J. Phys. **31**, 1165 (1953).

[2] A. T. van Urk, W. H. Keesom, and H. Kamerlingh Onnes, Proc. Roy. Akad. (Amsterdam) **28**, 58 (1925).
[3] J. F. Allen and A. D. Misener, Proc. Cambridge Phil. Soc. **34**, 299 (1938).
[4] K. N. Zinoveva, Zh. Eksperim. i Teor. Fiz. **29**, 899 (1955) [English transl.: Soviet Phys.—JETP **2**, 774 (1956)].