Bose-Einstein Condensation in Narrow Channels*

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The problem of the transition temperature of liquid helium in narrow channels has been considered by using the ideal Bose-Einstein gas as a model. It has been found that for a channel of square cross section ($D\times D$) and length L, the transition temperature T_c is given by the expression $T_c/T_{cB}=(0.84\ \text{Å}^{-1})D^2/L$ under the conditions that $D^2/L\ll\lambda$, where λ is the thermal de Broglie wavelength, and T_{cB} is the transition temperature in the case of an infinite volume.

I. INTRODUCTION

MEASUREMENTS[1,2] on unsaturated liquid-He^4 films have shown that the superfluid transition temperature T_c decreases with decreasing thickness of the film. Similarly, when liquid He^4 is made to flow through narrow capillaries,[3,4] one finds that T_c decreases with decreasing diameter of the capillary. Based on the suggestion by London[5] and Tisza[6] that the transition in liquid helium and Bose-Einstein condensation are related phenomena, several calculations have been made in an effort to account for the depression in the transition temperature. Ziman[7] and Singh *et al.*[8] have made independent calculations for a model of an ideal Bose-Einstein gas confined to a thin film. Ziman succeeds in finding results which are in qualitative agreement with the observed depression, while Singh *et al.* do not. Mills,[9] on the other hand, considered the case of an ideal Bose-Einstein gas confined to narrow channels. His calculation, however, is not applicable to the prediction of the variation of T_c because of the approximations that were made.

In this paper, the problem of the transition temperature of liquid helium in narrow channels will be considered in an attempt to find an explicit expression for the size dependence of T_c. As an approximation to the He^4 system, the model of an ideal Bose-Einstein gas is used.

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II. THEORY

Consider a system of N noninteracting Bose-Einstein particles, each of mass M, confined in a narrow channel of dimensions $D\times D\times L$, where $L\gg D$. Assume the wave function of the particles to vanish at the boundaries along the smaller dimensions D, and to be periodic at the boundary of the dimension L. The periodic boundary condition is chosen at the ends of the channel because it corresponds more nearly to flow through a channel. Under these conditions, the energy eigenvalues are easily shown to be

$$E_{lmn}=(l^2+m^2)\Delta+n^2\delta\,, \tag{1}$$

where $\Delta=(h^2/8M)(1/D^2)$ and $\delta=(h^2/2M)(1/L^2)$, and the quantum numbers l, m, n can take on the values

$$l,m=1,\,2,\,3,\,\cdots \tag{2}$$

and

$$n=0,\,\pm1,\,\pm2,\,\cdots.$$

For considerations at very low temperatures, it is only important to consider the distribution of the lowest lying energy levels. The energy differences between the first low-lying levels and the ground state

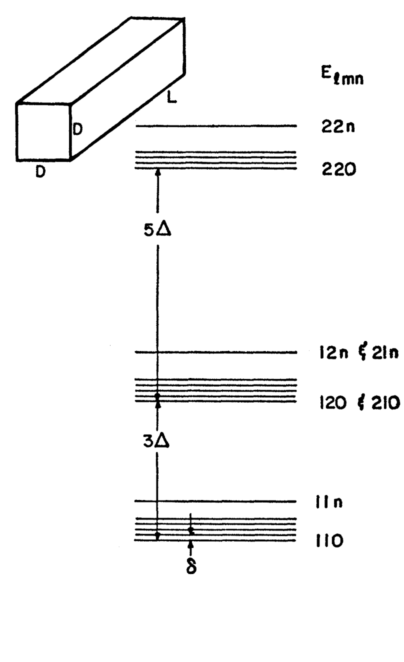

FIG. 1. Schematic of energy levels E_{lmn} for a particle in a box of size $D\times D\times L$, with $L\gg D$.

$l=1$, $m=1$, $n=0$ are

$$E_{11n}-E_{110}=n^2\delta \qquad (3)$$

and

$$E_{1m0}-E_{110}=E_{l10}-E_{110}=(m^2-1)\Delta. \qquad (4)$$

The energy-level diagram is indicated schematically in Fig. 1. For a typical channel with $D=50$ Å, and $L=50\,000$ Å, $\delta=1.3\times10^{-23}$ ergs, $\Delta=3.3\times10^{-18}$ ergs, and $\delta/\Delta=3\times10^{-6}$. Therefore, the $11n$ level series can be treated as a continuum of levels above the ground state. This fact will be used later to replace by integrations all summations over the quantum number n.

For an ideal Bose-Einstein assembly of particles, the average number of particles in an energy level E_{lmn} having degeneracy g_{lmn} is

$$N_{lmn}=\frac{g_{lmn}}{\exp[\alpha+(E_{lmn}-E_{110})/kT]-1}, \qquad (5)$$

where α is a parameter determined by the total number of particles. Therefore, the number of particles in the ground state is

$$N_{110}=1/(e^\alpha-1), \qquad (6)$$

where the permissible values of α are those for which $\alpha>0$, because N_{110} cannot be negative.

The number of particles in the excited states can be written as

$$N_{\text{exc}}=\sum_{n\neq0}N_{11n}+\sum_{n=-\infty}^{+\infty}(N_{12n}+N_{21n})+\sum_{l=2}^{\infty}\sum_{m=2}^{\infty}\sum_{n=-\infty}^{+\infty}N_{lmn}$$

$$=2\sum_{n=1}^{\infty}\left\{\exp\left[\alpha+\frac{n^2\delta}{kT}\right]-1\right\}^{-1}+2\sum_{n=-\infty}^{+\infty}\left\{\exp\left[\alpha+\frac{3\Delta+n^2\delta}{kT}\right]-1\right\}^{-1}$$

$$+\sum_{l=2}^{\infty}\sum_{m=2}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\exp\left[\alpha+(l^2+m^2-2)\frac{\Delta}{kT}+\frac{n^2\delta}{kT}\right]-1\right\}^{-1}. \qquad (7)$$

Since $\alpha>0$, Eq. (7) can be replaced, following London,[5] by an inequality

$$N_{\text{exc}}<2\sum_{n=1}^{+\infty}\left\{\exp\left[\frac{n^2\delta}{kT}\right]-1\right\}^{-1}+2\sum_{n=-\infty}^{+\infty}\left\{\exp\left[\frac{3\Delta}{kT}+\frac{n^2\delta}{kT}\right]-1\right\}^{-1}+\sum_{l=2}^{\infty}\sum_{m=2}^{\infty}\sum_{n=-\infty}^{\infty}\left\{\exp\left[(l^2+m^2-2)\frac{\Delta}{kT}+\frac{n^2\delta}{kT}\right]-1\right\}^{-1} \qquad (8)$$

$$\equiv\Sigma_1+\Sigma_2+\Sigma_3. \qquad (9)$$

The summations over n in Σ_1, Σ_2, and Σ_3 can be replaced by integrals. In Σ_1, it is an incomplete Bose-Einstein integral which is evaluated as a power series in $\xi_0=(\delta/kT)\ll1$ in the Appendix. Similarly, Σ_2 and Σ_3 are also evaluated in the Appendix. These summations from the Appendix [Eqs. (A4), (A6), (A10)] in the case when $6\Delta/kT\ll1$ reduce to

$$\Sigma_1=2\left(\frac{kT}{\delta}\right)\left[1-\pi\left(\frac{1}{96}+\frac{1}{\sqrt2}\right)\frac{\lambda}{L}+\frac{\pi}{2}\left(\frac{\lambda}{L}\right)^2+\cdots\right], \qquad (10)$$

$$\Sigma_2=2\left(\frac{kT}{\delta}\right)\left[\frac{2\pi}{\sqrt3}\frac{D}{L}-\frac{4}{3}\frac{D^2}{L^2}-1.46\pi\frac{\lambda}{L}+\cdots\right], \qquad (11)$$

and

$$\Sigma_3\leqq2\left(\frac{kT}{\delta}\right)\left[\frac{5.22\pi D^2}{\lambda L}+\cdots\right], \qquad (12)$$

where $\lambda=h/(2\pi MkT)^{1/2}$ is the thermal de Broglie wavelength. On substituting these into Eq. (9), one obtains

$$N_{\text{exc}}<2\left(\frac{kT}{\delta}\right)\left[1+\frac{5.22D^2}{\lambda L}+\frac{2\pi}{\sqrt3}\frac{D}{L}-\pi\left(\frac{1}{96}+\frac{1}{\sqrt2}-1.46\right)\frac{\lambda}{L}+\cdots\right]. \qquad (13)$$

So long as the second, third, etc., terms within the square brackets are much less than unity, Eq. (13) can be written as

$$N_{\text{exc}}<2(kT/\delta). \qquad (14)$$

When most of the particles are still in the states other than the ground state, $N_{\text{exc}}\approx N=\rho D^2L$, where N is the total number of particles and ρ the number density of particles. Therefore, the limiting temperature down to which the relation (14) can hold is

$$T_c=\frac{\delta}{2k}\rho D^2L=\frac{h^2\rho D^2}{4MkL}. \qquad (15)$$

On the other hand, the transition temperature for the bulk case is given by[5]

$$T_{cb}=\frac{h^2}{2\pi Mk}\left(\frac{\rho}{2.612}\right)^{2/3}. \tag{16}$$

Hence, the reduced critical temperature t_c is given by

$$t_c=T_c/T_{cb}=\tfrac{1}{2}\pi(2.612)^{2/3}\rho^{1/3}(D^2/L). \tag{17}$$

Substituting the value $\rho=2.25\times10^{22}$ particles/cm^3, which is typical of liquid He^4, one gets

$$t_c=0.84(D^2/L), \tag{18}$$

where D and L are expressed in angstroms. Equation (18) suggests that in the case of a channel having arbitrary cross-sectional shape, one would get

$$t_c=f(A/L), \tag{19}$$

where A = cross-sectional area of the channel and f is a form factor characteristic of the shape of the channel.

III. DISCUSSION

The main purpose of the present calculation is not to predict the precise transition temperature of a Bose-Einstein gas, but rather to find out its size dependence when confined in narrow channels. It is remarked in passing that even when vanishing boundary conditions at the ends of the channel are assumed, a similar dependence on geometry is obtained, namely $t_c\approx(0.84/\ln 3)(D^2/L)$.

The limits under which Eq. (18) is valid are

$$\text{(i)}\quad D^2/L\ll 6\times10^{-2}\lambda \quad\text{when}\quad D\gg\lambda$$

and

$$\text{(ii)}\quad \frac{D^2}{L}\ll\frac{\exp[(3\pi/2)(\lambda^2/D^2)]}{\pi}\lambda \quad\text{when}\quad D\ll\lambda.$$

Condition (i) is valid at high temperatures (i.e., $kT\gg6\Delta$) and is obtained from the neglect of terms other than the first in Eq. (13). Condition (ii) is valid at low temperatures (i.e., $kT\ll6\Delta$). These conditions assume that T_c is always less than T_{cB}. In addition, it is noted that the conditions are easier to satisfy at the lower temperatures.

The somewhat surprising result that the transition temperature varies inversely with length can be understood by examining Eq. (15) which shows a direct dependence on the product of the volume (D^2L) and the energy separation (δ) of the first excited state. Because the energy separation in the limit considered in this paper varies as L^{-2}, the product with the volume yields an L^{-1} dependence for T_c.

The previous attempt[9] at the solution of ideal Bose-Einstein gas in a narrow channel was confined to computing the occupation of the ground state as a function of temperature. No objective criterion for determining a transition temperature from such curves is possible because the ground state is partially occupied at all temperatures. The correct approach is that given by London. One computes the number of particles in the excited states which must be less than the total number. At a particular temperature (T_c) this inequality can only be satisfied by putting additional particles in the ground state over those already present.

Experiments on the flow of liquid He^4 through Vycor glass indicate that the transition temperature for superflow increases with the diameter of the pores. A quantitative comparison with Eq. (18) cannot be made because the geometry of the pores in the Vycor glass is extremely complex and only a rough estimate of the average pore diameter is reported. In addition, there is no possibility of making an estimate of the length L of the pores in Vycor.

Edwards *et al.*[10] have recently studied the superflow of liquid He^4 through closely packed Saran charcoal, another porous material having average pore diameter of 10–15 Å. They observe superflow through it to within 10 mdeg of the bulk transition temperature. This result is not inconsistent with the results of this paper because the effects of D and L on t_c in Eq. (18) tend to oppose each other and the expected depression in the transition temperature due to the area could be nullified by a short length.

In order to verify the size dependence predicted by Eq. (18), it is imperative to obtain materials with uniform channels, whose diameters and lengths can be estimated and controlled accurately.

Calculations based on ideal Bose-Einstein gas model have been attempted by Ziman[7] and Singh *et al.*[8] to account for the depression in transition temperature of liquid He^4 when it is confined to thin films. Ziman obtains a transcendental equation between t_c, D, and L, and uses an adjustable parameter to fit the solutions of this equation with the observed depression. On the other hand, Singh *et al.* get a result which is strongly dependent on the nature of boundary conditions they

[10] M. H. Edwards, A. S. McKirdy, and W. C. Woodbury, Proceedings of the Ninth International Conference on Low Temperature Physics, 1964, Columbus, Ohio (to be published).

choose. It is to be noted that if the methods used in the present calculation are applied for the case of a thin film of dimensions $D\times L\times L$, where $L\gg D$, it is found that the transition temperature is given by $t_c=(T_c/T_{cB})\approx 0.9\times 10^{-2}D(\text{Å})$. This result is in qualitative agreement with the experimental results far below the bulk transition temperature.

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Note added in proof. In a private communication, P. R. Zilsel has pointed out that the inequality (14) is exhausted by the first term of Σ_1, viz.,

$$\frac{2}{e^{\xi_0}-1}\rightarrow\frac{2}{\xi_0}=2\frac{kT}{\delta}.$$

Indeed, for small ξ_0,

$$\Sigma_1=2\sum_{n=1}^{\infty}\frac{1}{e^{n^2\xi_0}-1}\rightarrow\frac{2}{\xi_0}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=1.65,$$

while

$$\int_1^{\infty}\frac{1}{n^2}dn=1.$$

So in converting the sum Σ_1 into an integral, a certain factor has been lost, and the factor of 2 in (14) should be somewhat larger. However, this does not affect the dependence of t_c on D and L.

APPENDIX

Replacement of the summation over n in Σ_1 by an integral gives

$$\Sigma_1=\left(\frac{kT}{\delta}\right)^{1/2}\int_{\xi_0}^{\infty}\frac{\xi^{-1/2}d\xi}{e^{\xi}-1},\qquad\text{(A1)}$$

where $\xi_0=\delta/kT$,

$$=\left(\frac{kT}{\delta}\right)^{1/2}\int_{\xi_0}^{\infty}\xi^{-1/2}\sum_{p=1}^{\infty}e^{-p\xi}d\xi.\qquad\text{(A2)}$$

Using

$$\sum_{m=a}^{b}f(m)=\int_{a-1/2}^{b+1/2}f(m)dm-f'(m)/24|_a{}^b,$$

if $f(n+\frac{1}{2})-f(n-\frac{1}{2})-f'(n)\ll f(n)$, where $a<n<b$, (A2) reduces to

$$\Sigma_1=2\left(\frac{kT}{\delta}\right)\int_{\xi_0}^{\infty}\xi^{-1/2}\left(\xi^{-1}e^{-\xi/2}-\frac{1}{24}\xi e^{-\xi}\right)d\xi.\qquad\text{(A3)}$$

Expanding the incomplete gamma integrals as a power series[11] in ξ_0, and rearranging the terms, (A3) reduces to

$$\Sigma_1=2\left(\frac{kT}{\delta}\right)\left[1-\sqrt{\pi}\left(\frac{1}{96}+\frac{1}{\sqrt{2}}\right)\left(\frac{\delta}{kT}\right)^{1/2}+\frac{1}{2}\left(\frac{\delta}{kT}\right)+\cdots\right].\qquad\text{(A4)}$$

Again, the replacement in Σ_2 of the summation over n by an integral gives,

$$\Sigma_2=2\left(\frac{kT}{\delta}\right)^{1/2}\int_0^{\infty}\frac{\xi^{-1/2}d\xi}{\exp(\xi+3\Delta/kT)-1}-\frac{2}{e^{3\Delta/kT}-1}.\qquad\text{(A5)}$$

Using the power series expansion[12] of the Bose-Einstein integral when $3\Delta/kT\ll 1$, one obtains

$$\Sigma_2=2\left(\frac{kT}{\delta}\right)\left[\frac{\pi}{\sqrt{3}}\left(\frac{\delta}{\Delta}\right)^{1/2}-\frac{1}{3}\frac{\delta}{\Delta}-1.46\sqrt{\pi}\left(\frac{\delta}{kT}\right)^{1/2}+0.208\times 3\sqrt{\pi}\left(\frac{\delta}{kT}\right)^{1/2}\left(\frac{\Delta}{kT}\right)+\cdots\right].\qquad\text{(A6)}$$

Similarly, Σ_3 can be written as

$$\Sigma_3=\left(\frac{kT}{\delta}\right)^{1/2}\sum_{l=2}^{\infty}\sum_{m=2}^{\infty}\int_0^{\infty}\frac{\xi^{-1/2}d\xi}{\exp(\xi+\eta)-1}-\sum_{l=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{\exp\eta-1},\qquad\text{(A7)}$$

where $\eta=(l^2+m^2-2)(\Delta/kT)$. To estimate Σ_3, replace

[11] Jahke and Emde, *Tables of Higher Functions* (McGraw-Hill Book Company, Inc., New York, 1960), p. 14.

[12] F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1954), Vol. II, p. 203.

the double summation by an integral to get

$$\Sigma_3=\pi\frac{kT}{\Delta}\left(\frac{kT}{\delta}\right)^{1/2}\int_{6\Delta/kT}^{\infty}d\eta\int_0^{\infty}\frac{\xi^{-1/2}d\xi}{\exp(\xi+\eta)-1}-\frac{\pi kT}{\Delta}\int_{6\Delta/kT}^{\infty}\frac{d\eta}{\exp(\eta)-1}$$
$$=\pi\frac{kT}{\Delta}\left(\frac{kT}{\delta}\right)^{1/2}\int_0^{\infty}\xi^{-1/2}d\xi\int_0^{\infty}\frac{dx}{\exp[x+\xi+6\Delta/kT]-1}+\frac{\pi kT}{\Delta}\ln\left[1-\exp\left(-\frac{6\Delta}{kT}\right)\right]$$
$$=\pi\frac{kT}{\Delta}\left(\frac{kT}{\delta}\right)^{1/2}\int_0^{\infty}\xi^{-1/2}\left[-\ln\left(1-\exp\left[-\xi-\frac{6\Delta}{kT}\right]\right)\right]d\xi+\frac{\pi kT}{\Delta}\ln\left[1-\exp\left(-\frac{6\Delta}{kT}\right)\right]$$
$$=\pi\frac{kT}{\Delta}\left(\frac{kT}{\delta}\right)^{1/2}\int_0^{\infty}\xi^{-1/2}d\xi\sum_{p=1}^{\infty}\frac{\exp[-p(\xi+6\Delta/kT)]}{p}-\frac{\pi kT}{\Delta}\sum_{p=1}^{\infty}\frac{\exp(-p6\Delta/kT)}{p}$$
$$=\pi^{3/2}\frac{kT}{\Delta}\left(\frac{kT}{\delta}\right)^{1/2}\sum_{p=1}^{\infty}\frac{\exp[-p6\Delta/kT]}{p^{3/2}}-\frac{\pi kT}{\Delta}\sum_{p=1}^{\infty}\frac{\exp(-p6\Delta/kT)}{p}\leq\pi^{3/2}S\frac{kT}{\Delta}\left(\frac{kT}{\delta}\right)^{1/2}, \quad \text{(A8)}$$

where

$$S=\sum_{p=1}^{\infty}\frac{\exp(-p6\Delta/kT)}{p^{3/2}}. \quad \text{(A9)}$$

If $6\Delta/kT\ll1$,

$$S\approx\zeta(\tfrac{3}{2})=2.61. \quad \text{(A10)}$$

On the other hand, when $(3\Delta/kT)\gg1$, appropriate series expansions[12] yield

$$\Sigma_2=2\left(\frac{kT}{\delta}\right)^{1/2}\pi^{1/2}\sum_{p=1}^{\infty}\frac{\exp(-p3\Delta/kT)}{p^{1/2}}\approx2\left(\frac{kT}{\delta}\right)\sqrt{\pi}\left(\frac{\delta}{kT}\right)^{1/2}\exp\left(-\frac{3\Delta}{kT}\right) \quad \text{(A11)}$$

and

$$\Sigma_3=2\left(\frac{kT}{\delta}\right)\left[\frac{\pi^{3/2}}{2}\left(\frac{\delta}{kT}\right)^{1/2}\frac{kT}{\Delta}\exp\left(-\frac{6\Delta}{kT}\right)\right]. \quad \text{(A12)}$$

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Electron Interactions in Cryogenic Helium Plasmas*

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Microwave-propagation and microwave-interaction techniques have been used to determine the electron collision frequency for momentum transfer in helium for electron energies in the vicinity of 0.001 eV. Measurements of the complex microwave conductivity and electron-energy relaxation rates have been performed in the afterglow of a pulsed discharge in helium in the pressure range 0.1 to 5 Torr submerged in a bath of liquid helium at 4.2°K. Electron-radiation temperature measurements during plasma decay have demonstrated monotonically decreasing electron temperatures as a function of time. For times when an extrapolation of the electron temperature decay indicated near thermal equilibrium with the parent gas, a momentum-transfer cross section in the range 10×10^{-16} to 19×10^{-16} cm^2 was determined. Measurements of electron-energy relaxation rates for atomic densities exceeding 2.3×10^{18} cm^{-3}, where the electron de Broglie wavelength is becoming long in comparison to the average inter-scatterer spacing, indicate the limit of validity of binary-collision concepts.

I. INTRODUCTION

THE elastic scattering of low-energy electrons by helium atoms in the ground state has received considerable attention in the past with a number of authors attempting to predict the behavior of the scattering cross section in the limit of zero electron energy. The calculations of Morse and Allis,[1] using an exchange approximation, have been shown to be in good agreement with the experimentally determined cross section over the energy range 1 to 40 eV. Their calculations indicate a cross section rising with decreasing electron

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