PHYSICAL REVIEW VOLUME 138, NUMBER 2A 19 APRIL 1965

Two-Particle Collisions. I. General Relations for Collisions in the Laboratory System

MICHAŁ GRYZIŃSKI
Institute of Nuclear Research, Świerk k.Otwocka, Poland
(Received 1 July 1964)

The relations between the energies and angles for two-particle collisions are given for the case of conservative central forces. They have been derived in the laboratory system of coordinates, since this system is of prime importance for the interpretation of the experimental results. The dependence on the law of interaction enters the obtained relations through the function for the c.m.-system scattering angle. Some remarks on the statistics of encounters are given, and the cross sections for collisions in which the colliding particles experience given changes in energy and direction are derived. These cross sections can be used to determine the cross sections describing various physical processes. On the basis of the cross sections derived for the scattering problem, one can suggest that the "diffraction" pattern of scattered particles is due to quantized states of the scattering particles, rather than to the law of interaction.

I. INTRODUCTION

DESPITE the fact that the two-body problem is fundamental for physical science, it has not been solved in the general case. Moreover, in the relatively simple case of Coulomb interaction, there are always great difficulties involved in the description of phenomena connected with two-particle collisions if they are not considered in the center-of-mass system. The laboratory system plays an important role in the interpretation of the experimental results, since all the observations of collision processes are made in that frame of reference.

The peculiar role of a fixed frame of reference was pointed out by Chandrasekhar, who derived some relations for the two-body problem in the laboratory system.[1–3] Nevertheless, the complete set of explicit relations between the geometrical variables (determining the motion of noninteracting particles) and the dynamical variables (determining the changes in their motion due to interaction forces) have not been given. These relations alone are not sufficient to describe all phenomena. It is necessary to know in addition the probabilities (cross sections) of a collision with a given change in the state of both particles. A partial solution of this problem has been given by the author for the Coulomb interaction.[4] Our problem now is to find the relations between the geometrical and dynamical variables in the general case of conservative central forces on the basis of the laws of conservation of energy and momentum, and to use them to derive the cross sections as the functions of the dynamical variables.

II. FORMULATION OF THE PROBLEM

Consider an encounter between two particles of masses m_1 and m_2. As a result of the collision, the velocities of both particles change in direction and in magnitude. We denote their velocity vectors before the encounter by $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, and after the encounter by $\mathbf{v}_1'$ and $\mathbf{v}_2'$.

We take the z axis of the laboratory system to coincide with the initial direction of particle 2, which, following the terminology of Chandrasekhar,[5] will be called the *test particle*. As a result of the interaction with particle 1, which we describe as the *field particle*, the velocity of particle 2 changes in magnitude (the particle energy changes) and in direction (the particle is scattered). We denote the change in energy of the test particle by ΔE, and the direction of the velocity after the collision by the angles ϑ and ϕ (Fig. 1). Similarly, we can assign the values $\Delta\tilde{E}, \tilde{\vartheta}, \tilde{\phi}$ for the field particle. It is quite obvious that the result of the collision will depend both on the interaction law between the particles and on the geometry of the encounter. To describe the geometry of the encounter, we need four geometrical quantities (one linear quantity and three angular ones). These are (Fig. 2):

(a) the collision parameter D, which would be the minimum distance between the particles had they not interacted (it should be stressed that this quantity does not coincide with the distance between the trajectories of the two particles);

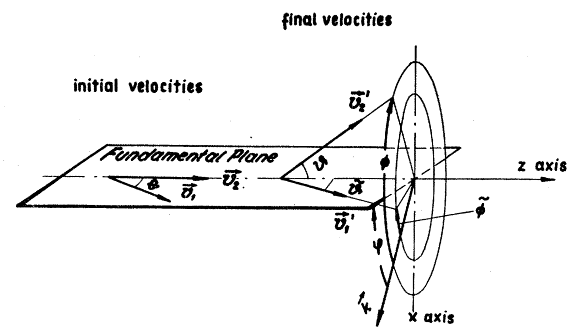

FIG. 1. Orientation in space (with respect to the laboratory system determined by the z axis and the vector $\mathbf{k}$) of initial and final velocities of colliding particles. The initial velocity vectors of the two particles define the plane which is called the fundamental plane.

[1] S. Chandrasekhar, Astrophys. J. **93**, 285 (1941).
[2] R. E. Williamson and S. Chandrasekhar, Astrophys. J. **93**, 305 (1941).
[3] S. Chandrasekhar, Astrophys. J. **93**, 323 (1941).
[4] M. Gryziński, Phys. Rev. **115**, 374 (1959).
[5] S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, 1942), p. 89.

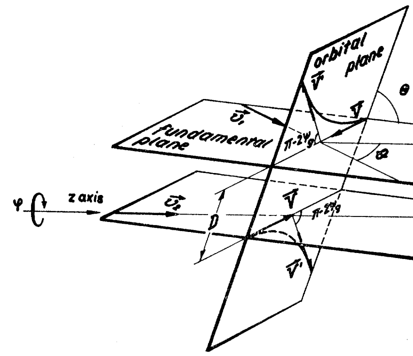

FIG. 2. Space diagram of an encounter. The relative motion of both particles takes place in the plane which is called the orbital plane. The dependence on the law of interaction is completely determined by the scattering angle in the orbital plane.

(b) the angle θ, which is the angle between the initial velocity vectors of both particles produced from one point (the plane determined by these two vectors is called the fundamental plane);

(c) the angle Θ, which is the angle formed by the segment D with the fundamental plane;

(d) the angle φ, which is the angle describing the position of the fundamental plane with respect to rotation about the z axis.

With the aid of Fig. 2, we can readily establish the range of variability of these quantities:

$$\begin{aligned} &0\leqslant D\leqslant \infty\,,\\ &0\leqslant \theta\leqslant \pi\,,\\ &0\leqslant \Theta\leqslant 2\pi\,,\\ &0\leqslant \varphi\leqslant 2\pi\,. \end{aligned} \tag{1}$$

Now we shall attempt to derive the dependence of the dynamical variables ΔE, ϑ, ϕ, $\Delta\tilde{E}$, $\tilde{\vartheta}$, $\tilde{\phi}$ which describe the changes in states of colliding particles upon the initial conditions expressed by the geometrical variables θ, Θ, φ, and D.

III. DYNAMICS OF A TWO-PARTICLE ENCOUNTER

According to the elementary theory of the two-body problem (Fig. 3),[6] the velocity of the center of mass of the colliding particles remains constant during the encounter

$$m_1\mathbf{v}_1+m_2\mathbf{v}_2=(m_1+m_2)\mathbf{V}_g=m_1\mathbf{v}_1'+m_2\mathbf{v}_2'\,. \tag{2}$$

Hence we can write

$$V_g^2=M_1^2v_1^2+M_2^2v_2^2+2M_1M_2v_1v_2\cos\theta\,, \tag{3}$$

where the following notation has been introduced:

$$M_1=m_1/(m_1+m_2),\quad M_2=m_2/(m_1+m_2)\,. \tag{4}$$

[6] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1963), pp. 53–58.

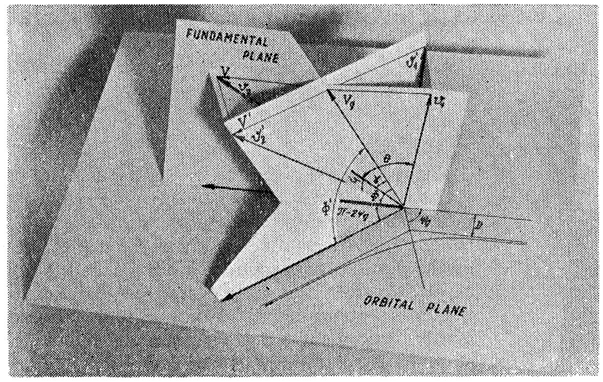

FIG. 3. Vector model of two-particle collision.

Let $\mathbf{V}$ and $\mathbf{V}'$ denote the initial and final velocities of the second particle relative to the first:

$$\mathbf{V}=\mathbf{v}_2-\mathbf{v}_1,\quad \mathbf{V}'=\mathbf{v}_2'-\mathbf{v}_1'\,; \tag{5}$$

then by means of (2) and (5) we can express $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_1'$, and $\mathbf{v}_2'$ in terms of $\mathbf{V}_g$, $\mathbf{V}$, $\mathbf{V}'$. Thus,

$$\mathbf{v}_1=\mathbf{V}_g-M_2\mathbf{V},\quad \mathbf{v}_1'=\mathbf{V}_g-M_2\mathbf{V}'\,; \tag{6}$$

$$\mathbf{v}=\mathbf{V}_g+M_1\mathbf{V},\quad \mathbf{v}_2'=\mathbf{V}_g+M_1\mathbf{V}'\,. \tag{7}$$

Since the potential energy of the two particles is zero, both before and after the encounter, the law of conservation of energy gives us

$$\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2=\tfrac{1}{2}m_1v_1'^2+\tfrac{1}{2}m_2v_2'^2\,.$$

Hence, using (6) and (7), it is easy to show that $V=V'$, so that the relative velocity *is changed* by the encounter only *in direction* and *not in magnitude*. The dynamical effect of the encounter is therefore known when the change in direction of V is determined.

a. Calculation of ΔE

From the definition we have

$$\Delta E=\tfrac{1}{2}m_2v_2'^2-\tfrac{1}{2}m_2v_2^2\,. \tag{8}$$

Hence, ΔE is positive if the test particle gains energy in collision, and is negative if it loses energy. Squaring relations (6) and (7), we readily obtain

$$v_2^2=V_g^2+2M_1V_gV\cos\Phi+M_1^2V^2\,, \tag{9}$$

$$v_1^2=V_g^2-2M_2V_gV\cos\Phi+M_2^2V^2\,, \tag{10}$$

where Φ is the angle between $\mathbf{V}_g$ and $\mathbf{V}$ (see Fig. 3). Similarly, after the encounter, we have

$$v_2'^2=V_g^2+2M_1V_gV'\cos\Phi'+M_1^2V'^2\,, \tag{11}$$

where Φ' is the angle between $\mathbf{V}_g$ and $\mathbf{V}'$. Also

$$\Delta E=\frac{m_1m_2}{m_1+m_2}V_gV(\cos\Phi'-\cos\Phi)\,. \tag{12}$$

Solving (10) with respect to $\cos\Phi$ and using (3) and (5), we can write

$$\cos\Phi=\frac{m_2v_2^2-m_1v_1^2+(m_1-m_2)v_1v_2\cos\theta}{(m_1+m_2)V_gV}, \quad (13)$$

$$\sin\Phi=v_1v_2\sin\theta/V_gV. \quad (14)$$

Since (see Fig. 3)

$$\cos\Phi'=\cos\Phi\cos(\pi-2\Psi_g)+\sin\Phi\sin(\pi-2\Psi_g)\cos\Theta, \quad (15)$$

we finally obtain

$$\Delta E=-2\frac{m_1m_2}{m_1+m_2}V_gV\cos^2\Psi_g(\cos\Phi-\sin\Phi\cos\Theta\tan\Psi_g) \quad (16)$$

or, in implicit form,

$$h_{\Delta E}(\theta,\Theta,\Psi_g)\equiv\Delta E+b\cos^2\Psi_g-2a\sin\Psi_g\cos\Psi_g\cos\Theta=0, \quad (17)$$

where

$$a=\mu v_1v_2\sin\theta,\qquad b=K_{12}[\tfrac{1}{2}m_2v_2^2-\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}(m_1-m_2)v_1v_2\cos\theta], \quad (18)$$

and

$$\mu=(m_1m_2)/(m_1+m_2),\qquad K_{12}=4(m_1m_2)/(m_1+m_2)^2. \quad (19)$$

The subscript of h, in formula (17) and hereafter, indicates the variable for which the equation $h=0$ is to be solved.

Expressing $\sin\theta$ and $\cos\theta$ in terms of relative velocity V, (18) can be rewritten in the alternative form

$$a=\tfrac{1}{2}\mu(-V^4+2V^2(v_1^2+v_2^2)-(v_2^2-v_1^2)^2)^{1/2}, \quad (20)$$

$$b=\mu\left(v_2^2-v_1^2+\frac{m_2-m_1}{m_2+m_1}V^2\right), \quad (21)$$

which is of prime importance for the interpretation of some relations and can also facilitate some transformations.

b. Calculation of the Scattering Angle ϑ

From the definition of the scattering angle ϑ we have $\cos\vartheta=\mathbf{V}_2\cdot\mathbf{V}_2'/V_2V_2'$ taking into account (7) and using the vector model of the collision (Fig. 3), we can write

$$v_2'\cos\vartheta=M_1V\cos(\mathbf{V}'\mathbf{v}_2)+V_g\cos\gamma. \quad (22)$$

From the spherical triangle $\mathbf{V}'\mathbf{V}\mathbf{v}_2O$ we obtain

$$\cos(\mathbf{V}'\mathbf{v}_2)=\cos(\Phi-\gamma)\cos(\pi-2\Psi_g)+\sin(\Phi-\gamma)\sin(\pi-2\Psi_g)\cos\Theta. \quad (23)$$

It follows from triangle $\mathbf{V}O\mathbf{v}_2$ that

$$\cos(\Phi-\gamma)=(v_2-v_1\cos\theta)/V, \quad (24)$$

$$\sin(\Phi-\gamma)=v_1\sin\theta/V. \quad (25)$$

Taking the scalar product of $\mathbf{v}_2$ and $\mathbf{V}_g$, we obtain

$$\cos\gamma=(1/V_g)(M_1v_1\cos\theta+M_2v_2),\qquad \sin\gamma=(1/V_g)M_1v_1\sin\theta. \quad (26)$$

Equation (22), (23), (24), (25), and (26) finally yield

$$v_2'\cos\vartheta=v_2-2M_1(v_2-v_1\cos\theta)\cos^2\Psi_g+2M_1v_1\sin\theta\sin\Psi_g\cos\Psi_g\cos\Theta \quad (27)$$

or, in implicit form,

$$h_{\cos\vartheta}(\theta,\Theta,\Psi_g)\equiv\cos\vartheta-\frac{1-2(m_1/(m_1+m_2))[(1-(v_1/v_2)\cos\theta)\cos^2\Psi_g-(v_1/v_2)\sin\theta\sin\Psi_g\cos\Psi_g\cos\Theta]}{\left\{1+K_{12}\left[\frac{m_1v_1^2}{m_2v_2^2}-1+\left(1-\frac{m_1}{m_2}\right)\frac{v_1}{v_2}\cos\theta\right]\cos^2\Psi_g+4\frac{m_1}{m_1+m_2}\frac{v_1}{v_2}\sin\theta\sin\Psi_g\cos\Psi_g\cos\Theta\right\}^{1/2}}=0. \quad (28)$$

It is sometimes more convenient to have the scattering angle as a function of ΔE. Solving (17) for $\cos\Theta$ and substituting into (28), we find that

$$h_{\cos\vartheta}(\theta,\Delta E,\Psi_g)\equiv\cos\vartheta-\frac{1}{(1+\Delta E/E_2)^{1/2}}\times\left[1+\frac{1}{2}\frac{\Delta E}{E_2}-2\left(\frac{m_1}{m_1+m_2}\right)^2\left(\frac{V}{v_2}\right)^2\cos^2\Psi_g\right]=0, \quad (29)$$

where $E_2=\frac{1}{2}m_2v_2^2$.

c. Calculation of the Azimuthal Scattering Angle ϕ

The spherical triangle $\mathbf{v}_2'\mathbf{v}_2\mathbf{V}_gO$ gives us

$$\cos(\mathbf{V}_g\mathbf{v}_2')=\cos\gamma\cos\vartheta+\sin\gamma\sin\vartheta\cos(\phi-\varphi). \quad (30)$$

On the other hand, we have

$$\cos(\mathbf{V}_g\mathbf{v}_2')=V_g/v_2'+M_1(V/v_2')\cos\Phi'. \quad (31)$$

Taking into account (12), (30), and (31), we can write

$$\frac{V_g}{v_2'}+M_1\frac{V}{v_2'}\left(\frac{m_1+m_2}{m_1m_2}\frac{\Delta E}{V_gV}+\cos\Phi\right)=\cos\gamma\cos\vartheta+\sin\gamma\sin\vartheta\cos(\phi-\varphi). \quad (32)$$

Finally, after substitution of the known expressions for V_g, $\cos\gamma$, $\sin\gamma$, and $\cos\Phi$, we have

$$h_\phi(\theta,\varphi,\Delta E,\vartheta)\equiv\phi-\varphi-\text{arc cos}\left\{\frac{1}{(1+\Delta E/E_2)^{1/2}}\frac{1}{\sin\theta\,\sin\vartheta}\left[\left(\frac{m_2v_2}{m_1v_1}+\cos\theta\right)\left(1-\left(1+\frac{\Delta E}{E_2}\right)^{1/2}\cos\vartheta\right)+\frac{1}{2}\frac{m_2v_2}{m_1v_1}\left(1+\frac{m_1}{m_2}\right)\frac{\Delta E}{E_2}\right]\right\}=0; \quad (33)$$

or, using (17) and (28), the azimuthal scattering angle ϕ can be expressed in terms of variables Θ and Ψ_g

$$h_\phi(\theta,\Theta,\varphi,\Psi_g)\equiv\phi-\varphi-\text{arc cos}\frac{(v_1/v_2)\sin\theta\cos\Psi_g+(1-(v_1/v_2)\cos\theta)\sin\Psi_g\cos\Psi_g\cos\Theta}{\left\{\left(\frac{V}{v_2}\right)^2-\left\{\frac{m_1}{m_1+m_2}\left[1-\frac{v_1^2}{v_2^2}+2\frac{m_2}{m_1}\left(1-\frac{v_1}{v_2}\cos\theta\right)\right]\cos\Psi_g-2\frac{v_1}{v_2}\sin\theta\sin\Psi_g\cos\Theta\right\}^2\right\}^{1/2}}=0. \quad (34)$$

d. Calculation of $\Delta\tilde{E}$ and the Angles $\tilde{\vartheta}$ and $\tilde{\phi}$

Having obtained the relations for the energy and the direction of the scattered test particle, we can derive the relations for the energy and the direction of the recoil field particle.

From the law of conservation of energy we at once obtain $\Delta E=-\Delta\tilde{E}$. Hence, $\Delta\tilde{E}$ is given by Eq. (17) with the opposite sign.

From the definition of the angle $\tilde{\vartheta}$ we have $\cos\tilde{\vartheta}=\mathbf{v}_2\cdot\mathbf{v}_1'/v_2\cdot v_1'$. Taking into account (6) and using the known formulas for V_g, $\cos(\mathbf{V}'\mathbf{v}_2)$, and $\cos\gamma$, we can write

$$h_{\cos\tilde{\vartheta}}(\theta,\Delta\tilde{E},\Psi_g)\equiv\cos\tilde{\vartheta}-\frac{1}{(1+\Delta\tilde{E}/E_1)^{1/2}}\left[\cos\theta+\frac{1}{2}\frac{v_1}{v_2}\frac{\Delta\tilde{E}}{E_1}+2\left(\frac{m_2}{m_1+m_2}\right)^2\frac{m_1v_1}{m_2v_2}\left(\frac{V}{v_1}\right)^2\cos^2\Psi_g\right]=0; \quad (35)$$

or, in terms of the variables Θ and Ψ_g

$$h_{\cos\tilde{\vartheta}}(\theta,\Theta,\Psi_g)\equiv\cos\tilde{\vartheta}-\frac{\cos\theta+2(m_2/m_1+m_2)[(v_2/v_1-\cos\theta)\cos^2\Psi_g-\sin\theta\sin\Psi_g\cos\Psi_g\cos\Theta]}{\left\{1+K_{12}\left[\frac{m_2v_2^2}{m_1v_1^2}-1+\left(1-\frac{m_2}{m_1}\right)\frac{v_2}{v_1}\cos\theta\right]\cos^2\Psi_g-4\frac{m_2}{m_1+m_2}\frac{v_2}{v_1}\sin\theta\sin\Psi_g\cos\Psi_g\cos\Theta\right\}^{1/2}}=0. \quad (36)$$

To derive the relations for the angle $\tilde{\phi}$, we consider the spherical triangle $\mathbf{v}'_1\mathbf{V}_g\mathbf{v}_2 0$. The relation between its angles is

$$\cos(\mathbf{V}_g\cdot\mathbf{v}_1')=\cos\gamma\cos\tilde{\vartheta}+\sin\gamma\sin\tilde{\vartheta}\cos(\tilde{\phi}-\varphi); \quad (37)$$

on the other hand, we can write

$$\cos(\mathbf{V}_g\cdot\mathbf{v}_1')=V_g/v_1'-M_2(V/v_1)\cos\Phi', \quad (38)$$

and, after some transformations similar to these made previously, we obtain

$$h_{\tilde{\phi}}(\theta,\Delta\tilde{E},\tilde{\vartheta},\varphi)\equiv\tilde{\phi}-\varphi-\text{arc cos}\left(\frac{1}{\sin\theta\,\sin\vartheta}\left\{\frac{1}{(1+\Delta\tilde{E}/E_1)^{1/2}}\left[1+\frac{1}{2}\left(1+\frac{m_2}{m_1}\right)\frac{\Delta\tilde{E}}{E_1}+\left(1+2\frac{m_1-m_2}{m_1+m_2}\right)\frac{m_2v_2}{m_1v_1}\cos\theta\right]-\left(1+\frac{m_2v_2}{m_1v_1}\right)\cos\tilde{\vartheta}\right\}\right). \quad (39)$$

The relations derived above are in the most general form, and the dependence on the interaction law enters only through the trigonometric functions of the angle Ψ_g, i.e., the angle describing the scattering of the particle in the c.m. system. If the interaction reduces to a central force dependent only on the distance between the particles, then the relation describing the angle Ψ_g is relatively simple[7]:

$$\Psi_g=\int_{r_{\min}}^{\infty}\frac{dr}{r\{(r/D)^2[1-U(r)/\frac{1}{2}\mu V^2]-1\}^{1/2}}, \quad (40)$$

where μ, V, D have the same meaning as before, while

[7] See Ref. 6, pp. 170–171.

$U(r)$ is the potential function of the two particles and $r_{\min}$ is the distance of closest approach of the particle of mass μ to the center of force.

In the special case in which the interaction force can be written in the form

$$F(r)=-dU(r)/dr=\pm k/r^{\nu}, \tag{41}$$

the relation (40), after the substitution

$$\begin{aligned} y&=D/r, \\ y_0&=D(\mu V^2/k)^{1/(\nu-1)} \end{aligned} \tag{42}$$

has been introduced, results in

$$\Psi_g=\int_0^{y_{00}}\left[1-y^2\mp\frac{2}{\nu-1}\left(\frac{y}{y_0}\right)^{\nu-1}\right]^{-1/2}dy, \tag{43}$$

where y_{00} is the positive root of the expression in square brackets. In this case, Ψ_g depends only on ν and the dimensionless quantity y_0.

If, in Eqs. (42) and (43), we set $\nu=2$ (Coulomb interaction), we obtain at once

$$\begin{aligned} \sin\Psi_g&=\frac{(D/k)\mu V^2}{\{1+((D/k)\mu V^2)^2\}^{1/2}}, \\ \cos\Psi_g&=[1+((D/k)\mu V^2)^2]^{-1/2}. \end{aligned} \tag{44}$$

If we set $\nu=3$, we have

$$\Psi_g=\frac{\pi}{2}\frac{(D/k)\mu V^2}{\{1+((D/k)\mu V^2)^2\}^{1/2}}. \tag{45}$$

In the case of a collision of two rigid impenetrable spheres of radii R_1 and R_2, (43) gives

$$\begin{aligned} \sin\Psi_g&=2D/(R_1+R_2), \\ \cos\Psi_g&=\{1-(2D/(R_1+R_2))^2\}^{1/2}. \end{aligned} \tag{46}$$

The rigid-sphere case is the only model of a collision in which the scattering angle depends on the collision parameter and not on the relative velocity of the particles; we stress, however, that this model is a mathematical abstraction.

IV. COLLISION STATISTICS. CROSS SECTIONS

In order to define the concept of the cross section, we shall analyze the motion of two points moving with inertia in space. We take their position in space to be completely arbitrary. Then the probability that the particles, as a result of their relative motion, pass each other at a distance D, where the plane of their relative motion creates an angle Θ with the fundamental plane, is proportional to the surface element $2\pi DdDd\Theta$ (Fig. 2). The probability that the particles pass each other (by the time of their passing we understand the instant at which the distance between them is smallest) in the interval of time dt is proportional to the change in distance between the two points in their relative motion, Vdt. Taking into account the fact that in the time dt point 2 traverses a distance $dx=v_2dt$ in the laboratory system, and denoting by $f(\theta,\varphi)d\theta d\varphi$ the probability that the velocity vector of point 1 has a direction determined by angles θ and φ (see Fig. 2), we find that the probability of the "collision" determined by the four geometrical parameters θ, Θ, φ, D on the path dx is

$$(1/V_0)f(\theta,\varphi)d\theta d\varphi(d\Theta/2\pi)2\pi DdD(V/v_2)dx, \tag{47}$$

where the proportionality constant V_0, having the dimensions of volume, denotes the region in which the points are found. If N_1 particles of type 1 are in the volume V_0, then the number of collisions will be N_1 times as great; thus

$$\begin{aligned} \nu_{\text{coll}}&=n_1(V/v_2)f(\theta,\varphi)d\theta d\varphi(d\Theta/2\pi)2\pi DdDdx \\ &\equiv n_1\sigma(\theta,\Theta,\varphi,D)dx, \end{aligned} \tag{48}$$

where $n_1=N_1/V_0$ is the spatial density of the field particles. The quantity σ, having the dimensions of area, is called the collision cross section (in this case defined by the geometrical variables θ, Θ, φ, D).

Sometimes the collision problem is given differently. Namely, we are not interested in the frequency of collisions of the test particle with the field particles, but in the probability that the collision between two definite particles took place at given geometric variables θ, φ, Θ, and D. In this case the cross section does not depend on the velocities of colliding particles, and is simply

$$\sigma_V(\theta,\Theta,\varphi,D)=f(\theta,\varphi)d\theta d\varphi(d\Theta/2\pi)2\pi DdD. \tag{49}$$

With the aid of the relations found above and Eqs. (17), (28), (34), (36), (39), we can go on to cross sections characterized not by geometrical variables θ, Θ, φ, D, but by the dynamical variables ΔE, ϑ, ϕ, $\Delta\tilde{E}$, $\tilde{\vartheta}$, $\tilde{\phi}$.

If, for example, we integrate $\sigma(\theta,\Theta,\varphi,D)$ over the geometrical variables in the region in which the condition $h_{\Delta E}[\theta,\Theta,\Psi_g,(\theta,D)]=0$ is fulfilled, we obtain the cross section for a collision in which particle 2 experiences a change in energy of ΔE:

$$\sigma_{\Delta E}=\int\int\int\int\frac{V}{v_2}f(\theta,\varphi)\delta[h_{\Delta E}(\theta,\Theta,\Psi_g)]2\pi DdD\frac{d\Theta}{2\pi}d\theta d\varphi. \tag{50}$$

Integrating over the region in which the conditions $h_{\Delta E}=0$ and $h_{\cos\vartheta}=0$ are fulfilled, we obtain the cross section for the scattering of particle 2 in the direction ϑ with a simultaneous change in energy ΔE:

$$\begin{aligned} \sigma_{\Delta E,\cos\vartheta}\equiv&\int\int\int\int\frac{V}{v_2}f(\theta,\varphi)\delta[h_{\Delta E}(\theta,\Theta,\Psi_g)] \\ &\times\delta[h_{\cos\vartheta}(\theta,\Theta,\Psi_g)]2\pi DdD\frac{d\Theta}{2\pi}d\theta d\varphi; \end{aligned} \tag{51}$$

and similarly

$$\sigma_{\Delta E,\cos\vartheta,\phi}=\int\int\int\int\frac{V}{v_2}f(\theta,\varphi)\delta[h_{\Delta E}(\theta,\Theta,\Psi_g)]$$
$$\times\delta[h_{\cos\vartheta}(\theta,\Theta,\Psi_g)]\delta[h_\phi(\theta,\varphi,\Theta,\Psi_g)]$$
$$\times 2\pi DdD\frac{d\Theta}{2\pi}d\theta d\varphi. \quad (52)$$

In the same way, we determine the other cross sections ($\sigma_{\cos\vartheta},\sigma_{\cos\vartheta,\phi}$, etc.). Of course, between the cross sections mentioned here there occur relations of the type $\sigma_{\Delta E}=\int\sigma_{\Delta E,\cos\vartheta}d(\cos\vartheta)$, which, under certain conditions, can facilitate the calculations and the physical interpretation.

On the basis of formulas (17), (36), and (39), we can give an analogous set of cross sections for the recoil particle. Combining (17), (28), (34), (36), and (39), we can determine the cross sections of the type $\sigma_{\cos\vartheta,\cos\tilde{\vartheta}}$ describing a simultaneous change of state of both particles.

The function $f(\theta,\varphi)$ describing the spatial motion of the field particles in the l.s. has the following form in some particular cases:

(a) isotropic motion of the field particles:

$$f(\theta,\varphi)=(\tfrac{1}{2}\sin\theta)(1/2\pi); \quad (53)$$

(b) the velocity vectors of the field particles parallel to the plane passing through the z axis at the angle φ_0, the position of the vector in the plane being entirely arbitrary:

$$f(\theta,\varphi)=\tfrac{1}{2}\sin\theta\delta(\varphi-\varphi_0); \quad (54)$$

(c) the motion of the field particles in one given direction

$$f(\theta,\varphi)=\delta(\theta-\theta_0)\delta(\varphi-\varphi_0). \quad (55)$$

The cross sections defined above are exact insofar as they refer to the abstract problem of the collision of two isolated particles. In actual physical processes, we always have to do with the problem of many bodies, and the analysis of these processes on the basis of the two-body collision theory always has an approximate character. Hence, in order to be able to use the two-body approximation for physical processes, we must determine its range of applicability. In this case, we introduce the concept of collision time. Since the entire effect of the interaction is involved in the relative motion of the particles and is uniquely determined by the c.m.-system scattering angle, we define the collision time as the time after which this angle attains a value close to that corresponding to the interaction after an infinite time.

If we note by r_0 the distance at which the potential energy of both particles is equal to their relative kinetic energy, then all collisions can be divided into two groups, depending on whether the collision parameter is less than r_0—these are the so-called "close collisions"—or greater than r_0—these are the so-called "distant collisions"—; this division refers only to interactions described by monotonic functions. For close collisions, in which the direction of the relative velocity undergoes a considerable change ($\pi-2\Psi_g\rightarrow\pi$), the main effect of the scattering involves the region between the radii r_0 and $r_{1/2}$, where $r_{1/2}$ is the distance at which the potential energy of the particles is equal to half their relative kinetic energy (Fig. 4). We can therefore assume that the time for a close collision is

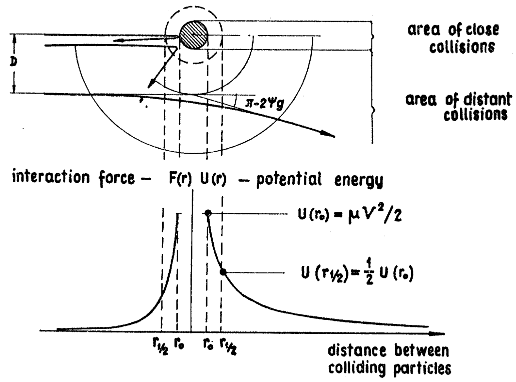

FIG. 4. Collision in the center-of-mass system. Illustrates the difference between the close collisions (the impact parameter is smaller than the distance of closest approach at central collision) and the distant collisions (it is greater).

$$t_{\mathrm{coll}}{}^{\mathrm{cl}}=2(r_{1/2}-r_0)/V. \quad (56)$$

For distant collisions ($D\gg r_0$), in which the direction of the relative velocity undergoes only a slight change, the basic effect of the scattering is manifested in the region $D-D_{1/2}$, where $D_{1/2}$ is defined by the relation $U(D_{1/2})=\frac{1}{2}U(D)$. We can define the distant collision time in a similar way:

$$t_{\mathrm{coll}}{}^{\mathrm{dist}}=2(D_{1/2}-D)/V. \quad (57)$$

The quantities $r_{1/2}$ and $D_{1/2}$ depend on the interaction law between the particles; for forces $F(r)\propto r^{-\nu}$ the expressions defining the collision time can be written as follows:

$$t_{\mathrm{coll}}{}^{\mathrm{cl}}=(2r_0/V)(2^{1/(\nu-1)}-1)\rightarrow$$
$$(2/V)(2k/\mu V^2)^{1/(\nu-1)}(2^{1/(\nu-1)}-1), \quad (58)$$

$$t_{\mathrm{coll}}{}^{\mathrm{dist}}=(2D/V)(2^{1/(\nu-1)}-1). \quad (59)$$

Relations (58) and (59) may be put together and written generally:

$$t_{\mathrm{coll}}\simeq 2((r_0+D)/V)(2^{1/(\nu-1)}-1), \quad (60)$$

or in the case of a Coulomb interaction ($\nu=2$)

$$t_{\mathrm{coll}}{}^{\mathrm{cl}}=2r_0/V\rightarrow(2/V)(2k/\mu V^2)^2\propto 1/V^5, \quad (61)$$

$$t_{\mathrm{coll}}{}^{\mathrm{dist}}=2D/V, \quad (62)$$

$$t_{\mathrm{coll}}=2((r_0+D)/V). \quad (63)$$

Hence, for the two-body collision approximation to have sense, we must consider only those processes for which the duration of the act of collision is considerably shorter than the time in which the field or test particle undergoes an appreciable change in velocity due to the interaction with other particles or external fields.

V. CALCULATION OF THE BASIC CROSS SECTIONS

From among the many cross sections defined by arbitrary combinations of the dynamical variables ΔE, $\Delta\tilde{E}$, ϑ, $\tilde{\vartheta}$, ϕ, $\tilde{\phi}$, expressions (50), (51), and (52) are the most important for the interpretation of physical processes. Proceeding with their calculation, we note that the integration over the collision parameter D can be replaced by an integration over the scattering angle Ψ_g. Solving (40) with respect to D^2, we can write

$$\begin{aligned} D^2 &= F(1/\cos^2\Psi_g,\theta)\,, \\ 2DdD &= F'(1/\cos^2\Psi_g,\theta)d(1/\cos^2\Psi_g)\,, \end{aligned} \tag{64}$$

where the form of the function F depends on the law of interaction between the particles. If the interaction comes from forces $\propto k/r^\nu$, then the function $F(1/\cos^2\Psi_g,\theta)$ has the form

$$F(1/\cos^2\Psi_g,\theta)=(k/\mu V^2)^{2/\nu-1}y_0^2(1/\cos^2\Psi_g)\,, \tag{65}$$

where y_0, now given by integral (43), is a function of the angle Ψ_g only. In the case of forces decreasing with the square of the distance, we can write at once on the basis of (44)

$$\begin{aligned} 2DdD &= (k/\mu V^2)^2 d(1/\cos^2\Psi_g)\,, \\ F'(1/\cos^2\Psi_g,\theta) &= (k/\mu V^2)^2\,, \end{aligned} \tag{66}$$

or in the case of perfectly rigid spheres

$$\begin{aligned} 2DdD &= \left(\frac{R_1+R_2}{1/\cos^2\Psi_g}\right)^2 d(1/\cos^2\Psi_g)\,, \\ F'(1/\cos^2\Psi_g) &= \left(\frac{R_1+R_2}{1/\cos^2\Psi_g}\right)^2. \end{aligned} \tag{67}$$

Taking into account the fact that[8]

$$\int_a^b f(x)\delta[\varphi(x)]dx=\sum_s \frac{f(x_s)}{|\varphi_x'(x_s)|}\,,$$

where the sum is taken over all roots of the equation $\varphi(x)=0$ in the interval (a,b), and integrating (50), (51), and (52) with respect to the angle Θ, we obtain

$$\sigma_{\Delta E}=\int\int\frac{V}{v_2}f(\theta)\frac{F'(1/\cos^2\Psi_g,\theta)}{[W_{\Psi_g}(\Delta E,1/\cos^2\Psi_g,\theta)]^{1/2}}d(1/\cos^2\Psi_g)d\theta\,, \tag{68}$$

$$\sigma_{\Delta E,\cos\vartheta}=\int\int\frac{V}{v_2}f(\theta)\frac{F'(1/\cos^2\Psi_g,\theta)}{[W_{\Psi_g}(\Delta E,1/\cos^2\Psi_g,\theta)]^{1/2}}\delta[h_{\cos\vartheta}(\Delta E,1/\cos^2\Psi_g,\theta)]d(1/\cos^2\Psi_g)d\theta\,, \tag{69}$$

$$\sigma_{\Delta E,\cos\vartheta,\phi}=\int\int\int\frac{V}{v_2}f(\theta,\varphi)\frac{F'(1/\cos^2\Psi_g,\theta)}{[W_{\Psi_g}(\Delta E,1/\cos^2\Psi_g,\theta)]^{1/2}}\delta[h_{\cos\vartheta}(\Delta E,1/\cos^2\Psi_g,\theta)]\delta[h_\phi(\Delta E,1/\cos^2\Psi_g,\theta,\varphi)]\times d(1/\cos^2\Psi_g)d\theta d\varphi\,, \tag{70}$$

where we have denoted

$$W_{\Psi_g}=(2a\sin\Psi_g\cos\Psi_g)^2-(\Delta E+b\cos^2\Psi_g)^2\,. \tag{71}$$

In the expressions for $\sigma_{\Delta E}$, $\sigma_{\Delta E,\cos\vartheta}$ we have also integrated over the angle φ after taking into account the fact that $\int f(\theta,\varphi)d\varphi=f(\theta)$. Without any assumption as to the velocity distribution of the field particles $f(\theta,\varphi)$, only the cross sections $\sigma_{\Delta E,\cos\vartheta}$ and $\sigma_{\Delta E,\cos\vartheta,\phi}$ can be calculated.

Integrating over the angle Ψ_g and performing some transformations, we obtain

$$\sigma_{\Delta E,\cos\vartheta}=\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{5/2}}\frac{1}{2\sqrt{2}m_2v_1v_2}\int\frac{\mu^2V^4}{E_2^2}\times\frac{F'(u_\xi,\theta)}{(W_\xi(\Delta E,\cos\vartheta,\theta))^{1/2}}f(\theta)d\theta\,, \tag{72}$$

$$\sigma_{\Delta E,\cos\vartheta,\phi}=\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{5/2}}\frac{1}{2\sqrt{2}m_2v_1v_2}\int\int\frac{\mu^2V^4}{E_2^2}\times\frac{F'(u_\xi,\theta)}{(W_\xi(\Delta E,\cos\vartheta,\theta))^{1/2}}\delta[h_\phi(\Delta E,\cos\vartheta,\theta,\varphi)]\times f(\theta,\varphi)d\theta d\varphi\,, \tag{73}$$

where

$$W_\xi=-\cos^2\theta+2AB\cos\theta+1-A^2-B^2\,, \tag{74}$$

$$A=\left(\frac{\xi}{2}\right)^{1/2}\left(\frac{\Delta E/E_2}{2\xi}-1\right), \tag{75}$$

$$B=\left(\frac{\xi}{2}\right)^{1/2}\frac{v_2}{v_1}\left(\frac{\Delta E/E_2}{2\xi}+\frac{m_2}{m_1}\right), \tag{76}$$

[8] D. Ivanenko and A. Sokolov, *Classical Theory of Fields* (GITTL, Moscow-Leningrad, 1951), pp. 31–33.

$$\xi=1+\frac{1}{2}\frac{\Delta E}{E_2}-\left(1+\frac{\Delta E}{E_2}\right)^{1/2}\cos\vartheta, \tag{77}$$

$$1/\cos^2\Psi_g\to u_\xi=2\left(\frac{m_1}{m_1+m_2}\right)^2\left(\frac{V}{v_2}\right)^2\frac{1}{\xi}. \tag{78}$$

Without detracting from the generality of the considerations, we can still integrate expression (73) with respect to the angle φ. This integration yields

$$\sigma_{\Delta E,\cos\vartheta,\phi}=\frac{(1+\Delta E/E_2)^{1/2}}{\xi^{5/2}}\frac{1}{2\sqrt{2}m_2v_1v_2}\int\frac{\mu^2V^4}{E_2{}^2}$$
$$\times\frac{F'(u_\xi,\theta)}{(W_\xi(\Delta E,\cos\varphi,\theta))^{1/2}}[\theta,\,\phi-\varphi(\Delta E,\vartheta,\theta)]d\theta, \tag{79}$$

where $\phi-\varphi(\Delta E,\vartheta,\theta)$ is defined by the relation $h_\phi\times(\theta,\varphi,\Delta E,\varphi)=0$.

Taking into consideration that $\int\sigma_{\Delta E,\cos\vartheta}d(\Delta E)=\sigma_{\cos\vartheta}$ and using (69), we obtain the cross section for the scattering of the test particle through the angle ϑ:

$$\sigma_{\cos\vartheta}=\sum_{i=1}^{2}\int\int\frac{V}{v_2}f(\theta)\frac{F'(1/\cos^2\Psi_g,\theta)}{(W_{\Psi_g}(\Delta E_i,1/\cos^2\Psi_g,\theta))^{1/2}}2E_2$$
$$\times\frac{(1+\Delta E_i/E_2)^{3/2}}{|2M_1{}^2(V/v_2)^2\cos^2\Psi_g+\frac{1}{2}(\Delta E_i/E_2)|}$$
$$\times d(1/\cos^2\Psi_g)d\theta, \tag{80}$$

where

$$\Delta E_i=E_2\{[|\cos\vartheta|+(-1)^i(4M_1{}^2(V/v_2)^2\cos^2\Psi_g-\sin^2\vartheta)^{1/2}]^2-1\} \tag{81}$$

are the roots of equation $h_{\cos\vartheta}(\Delta E,1/\cos\Psi_g,\theta)=0$.

From the direct substitution of (81) in (29), it follows that if $4M_1{}^2(V/v_2)^2\cos^2\Psi_g>1$, then for two values of i ($i=1$, 2) there exist two different scattering angles ϑ and $\pi-\vartheta$, and therefore in (80) instead of the sum over i we have $|\cos\vartheta|\to\cos\vartheta$. If $4M_1{}^2(V/v_2)^2\cos^2\Psi_g<1$, then $\cos\vartheta$, independently of i, is always positive (the particle cannot be back-scattered) and thus in expression (80) there is the sum over i.

VI. ENERGY-EXCHANGE RELATIONS

The above relations, derived in most general form, already allow us to draw some specific conclusions.

Thus, since the cross section must be a real quantity, we obtain at once from (68) the condition $W_{\Psi_g}\geqslant 0$ which is fulfilled if the determinant of the equation $W_{\Psi_g}(1/\cos^2\Psi_g)=0$ is non-negative, or explicitly

$$-\Delta E^2-\Delta Eb+a^2\geqslant 0. \tag{82}$$

This inequality does not depend on the character of the interaction but is a consequence of the laws of conservation of energy and momentum; it determines the limits of the exchange of energy between the colliding particles. Hence, solving for ΔE, we obtain

$$\Delta E^{\pm}=-b/2\pm(b^2/4+a^2)^{1/2}, \tag{83}$$

where ΔE^+, being always positive, determines the upper limit for the gain of energy and ΔE^-, being always negative, determines the upper limit for the loss of energy of the test particle. Inequality (82) is therefore fulfilled if $\Delta E^-\leqslant\Delta E\leqslant\Delta E^+$.

In the two limiting cases of greatest interest for us, $m_1=m_2$ and $m_1\ll m_2$, expression (83) takes the form

$$\frac{\Delta E^\pm}{E_2}=-\frac{1}{2}\left[\left(\frac{E_1}{E_2}-1\right)^2\pm\left\{\left(\frac{E_1}{E_2}-1\right)^2+4\frac{E_1}{E_2}\sin^2\theta\right\}^{1/2}\right]$$
$$\text{for}\quad m_1=m_2, \tag{84}$$

$$\frac{\Delta E^\pm}{K_{12}E_2}\simeq+\frac{1}{2}\left[-\left(1-\frac{v_1}{v_2}\cos\theta\right)\right.$$
$$\left.\pm\left\{\left(1-\frac{v_1}{v_2}\cos\theta\right)^2+\left(\frac{v_1}{v_2}\right)^2\sin^2\theta\right\}^{1/2}\right]$$
$$\text{for}\quad m_1\ll m_2\quad\text{and}\quad E_1<E_2. \tag{85}$$

Differentiating (83) with respect to the angle θ and equating the derivative to zero, we find the angle which must be formed by the vectors of the initial velocities of the two particles, so that the energy exchange between the colliding particles is a maximum. Simple calculation leads to the relations

$$\text{if}\ \frac{1}{2}\frac{v_2}{v_1}\left|1-\frac{m_2}{m_1}\right|\leqslant 1,\ \text{then}\ \cos\theta_{\Delta E^-{}_{\max\max}}=\frac{1}{2}\frac{v_2}{v_1}\left(1-\frac{m_2}{m_1}\right); \tag{86}$$

$$\text{if}\ \frac{1}{2}\frac{v_1}{v_2}\left|1-\frac{m_1}{m_2}\right|\leqslant 1,\ \text{then}\ \cos\theta_{\Delta E^+{}_{\max\max}}=\frac{1}{2}\frac{v_1}{v_2}\left(1-\frac{m_1}{m_2}\right). \tag{87}$$

The energy changes corresponding to these angles are then $\Delta E^-=-E_2$ and $\Delta E^+=E_1$, respectively. We have obtained a very surprising result, from which it follows that even in the case where the masses of the colliding particles are unequal, collisions with a full momentum transfer are possible in a certain energy range. Thus, if inequality (86) is fulfilled, then for the angle $\theta=\theta_{\Delta E^-{}_{\max\max}}$ the test particle can lose its entire energy in the collision. If, however, equality (87) is fulfilled, then for the angle $\theta=\theta_{\Delta E^+{}_{\max\max}}$ the test particle can acquire the entire energy of the field particle. In the case of colliding particles of equal masses, full momentum transfer is possible independently of the energy of the colliding particles, and can occur if the velocity vectors of both particles are perpendicular to each other. If the mass of the test particle is considerably greater than that of the field particle, it can lose its entire energy if the momentum is smaller than half the momentum of the field particle; a necessary condition for the acquisition of the entire energy from the field particle is fulfilled if the test-particle velocity is no greater than half the velocity of the field particle. If inequalities (86) and (87) are not fulfilled, then it is not possible for a collision to occur in which one of the particles loses its entire energy. The greatest energy transfer then occurs at the angle $\theta=0$ or $\theta=\pi$, i.e., when the velocity vectors of both particles are parallel or antiparallel.

Setting in (83) $\theta=0$ (the particles move in the same direction), we obtain at once

$$\Delta E^{\pm}{}_{\theta=0}=K_{12}E\left(1+\frac{m_1v_1}{m_2v_2}\right)\frac{1}{2}\left[-\left(1-\frac{v_1}{v_2}\right)\pm\left|1-\frac{v_1}{v_2}\right|\right] \tag{88}$$

and therefore:

if $v_2>v_1$, then $\Delta E<0$ (the test particle overtakes the field particle and transfers part of its energy to it);
if $v_2<v_1$, then $\Delta E>0$ (the test particle acquires energy from the field particle which overtakes it).

In the case of antiparallel vectors $\theta=\pi$, (83) gives us

$$\Delta E^{\pm}{}_{\theta=\pi}=K_{12}E_2\left(1+\frac{v_1}{v_2}\right)\frac{1}{2}\left[-\left(1-\frac{m_1v_1}{m_2v_2}\right)\pm\left|1-\frac{m_1v_1}{m_2v_2}\right|\right]. \tag{89}$$

Now, the gain or loss of energy of the particle depends not on the velocities of the particles but on their momenta. Thus,

$$\begin{aligned}&\text{if}\quad m_2v_2>m_1v_1,\quad\text{then}\quad \Delta E<0;\\ &\text{if}\quad m_2v_2<m_1v_1,\quad\text{then}\quad \Delta E>0.\end{aligned}$$

In a collision with antiparallel velocities, the particle which has the greater momentum loses energy. In the case $m_1\ll m_2$ and $E_1<E_2$ (then automatically $m_1v_1\ll m_2v_2$) particle 2 undergoes the greatest possible loss in energy if the velocities of both particles are antiparallel:

$$\Delta E^{-}{}_{\max}\simeq -K_{12}E_2(1+v_1/v_2). \tag{90}$$

Particle 2 can gain the energy if its velocity is smaller than the velocity of particle 1. The maximum gain in energy occurs when the velocities of both particles are parallel. Then this gain is

$$\Delta E^{+}{}_{\max}\simeq K_{12}E_2(1-v_1/v_2). \tag{91}$$

At this point, it is worth drawing attention to the fact that the maximum change in energy does not at all correspond to a central collision, as might appear at first glance. This can be readily shown if we differentiate (17) with respect to the angle Ψ_g and equate the derivative to zero. It then turns out that the c.m.-system scattering angle corresponding to a collision with a maximum change of energy is different from zero and equal to

$$\tan\Psi_g=\frac{b}{2a}\pm\left\{\left(\frac{b}{2a}\right)^2+1\right\}^{1/2}, \tag{92}$$

where the sign before the root is used according to the sign of b, so that $\tan\Psi_g\geqslant 0$; and only when $a\to 0$, which occurs if $v_1\to 0$ (scattering on particles at rest) or $\sin\theta\to 0$ (scattering on field particles moving parallel or antiparallel), does $\tan\Psi_g\to 0$, and hence the collision parameter D also tends to zero. Inserting the obtained value of the angle Ψ_g in expression (17), we obviously obtain relation (83).

If we regard ΔE as an independent variable, then condition (82) determines the limits of integration in Eq. (68) with respect to angle θ. Solving (82) for $\cos\theta$, we then obtain

$$\cos\theta_{1,2}=\begin{cases}x_0\pm x_1 & \text{if}\quad -1\leqslant x_0\pm x_1\leqslant 1,\\ +1 & \text{if}\quad x_0\pm x_1\geqslant 1,\\ -1 & \text{if}\quad -1\geqslant x_0\pm x_1,\end{cases} \tag{93}$$

where we have denoted

$$x_0\pm x_1=\frac{1}{2}\left(1-\frac{m_1}{m_2}\right)\frac{\Delta E}{E_1}\frac{v_1}{v_2}\pm\left\{\left(1-\frac{\Delta E}{E_1}\right)\left(1+\frac{\Delta E}{E_2}\right)\right\}^{1/2}. \tag{94}$$

Diagrams of the integration of the cross section $\sigma_{\Delta E}$ with respect to the angle θ, which is equivalent to defining the region in which inequality (82) is fulfilled for several special cases of the masses of the colliding particles, are shown in Figs. 5, 6, 7, 8, 9, and 10.

VII. SCATTERING RELATIONS

Similarly, as the condition $W_{\Psi_g}\geqslant 0$ determines the range of variability of the variables $\Delta E, \theta, \Psi_g$, the condition $W_\xi\geqslant 0$ determines the range of variability of variables $\Delta E, \vartheta, \theta$. Reducing W_ξ given by (74) to the form

$$W_\xi=\frac{1}{2\xi}\left\{-\xi^2\left(1+2\frac{m_2v_2}{m_1v_1}\cos\theta+\frac{m_2{}^2v_2{}^2}{m_1{}^2v_1{}^2}\right)+2\xi\left[\sin^2\theta+\frac{\Delta E}{2E_2}\left(1-\frac{E_2}{E_1}\frac{v_2}{v_1}\cos\theta+\frac{m_2v_2}{m_1v_1}\cos\theta\right)\right]-\left(\frac{\Delta E}{2E_2}\frac{V}{v_1}\right)^2\right\}; \tag{95}$$

it can be easily seen that the condition $W_\xi\geqslant 0$ gives us $\xi_1\leqslant\xi\leqslant\xi_2$, where $\xi_{1,2}$ are the roots of the equation $W_\xi=0$. Therefore, the range of variability of the $\cos\vartheta$ is determined by

$$\cos\vartheta_{1,2}=\left(1+\frac{\Delta E}{E_2}\right)^{-1/2}\left\{1+\frac{1}{2}\frac{\Delta E}{E_2}-\xi_{1,2}(\cos\theta)\right\}, \tag{96}$$

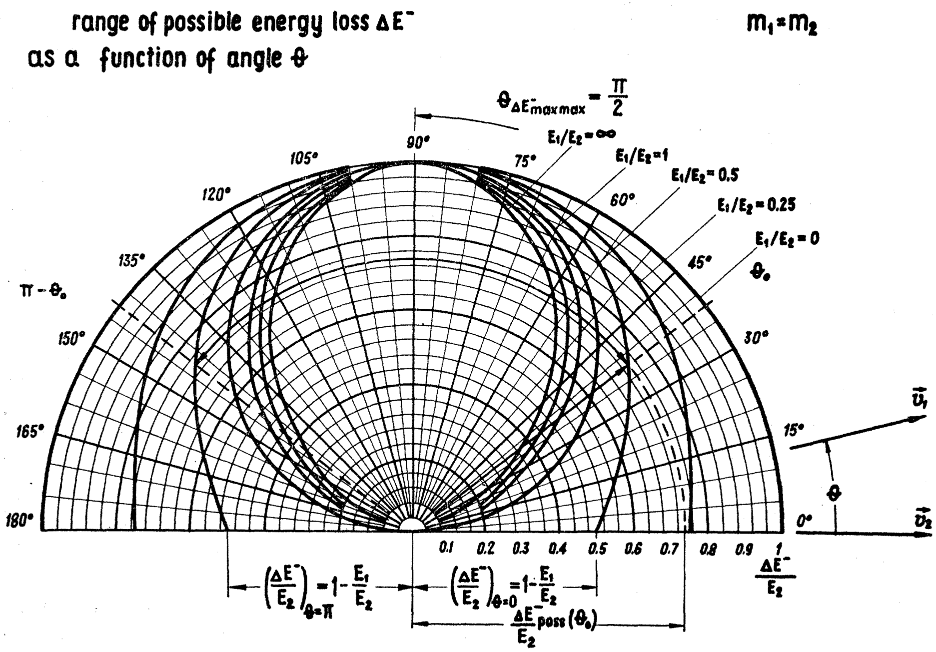

FIG. 5. Possible energy loss as a function of angle between initial velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for equal masses of colliding particles. The parameter is the ratio of the energies of the two particles. At the angle $\theta = \theta_{\Delta E^-\max\max} = \frac{1}{2}\pi$, the test particle can lose its total energy independently of the energy of the field particle. The range of the possible loss of energy for a given angle and for a given ratio of E_1/E_2 is represented in the graph by the radial arrow. The azimuthal arrow represents the range of angles θ at which the test particle can lose the given amount of energy ΔE.

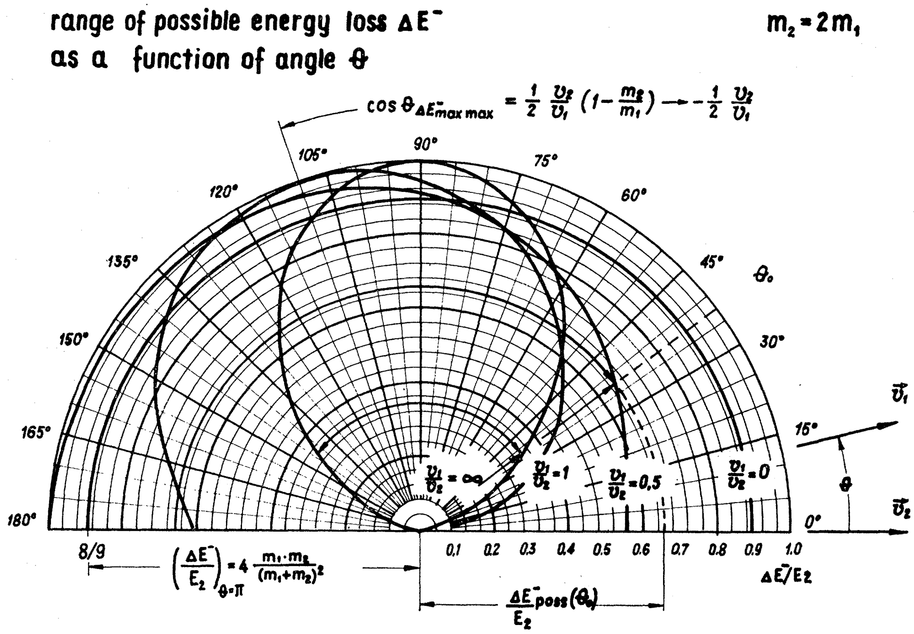

FIG. 6. Range of possible energy loss as a function of angle θ for $m_2 = 2m_1$. Now, the angle $\theta_{\Delta E^-\max\max}$ depends on the ratio of velocities of both particles, and changes from $\frac{1}{2}\pi$ for infinite velocity of the field particle to π for $v_1 \leqslant \frac{1}{2}v_2$. The meaning of the arrows is the same as in Fig. 5.

where

$$\xi_{1,2} = \left(\frac{p_1}{P_C}\right)^2 \left\{ \sin^2\theta + \frac{1}{2}\frac{\Delta E}{E_2}\left[1 - \frac{E_2}{E_1} + \frac{v_2}{v_1}\left(\frac{m_2}{m_1} - 1\right)\cos\theta\right] \right.$$

$$\left. \pm \left(\left\{ \sin^2\theta + \frac{1}{2}\frac{\Delta E}{E_2}\left[1 - \frac{E_2}{E_1} + \frac{v_2}{v_1}\left(\frac{m_2}{m_1} - 1\right)\cos\theta\right] \right\}^2 - \left(\frac{1\Delta E}{2E_2}\right)^2 \left(\frac{V}{v_1}\right)^2 \left(\frac{P_C}{p_1}\right)^2 \right)^{1/2} \right\}, \quad (97)$$

$\mathbf{p}_1$, $\mathbf{p}_2$ are the momenta of both particles before encounter, and $\mathbf{P}_C$ is the momentum of the center of mass:

$$P_C^2 = p_1^2 + p_2^2 + 2p_1p_2\cos\theta. \quad (98)$$

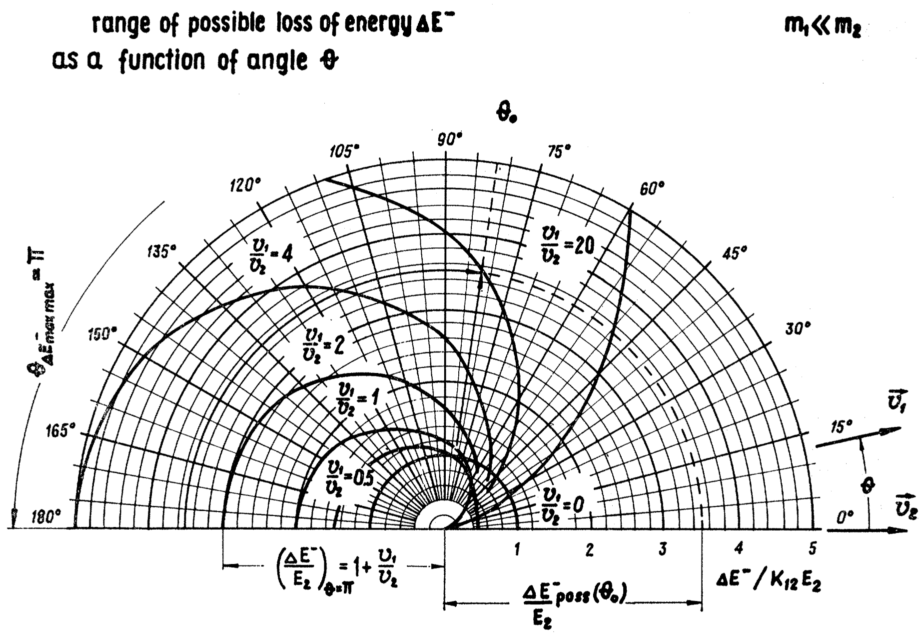

FIG. 7. Range of possible energy loss as a function of angle θ in the case of heavy particles colliding with light ($m_2 \gg m_1$). The greatest possible loss of energy takes place for antiparallel moving particles ($\theta_{\Delta E^-_{\max\,\max}} = \frac{1}{2}\pi$).

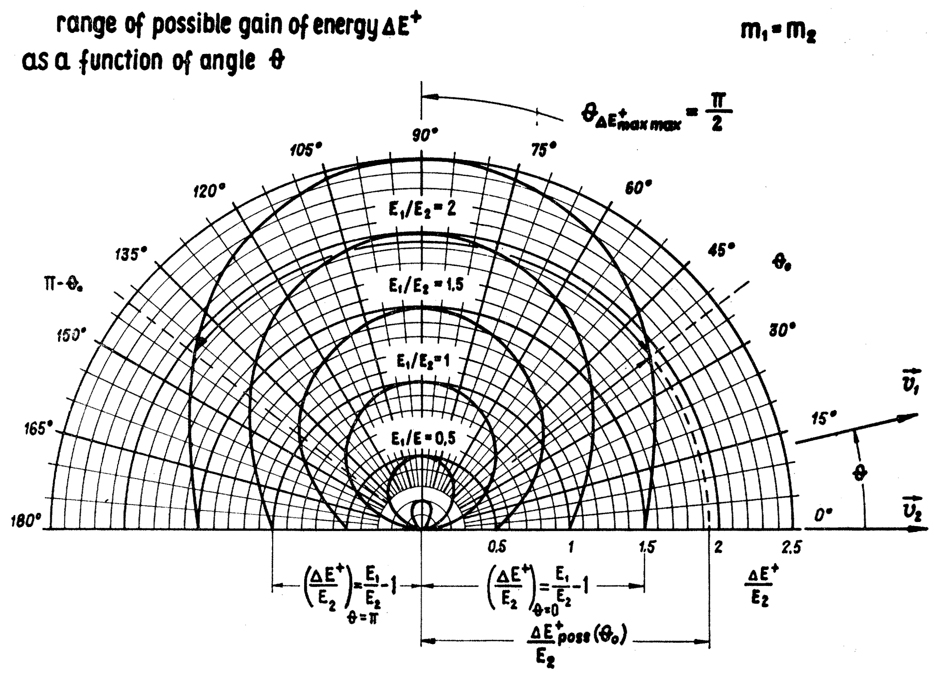

FIG. 8. Possible gain of energy as a function of angle θ for equal masses of colliding particles. The maximum maximorum gain of energy of the test particle takes place at an angle $\theta = \frac{1}{2}\pi$.

For parallel or antiparallel velocities of colliding particles ($\theta = 0$ or $\theta = \pi$), the relation (96) takes the form

$$\cos\vartheta_{1,2} \to \cos\vartheta = \left(1 + \frac{\Delta E}{E_2}\right)^{-1/2} \left\{1 + \frac{\Delta E}{2E_2}\left(1 - \frac{1 \mp v_2/v_1}{1 \pm m_2 v_2/m_1 v_1}\right)\right\}, \tag{99}$$

where the upper sign is for the parallel and the lower one for the antiparallel velocity (Fig. 11). In this case there is a unique relation between the change of energy of the colliding particles ΔE, the scattering angle ϑ, and the impact parameter D (this can be easily shown putting $\theta = 0$ or $\theta = \pi$ in the equation $W_{\Psi_0} = 0$).

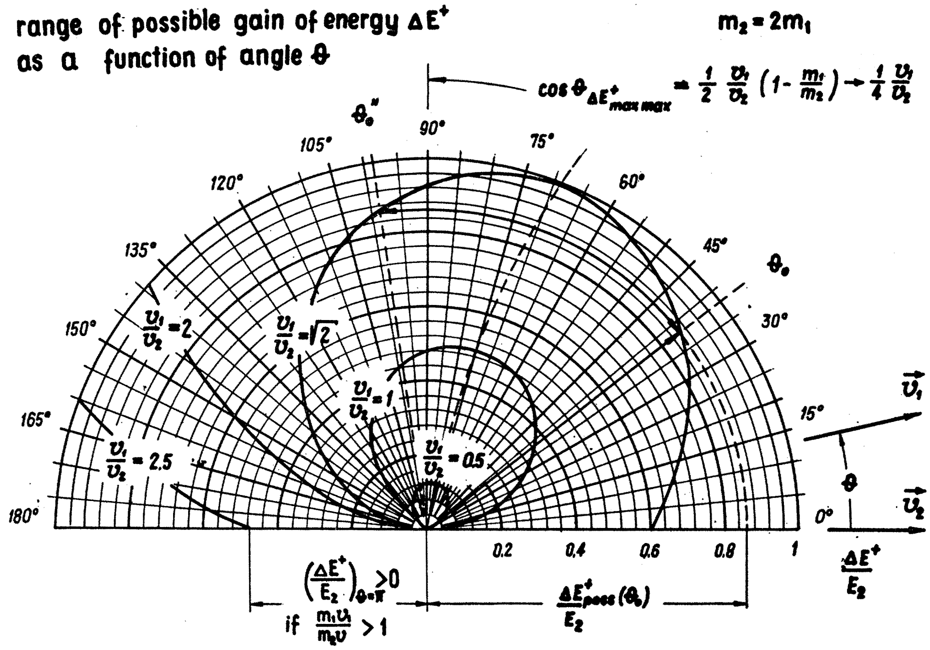

FIG. 9. Possible gain of energy as a function of angle θ for $m_2=2m_1$. The angle $\theta_{\Delta E^-\max\max}$ decreases with increase of velocity of the field particle.

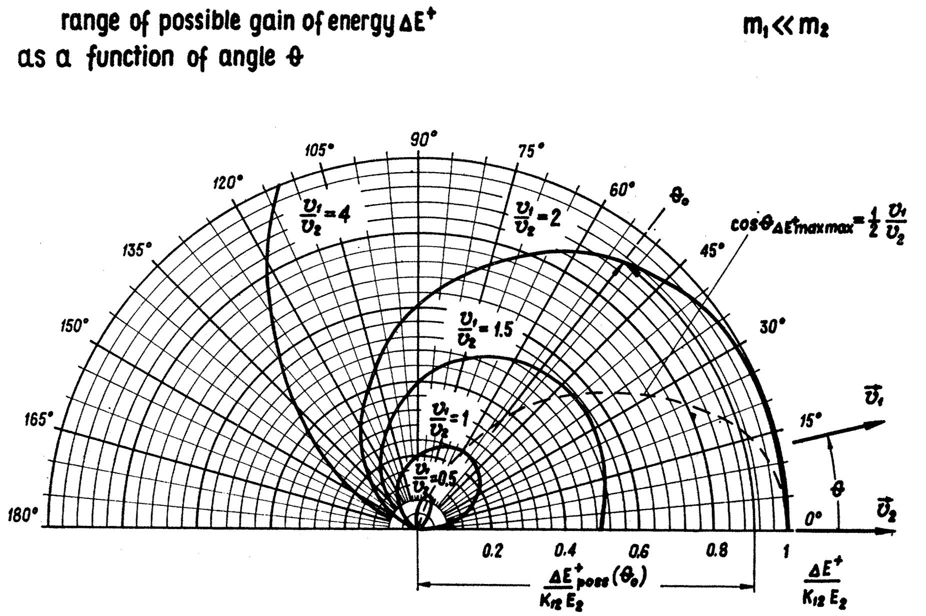

FIG. 10. Possible gain of energy as a function of angle θ for a mass of the test particle much greater than that of the field particle; if the velocity of the field particle tends to zero, the test particle can gain energy at an angle near to $\frac{1}{2}\pi$ only.

For perpendicular velocities ($\mathbf{v}_1\cdot\mathbf{v}_2=0$) the range of variability of the angle ϑ is determined by

$$\cos\vartheta_{1,2}=\left(1+\frac{\Delta E}{E_2}\right)^{-1/2}\left\{1+\frac{\Delta E}{2E_2}\,\frac{1+\dfrac{\Delta E}{2E_2}\left(1-\dfrac{E_2}{E_1}\right)\pm\left[\dfrac{\Delta E}{E_2}\left(1-\dfrac{E_2}{E_1}\right)-\left(\dfrac{\Delta E}{2E_2}\right)^2\dfrac{v_2^2}{v_1^2}\left(1+\dfrac{m_2}{m_1}\right)^2\right]^{1/2}}{1+m_2^2v_2^2/m_1^2v_1^2}\right\}, \tag{100}$$

which has been obtained from (96) after substituting $\theta=\frac{1}{2}\pi$ (Fig. 12).

The condition $W_\xi\geqslant 0$ examined with respect to the angle θ, after taking into account that for $\theta=0$ or $\theta=\pi$ $W_\xi=-(A-B)^2$ is always negative, gives us $\theta_1\leqslant\theta\leqslant\theta_2$, where

$$\cos\theta_{1,2}=AB\pm[(1-A^2)(1-B^2)]^{1/2}. \tag{101}$$

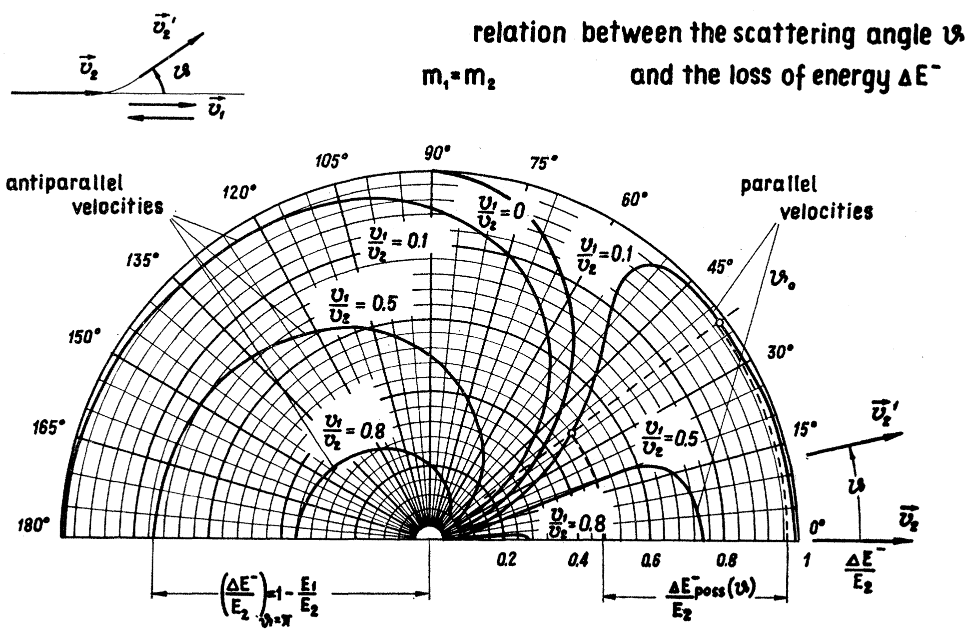

Fig. 11. Relation between the scattering angle ϑ and the loss of energy ΔE^- in the case of parallel and antiparallel velocities and for particles of equal masses. The parameter is the ratio of velocities of the colliding particles. There is a unique relation between the scattering angle and the loss of energy.

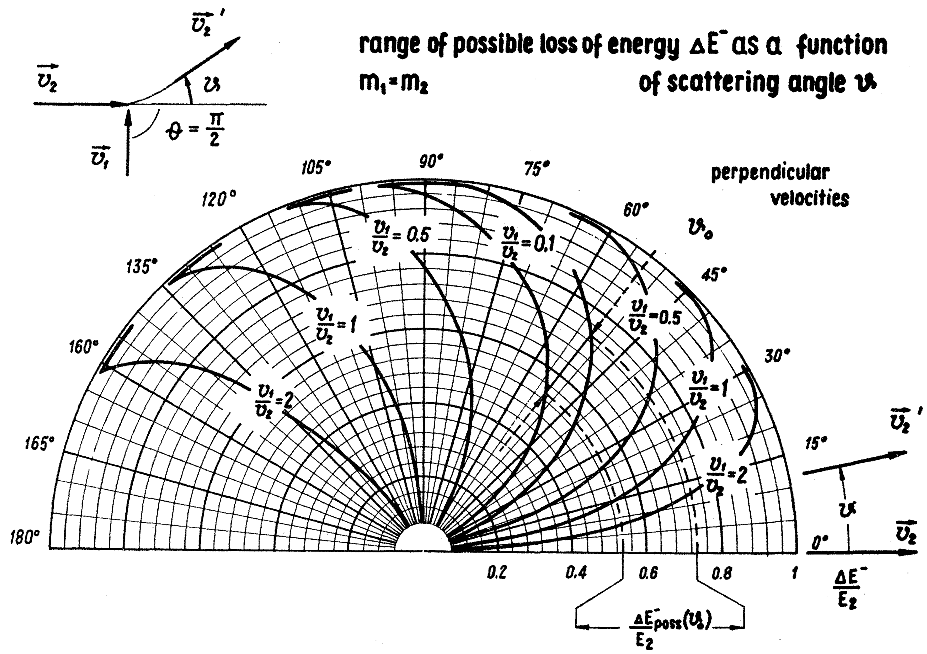

Fig. 12. Relation between the scattering angle ϑ and the loss of energy ΔE^- in the case of perpendicular velocities and for equal masses of both particles. In this case there is no unique relation between the loss of energy ΔE^- and the scattering angle ϑ. For instance, the test particle can be scattered at an angle ϑ_0 with the loss of energy in the interval shown in the figure by arrows.

Since the expression $[(1-A^2)(1-B)]^{1/2}$ has to be a real quantity, and $(1-A^2)$ can be reduced to the form $(1+\Delta E/E_2)\sin^2\vartheta/2\xi$ which is evidently always positive, we obtain the condition $B^2 \leqslant 1$, or explicitly

$$\xi^2+2\xi\frac{m_1}{m_2}\left(\frac{1}{2}\frac{\Delta E}{E_2}-\frac{E_1}{E_2}\right)+\left(\frac{\Delta E}{2E_2}\right)^2\left(\frac{m_1}{m_2}\right)^2\leqslant 0. \quad (102)$$

Solving the above inequality with respect to ξ, we obtain the limits for ξ

$$\xi_{1,2}=\tfrac{1}{2}[(m_1v_1/m_2v_2)((1-\Delta E/E_1)^{1/2}\pm 1)]^2 \quad (103)$$

and then the limits for the angle ϑ

$$\cos\vartheta_{1,2}=\left(1+\frac{\Delta E}{E_2}\right)^{-1/2}\left\{1+\frac{1}{2}\frac{\Delta E}{E_2}-\frac{1}{2}\left[\frac{m_1v_1}{m_2v_2}\left(\left(1-\frac{\Delta E}{E_1}\right)^{1/2}\pm 1\right)\right]^2\right\}. \quad (104)$$

Now, the limits do not depend on the angle θ. They determine the maximum range of variability for the scattering angle ϑ and for the variable ξ if θ changes from 0 to π (Fig. 13).

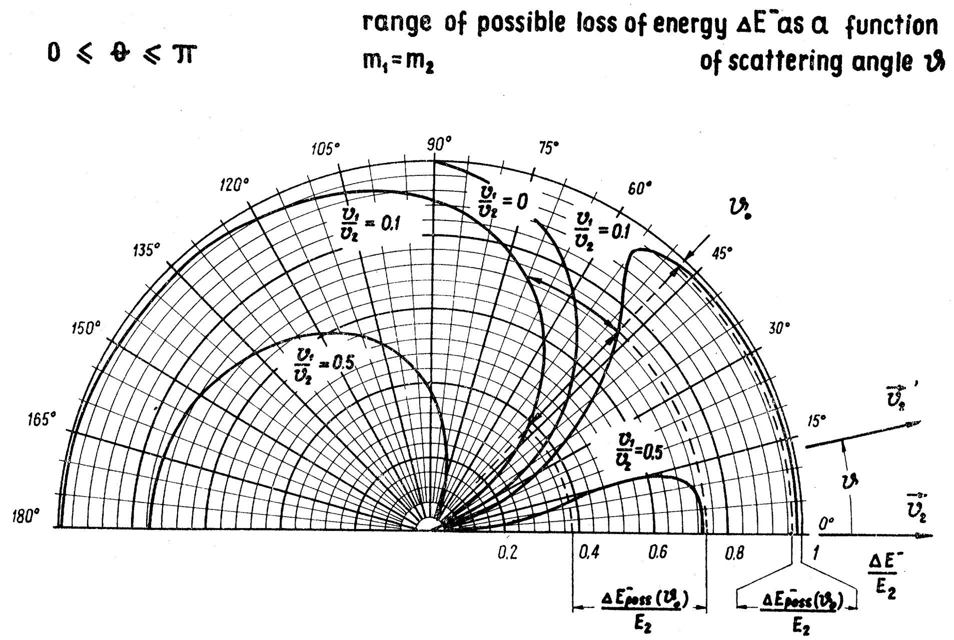

FIG. 13. Range of possible loss of energy as a function of the scattering angle ϑ for θ changing from 0 to π. For $v_1 \neq 0$ the test particle can be scattered at the same angle ϑ_0 with a small or large loss of energy. When $v_1 = 0$, then independently of the angle ϑ there is a unique relation between the scattering angle and the loss of energy.

The relation between the angles $\vartheta_{1,2}$ and the change in energy ΔE, for $\Delta E < 0$, $m_1 = m_2$ and also for some special values of the angle θ, is shown in Figs. 14, 15, and 16.

One could try to solve the condition $W_\xi \geqslant 0$ with respect to ΔE and determine the limits for ΔE as a function of the angles ϑ and θ. Unfortunately, solution of (95) with respect to ΔE in the general case is very difficult (a fourth-power equation with respect to ΔE); therefore, the analytical form of the scattering cross section can be obtained in some special cases only.

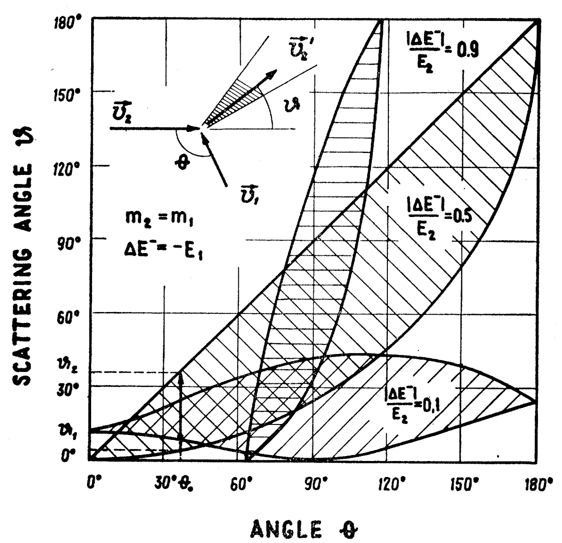

FIG. 14. Relation between scattering angle ϑ and the angle θ for particles of equal masses and for the loss of energy equal to the kinetic energy of the field particle. The shaded area represents the region of possible scattering angles ϑ for a given ratio $\Delta E^-/E_2$.

If the field particles can undergo discrete changes in energy only or the velocity distribution of field particles forms the set of discrete values for the angle θ, then from (72) it at once results that angular distributions will have sharp maxima corresponding to zero values of denominator ($W_\xi = 0$). They are responsible for the "diffraction" pattern of scattered particles. Therefore the spectrum of possible changes in energy and the orientation of the velocity vectors of the field particles determine completely the diffraction pattern of scattered particles, independently of the law of interaction, which influences the absolute value of the cross section only (Fig. 17).

In the case of large velocities of scattered particles ($v_2 \gg v_1$), Eq. (96), which determines the scattering

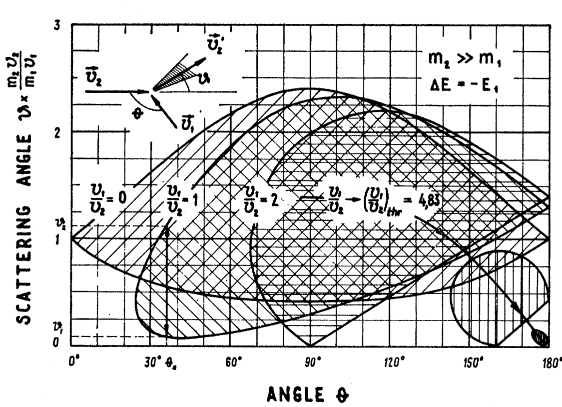

FIG. 15. Region of possible scattering angles ϑ for heavy particles ($m_2 \gg m_1$) as a function of angle θ. The parameter is the ratio of velocities of colliding particles.

maxima, can be approximately written:

$$\cos\vartheta_{1,2}\simeq\left(1+\frac{\Delta E}{E_2}\right)^{-1/2}\left\{1+\frac{1}{2}\frac{\Delta E}{E_2}-\frac{1}{2}\left(\frac{p_1}{p_2}\right)^2\left(\left(\sin^2\theta-\frac{\Delta E}{E_1}\right)^{1/2}\pm\sin\theta\right)^2\right\}; \quad (105)$$

or further, in the approximation $|\Delta E^-|<E_1(v_2/v_1)$,

$$\vartheta_{1,2}\simeq\frac{p_1}{p_2}\left(\left(\sin^2\theta+\frac{|\Delta E^-|}{E_1}\right)^{1/2}\pm\sin\theta\right)\propto\frac{1}{m_2v_2}. \quad (106)$$

Thus, we have obtained the very well known dependence of diffraction rings on the momentum of scattered particles.

The dependence of the diffraction pattern on the discrete states of the field particles is more evident if we examine the scattering of particles in a given direction. If there exist any favored directions in the velocity distribution of the field particles, then the velocity distribution has the form $f(\theta,\varphi)=\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)$ and from (79), we have

$$\sigma_{\Delta E,\cos\vartheta,\phi}\propto[\varphi_0(\Delta E,\vartheta,\theta_0)-\phi]; \quad (107)$$

therefore, by means of (33) the direction of a scattered particle is, for the given value of ΔE, exactly determined. As a result, the diffraction pattern has the form not of diffraction rings but of diffraction spots. If $v_2\gg v_1$ and $|\Delta E^-|<E_1(v_2/v_1)$, then the relation (33) with the help of (106) results in

$$\phi=\varphi. \quad (108)$$

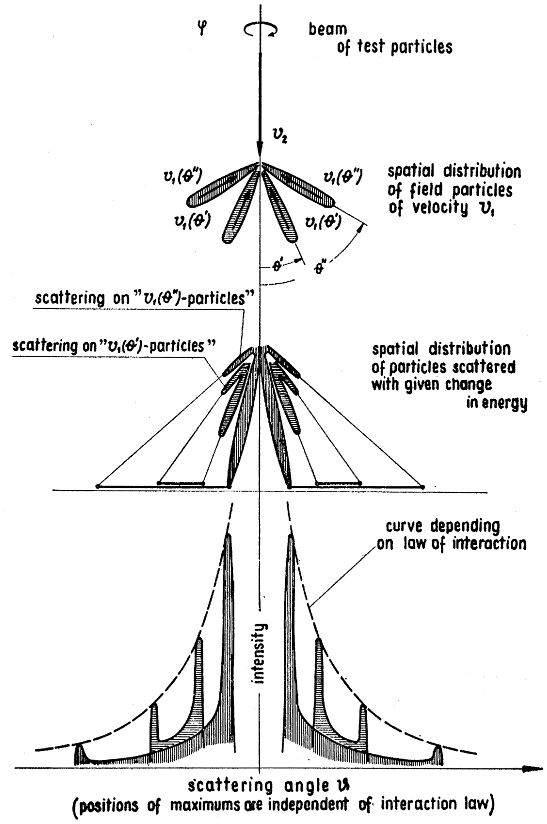

FIG. 17. Dependence of "diffraction" rings on anisotropy in the velocity distribution of field particles with respect to the angle θ.

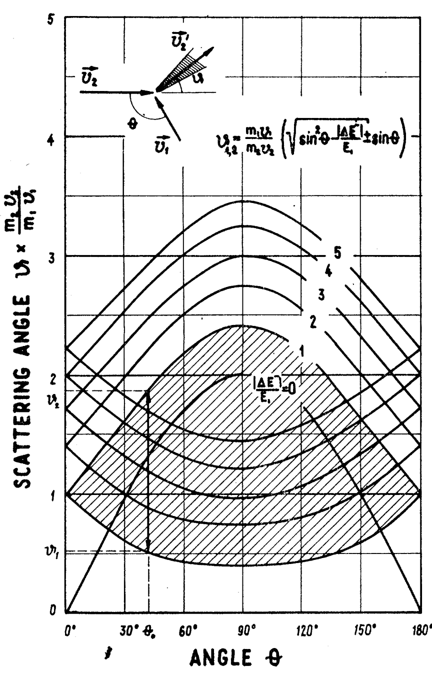

FIG. 16. Approximate dependence of the range for scattering angle ϑ as a function of angle θ for different values of the loss of energy.

In this approximation we have shown that the azimuthal scattering angle of the test particle is equal to the azimuthal angle of the velocity vector of the field particle with which the former has collided.

Summarizing, we can state that the diffraction pattern is due to discrete states of scattering particles. In the case of isotropic distribution of velocities of field particles with respect to the azimuthal angle φ, the diffraction pattern has the form of diffraction rings. In the case of full anisotropy in spatial distribution of velocity vectors, the diffraction pattern has the form of diffraction spots or sometimes of sectors of circles (see Fig. 18). Other forms of the diffraction pattern, such as of Kikuchi lines, can be interpreted as the result of the specific orientation of velocity vectors of field particles.

At last, it is necessary to stress that the diffraction pattern will exist too, although in simpler form, for continuous changes in energy of field particles, provided there exists a threshold for the gain of energy.

VIII. SOME SIMPLIFIED RELATIONS

Inserting $v_1=0$ in the expressions relating the dynamical variables with the geometrical variables (scat-

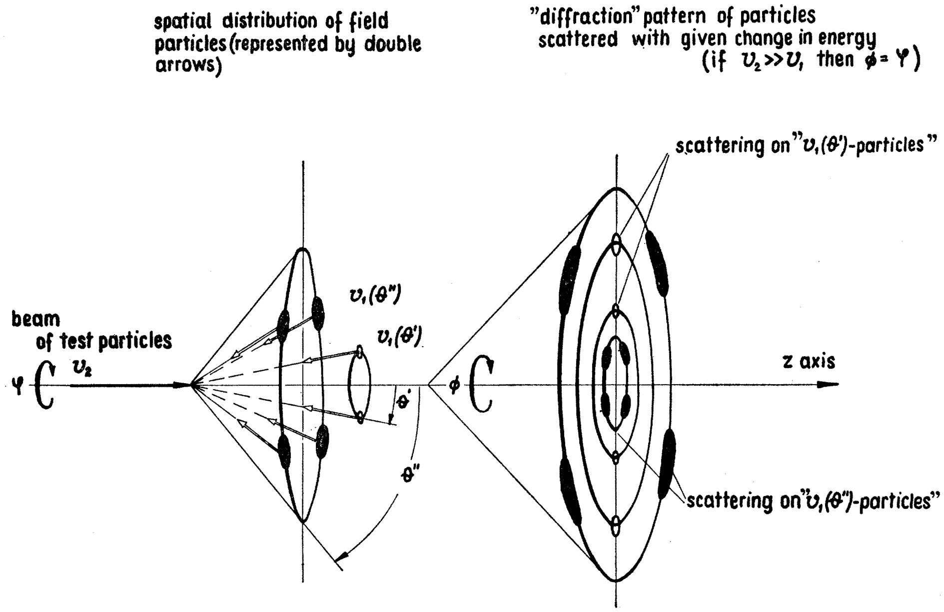

FIG. 18. Dependence of diffraction pattern on the anisotropy in the velocity distribution of field particles with respect both to the angle θ and to the azimuthal angle φ.

tering on field particles at rest), we obtain a number of already known relations. Thus, we obtain for the test particle,

$$\Delta E = -K_{12} \cos^2\Psi_g, \tag{109}$$

$$\cos\vartheta = \frac{1-2(m_1/(m_1+m_2))\cos^2\Psi_g}{(1-K_{12}\cos^2\Psi_g)^{1/2}}, \tag{110}$$

$$\cos\vartheta = \left(1+\frac{\Delta E}{E_2}\right)^{-1/2}\left[1+\frac{1}{2}\left(1+\frac{m_1}{m_2}\right)\frac{\Delta E}{E_2}\right], \tag{111}$$

and similarly for the field particle, which we can regard as a recoil particle,

$$\Delta\tilde{E} = K_{12}E_2\cos^2\Psi_g, \tag{112}$$

$$\cos\tilde{\vartheta} = \cos\Psi_g, \tag{113}$$

$$\cos\tilde{\vartheta} = (\Delta\tilde{E}/K_{12}E_2)^{1/2}. \tag{114}$$

Eliminating Ψ_g from (110) and (113), we obtain an obvious relation between the scattering angle of the test particle and the recoil angle of the field particle at rest:

$$\cos\vartheta = \frac{1-2(m_1/(m_1+m_2))\cos^2\tilde{\vartheta}}{(1-K_{12}\cos^2\tilde{\vartheta})^{1/2}}. \tag{115}$$

The relation between the scattering angle of the test particle and its change of energy in the collision merits particular attention. In the case of a field particle at rest, we have a unique relation between the scattering angle and the change of energy in the collision. Thus, if the field particles can experience only discrete energy changes (hence, if they are electrons bound in an atom or nucleons of a nucleus), then the scattering will have the character of "diffraction" rings independently of the interaction law between the particles.

Proceeding with the calculation of the basic cross sections, in the simple case where $v_1=0$, we will have

$$\sigma_{\Delta E} = \pi\frac{K_{12}E_2}{\Delta E^2}F'(1/\cos^2\Psi_g)_{1/\cos^2\Psi_g=-K_{12}E_2/\Delta E}, \tag{116}$$

$$\sigma_{\cos\vartheta} = 2\pi\frac{1}{\sin\vartheta}\frac{M_1^2}{(1-(m_2/m_1)^2\sin^2\vartheta)^{1/2}}\times\sum_{i=1}^{1}F'(1/\cos^2\Psi_g{}^i)[\cos\vartheta+(1-(m_2/m_1)^2\sin^2\vartheta)^{1/2}] \quad \text{if} \quad m_2\leqslant m_1$$

$$\times\sum_{i=1}^{2}F'(1/\cos^2\Psi_g{}^i)2[1+\cos^2\vartheta-(m_2/m_1)^2\sin^2\vartheta] \quad \text{if} \quad m_2\geqslant m_1, \tag{117}$$

where

$$\left(\frac{1}{\cos^2\Psi_g{}^i}\right) = 2M_1\frac{1-M_2\cos^2\vartheta+(-1)^i\cos\vartheta(1-2M_2+M_2{}^2\cos^2\vartheta)^{1/2}}{\sin^2\vartheta}, \tag{118}$$

and

$$\sigma_{\cos\vartheta}=2\pi(1/\cos^3\tilde{\vartheta})F'(1/\cos^2\Psi_g)_{1/\cos^2\Psi_g=1/\cos^2\tilde{\vartheta}}. \quad (119)$$

Taking into account that the cross section has to be a real as well as a non-negative quantity, we deduce from (117) that $0\leqslant\vartheta\leqslant\vartheta_{\max}$, where

$$\begin{aligned}\vartheta_{\max}&=\arcsin m_1/m_2 &&\text{for} \quad m_2>m_1,\\ \vartheta_{\max}&=\tfrac{1}{2}\pi &&\text{for} \quad m_2=m_1, \\ \vartheta_{\max}&=\pi &&\text{for} \quad m_2/m\to 0,\end{aligned} \quad (120)$$

and from (119) that $0\leqslant\tilde{\vartheta}\leqslant\frac{1}{2}\pi$ independently of the masses of the colliding particles.

On the other hand, by inserting $\Psi_g=0$ in the set of h functions, which corresponds to back scattering in the c.m. system, we obtain the relations for the central collisions. Now, these relations do not depend on the law of interaction, but only express in terms of geometrical and dynamical variables the conservation of energy and momentum; they have been examined very carefully by many authors.

IX. RELATIONS DERIVED FROM THE BASIC CROSS SECTIONS

With the aid of the previously defined basic cross sections, we can determine a number of derivative quantities describing various processes which accompany the collision of particles. Thus, we can define the quantity

$$S=\int\sigma_{\Delta E}\Delta Ed(\Delta E), \quad (121)$$

which we call the slowing-down cross section (slowing down as a result of dynamical friction), and which is directly related to the range of a particle in a medium:

$$R=\frac{1}{N}\int\frac{dE}{\langle S\rangle_{\text{av}}}, \quad (122)$$

where N is the density of the field particles, while $\langle S\rangle_{\text{av}}$ is the slowing-down cross section averaged over the velocities of the field particles.

Taking into account the fact that $\sigma_{\Delta E}$ is given by (68) and integrating with respect to ΔE over the entire region in which $M_{\Psi_g}(\Delta E)\geqslant 0$ [see (71)], we obtain

$$S=\pi\int\int\frac{V}{v_2}f(\theta)bF'\left(\frac{1}{\cos^2\Psi_g}\right)\cos^2\Psi_g d\left(\frac{1}{\cos^2\Psi_g}\right)d\theta. \quad (123)$$

The quantity S defined in this way, which applies to collisions with $\Delta E<0$ and $\Delta E>0$, is exact insofar as it refers to free field particles (for example, the slowing down of charged particles in a plasma); if, however, the slowing down takes place on a quantized set of particles (e.g., slowing down on electrons of atoms or molecules in the ground state), then the only collisions possible are those with a loss of energy equal to or greater than some minimum value (U). The stopping power will then be

$$S_U=\int_{-U}^{\Delta E^-_{\max}}\sigma_{\Delta E}\Delta Ed\Delta E, \quad (124)$$

where $\Delta E^-_{\max}$ is the maximum energy loss experienced by the test particle in the collision with the field particle.

Integrating $\sigma_{\Delta E}$ over the limits $-U$, $\Delta E^-_{\max}$, we obtain the cross section for a collision in which the test particle loses an energy equal to or greater than U:

$$Q=\int_{-U}^{\Delta E^-_{\max}}\sigma_{\Delta E}d\Delta E=\int\int\frac{V}{v_2}f(\theta)F'\left(\frac{1}{\cos^2\Psi_g}\right)\times\arccos\left(\frac{b\cos^2\Psi_g-U}{2a\sin\Psi_g\cos\Psi_g}\right)d\left(\frac{1}{\cos^2\Psi_g}\right). \quad (125)$$

In the case of atomic collisions, Q_u can be directly interpreted as the ionization cross section if U is the ionization potential.

In a similar way, we can construct a number of other derivative quantities, depending on the specific problem.

It should be noted that the total scattering cross section is divergent, a fact which is sometimes erroneously associated with the Coulomb interaction.[9,10] Actually, for each interaction described by a monotonic function, the total scattering cross section is always divergent,[11] which is entirely understandable, since the integration over small scattering angles always corresponds to the integration over a large collision parameter and, independently of the interaction law (excluding the abstract model of rigid spheres), the limit of angles $\vartheta\to 0$ corresponds to $D\to\infty$. All cross sections $\sigma_{\Delta E}$, $\sigma_{\Delta E,\cos\vartheta}$, $\sigma_{\cos\vartheta}$, etc., integrated over the entire range of variability of the variables are divergent, and there is no indication as to whether we are dealing with short-range or long-range forces. For such definitions, we can use, for convenience only, such quantities as $\int\sigma_{\Delta E}\Delta Ed(\Delta E)$ or $\int\sigma_{\cos\vartheta}\cos\vartheta d(\cos\vartheta)$ or other similar quantities.

[9] L. Granovskii, *Electrical Currents in Gases* (GITTL, Moscow-Leningrad, 1952), pp. 61–79.

[10] E. Segrè, *Experimental Nuclear Physics* (John Wiley & Sons, Inc., New York, 1953), Vol. I, p. 213.

[11] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Cambridge, 1953).

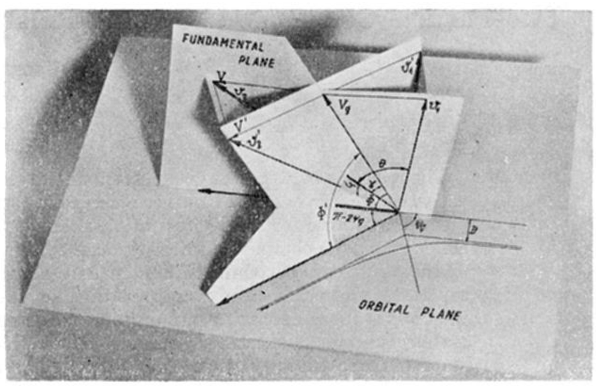

Fig. 3. Vector model of two-particle collision.