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Conductivity of Thin Metallic Films in a Longitudinal Magnetic Field

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The size-effect conductivity of thin metallic films in a longitudinal magnetic field is calculated following Chambers' kinetic formulation. We assume (i) that the electron Fermi surface is a single sphere in momentum space, (ii) purely diffuse scattering at the boundary surfaces, and (iii) an isotropic relaxation time in the bulk material. The calculation is based on a study of the trajectory of electrons scattered from the surfaces. The computed size-effect magnetoresistance exhibits a maximum in low magnetic fields and a monotonic decrease in higher magnetic fields. The resistivity maximum may be utilized to obtain information on the bulk mean free path or the cyclotron radius of the conduction electrons. The gross feature of the calculated size-effect variation of magnetoresistance (the galvanomagnetomorphic effect) is found in agreement with some experimental results.

IN the free-electron theory of metals, it is well known that a longitudinal magnetic field has no effect on the resistivity of an isotropic conductor.[1] However, in proving this statement, it is tacitly assumed that all the dimensions of the specimen are large compared with the mean free path of the conduction electrons. In the case of thin films or wires at low temperatures where boundary scattering of electrons prevails, this requirement breaks down and there may exist a significant longitudinal magnetoresistance effect due to the spiral motion of the electrons in the magnetic field. Previous work on sodium wires[2,3] has shown that the resistivity actually decreases in a longitudinal magnetic field. In this note, we report calculations of the magnetoresistance effect for the case of a thin metallic film placed in a magnetic field which is parallel to the current in the plane of the film. Theoretical study of this problem does not seem to appear in the literature. In the results, we shall see that the resistivity decreases in a longitudinal magnetic field after an initial increase. It is plausible that these features may be utilized to derive the mean free path and momentum of the conduction electrons in the metal.

Consider a thin metallic film of thickness d, with its surface parallel to the x-z plane (see Fig. 1). We assume

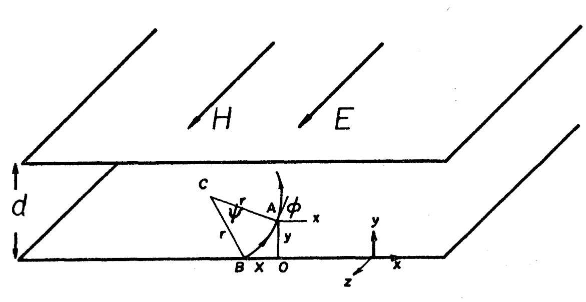

FIG. 1. The geometry of the thin film and the applied fields. Circular arc BA is the projection of an electron trajectory on the x-y plane.

that at the boundary planes $y=0$ and $y=d$, the surface scattering is purely diffuse, and that the electron Fermi surface in the metal is a single sphere. The mean free path l is defined by the product of the value of Fermi velocity v and an isotropic relaxation time τ. The dimensions of the specimen in the x-z plane are large compared with l so that the electronic distribution function is independent of x and z. The fields are parallel to each other and in the z direction ($\mathbf{H}\|\mathbf{E}\|\mathbf{J}\|\hat{z}$).

The general approach to the conduction problem of this type is based upon the Boltzmann equation. However, using a kinetic formulation, Chambers[3-5] has shown that the distribution function can be obtained without solving the Boltzmann equation, and the size-effect conductivity of a thin conductor in the absence of magnetic field is given by

$$\sigma=\frac{3\sigma_0}{4\pi s}\int_s ds\int_0^{2\pi} d\phi\int_0^{\pi} d\vartheta \sin\vartheta \cos^2\vartheta \times[1-\exp(-|\mathbf{R}-\mathbf{R}_0|/l)], \quad (1)$$

where σ_0 is the bulk conductivity, s the cross-sectional area of the specimen perpendicular to the current, $|\mathbf{R}-\mathbf{R}_0|$ the distance an electron has traversed from a point $\mathbf{R}_0$ on the boundary to a point $\mathbf{R}$ in the metal. It is clear that this result satisfies the requirement of diffuse scattering at the boundaries.

In the presence of a longitudinal magnetic field, which produces no drift current but does modify the electron trajectories, Eq. (1) is still applicable provided $|\mathbf{R}-\mathbf{R}_0|$ is taken to be the distance which an electron has traversed along its spiral trajectory in the magnetic field. This equation is appropriate for calculating the longitudinal magnetoconductivity in the present problem.

An electron traveling at an angle ϑ to the z axis will move in a helical path, while its projection on the x-y plane will describe a circle of radius $r=(m^*vc\sin\vartheta)/eH$, where m^* is the cyclotron effective mass of the electrons.

[1] See, for example, J. M. Ziman, *Electrons and Phonons* (Oxford University Press, London, 1960), p. 494.
[2] D. K. C. MacDonald and K. Sarginson, Proc. Roy. Soc. (London) **A203**, 223 (1950).
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[5] R. G. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952).

If, when an electron traverses from $\mathbf{R}_0$ to $\mathbf{R}$, its projection on the x-y plane moves an angle ψ around such a circle, then, following Chambers' formulation, the size-effect magnetoconductivity in the present case can be written as

$$\frac{\sigma}{\sigma_0}=1-\frac{3}{4\pi d}\int_0^d dy\int_0^{2\pi}d\phi\int_0^{\pi}d\vartheta\,\sin\vartheta\,\cos^2\vartheta \times\exp[-\psi(y,\phi,\vartheta)/\eta], \quad (2)$$

where $\eta=l/r_0$, $r_0=m^*vc/eH$.

We proceed to find ψ in terms of y, ϕ, ϑ and to evaluate the integrals in (2). One may note that the effect of H is independent of the direction of H, and by symmetry consideration, electrons scattered from either $y=0$ or $y=d$ will contribute equally to the conductivity. Thus, in evaluating the integrals, we shall consider only the electrons scattered from the lower boundary ($y=0$), and the result should be multiplied by 2 to give the total value of the integral.

In Fig. 1, the arc BA is the projection on the x-y plane of the trajectory of an electron that has been scattered from the lower boundary; we have $\cos\psi=(CB)\cdot(CA)/r^2$. For every point $A(0,y)$ in the metal, the point $B(X,0)$ is given by the algebraically larger root of the equation:

$$(X+r\sin\phi)^2+(y+r\cos\phi)^2-r^2=0, \quad (3)$$

where ϕ is the azimuth angle at A as shown in Fig. 1. From (3), it can be shown that $\psi(y,\phi,\vartheta)$ is given by

$$\cos\psi=1+\sin^2\phi\left[-1+\operatorname{sgn}(\sin\phi)\left(1-\frac{y^2+2yr\cos\phi}{r^2\sin^2\phi}\right)^{1/2}\right]+\frac{y\cos\phi}{r}, \quad (4)$$

where

$$\operatorname{sgn}(x)=+1 \quad x>0$$
$$=-1 \quad x<0.$$

Equation (3) also places limitations on the allowed ranges of ϕ for a given point A in Fig. 1 as follows:

$$0\leqslant y\leqslant r:\quad \phi_1\leqslant\phi\leqslant 2\pi-\phi_1$$
$$r\leqslant y\leqslant 2r:\quad \phi_2\leqslant\phi\leqslant 2\pi-\phi_2$$
$$2r<y:\quad \text{no real solution in (3)},$$

where $\phi_1=\arccos(1-y/r)$, $\phi_2=\pi-\arccos(y/r-1)$. Moreover, to take account of electrons scattered from the lower boundary only, the condition

$$r(1+\cos\phi)+y\leqslant d$$

must be satisfied for every $X\geqslant 0$ and $d\leqslant 2r$, $\sin\phi<0$.

Letting $\alpha=d/l$, $\mu=r_0/d$, $\xi=y/d$, and taking all the restrictive conditions into account, the size-effect conductivity can be written as

$$\begin{aligned}\frac{\sigma}{\sigma_0}=1-2\times\frac{3}{4\pi}\int_0^{\pi}d\vartheta\,\sin\vartheta\,\cos^2\vartheta\int_0^{1+(\mu\sin\vartheta-1)\Theta(1-\mu\sin\vartheta)}d\xi\Bigg[\int_{\phi_1}^{\pi}d\phi\,\exp(-\mu\alpha\psi)\\+\int_{\pi}^{2\pi-\phi_1}d\phi\,\exp(-\mu\alpha\psi)\Theta(1-\xi-\mu\sin\vartheta(1+\cos\phi))\Bigg]-2\times\frac{3}{4\pi}\int_0^{\pi}d\vartheta\,\sin\vartheta\,\cos^2\vartheta\,\Theta(1-\mu\sin\vartheta)\\\times\int_{\mu\sin\vartheta}^{1+(2\mu\sin\vartheta-1)\Theta(1-2\mu\sin\vartheta)}d\xi\Bigg[\int_{\phi_2}^{\pi}d\phi\,\exp(-\mu\alpha\psi)+\int_{\pi}^{2\pi-\phi_2}d\phi\,\exp(-\mu\alpha\psi)\Theta(1-\xi-\mu\sin\vartheta(1+\cos\phi))\Bigg]\end{aligned} \quad (5)$$

where

$$\psi=\arccos\left\{1+\sin^2\phi\left[-1+\operatorname{sgn}(\sin\phi)\left(1-\frac{\xi^2+2\mu\xi\sin\vartheta\cos\phi}{\mu^2\sin^2\vartheta\sin^2\phi}\right)^{1/2}\right]+\frac{\xi\cos\phi}{\mu\sin\vartheta}\right\},$$

$$\Theta(x)=1 \quad x>0$$
$$=0 \quad x<0.$$

The factor 2 appearing in front of $3/4\pi$ in (5) takes account of electrons scattered from both boundaries.

The integration in (5) is very complicated; in general we must resort to numerical methods. The triple integral has been evaluated by using an IBM 7040 computer at the Stony Brook Computing Center. Some calculated curves of $\rho/\rho_0(=\sigma_0/\sigma)$ versus $1/\mu$ are shown in Fig. 2.

It should be noted that $1/\mu=deH/m^*vc$ is proportional to H for a given specimen; therefore, theoretical curves such as those shown in Fig. 2 can be directly compared with experimental magnetoresistance data. With the approximations assumed in the present work, these curves may be applicable to some alkali metals. It is believed that the general aspect of these curves, if not swamped by the inherent bulk magnetoresistance, should also be found in other metals.

In Fig. 2, we observe that the resistivity decreases with increasing H beyond an initial maximum. This decrease in resistivity is due to the fact that for suffi-

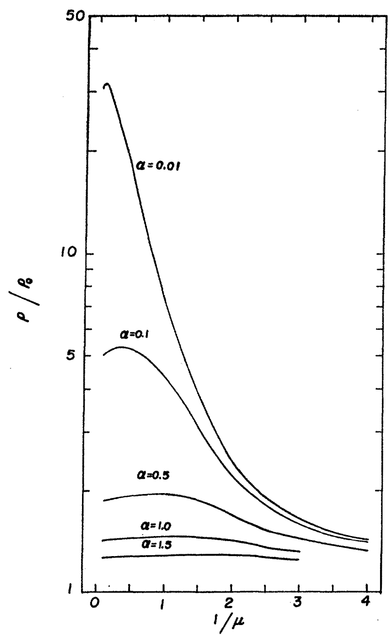

FIG. 2. Calculated curves of ρ/ρ_0 versus $1/\mu$ $(=d/r_0=deH/m^*vc)$ for various α $(=d/l)$.

ciently large H, the electrons may have their trajectories so curved as to reduce collisions with the boundaries. Thus, increasing the magnetic field will result in lengthening the effective free path of the conduction electrons. For very high H, the resistivity approaches the bulk value. This behavior of decreasing magnetoresistance has been observed in thin wires and films of some metals[2,3] and semimetals[6–8] in longitudinal or transverse magnetic fields. As Chambers[3] has pointed out, comparison of these curves with the experimental results will yield information on mean free path and momentum of the conduction electrons in the metal.

In contrast to the monotonic decrease of the longitudinal magnetoresistance found in thin wires,[2,3] we find in the present case that there is a resistivity maximum in the low-field region. This feature, similar to that discovered in transverse magnetoresistance of thin films,[2,9] has been observed in Sb[6] and Bi.[7,8] Detailed analysis of this particular effect is not available at the present time. Using elementary considerations, it seems that the initial increase of magnetoresistance in thin films is due to the fact that some electrons moving nearly parallel to the surface in zero magnetic field have been curved to collide with the boundaries after a magnetic field is turned on. If these electrons contribute most effectively to the conduction, then applying a magnetic field will cause shortening of the effective free path, thereby increasing the resistivity. Since the resistivity due to all the electrons decreases in higher fields, a maximum must exist.

As shown in Fig. 2, the value of $1/\mu$ at the resistivity maximum depends upon α. For higher α, that is, for lower l in a given specimen, the maximum shifts to a higher field. This behavior agrees with some experimental results.[6] Detailed calculation of the α dependence of this resistivity maximum is quite laborious. However, an approximate formula may be obtained from our calculated results. Within the range of our numerical computation, ($\alpha=0.01$ to $\alpha=1$), the $1/\mu$ value at the resistivity maximum can be fitted by

$$\left(\frac{1}{\mu}\right)_{\rho_{\max}}=1.26\alpha^{0.57}\quad(\text{error}\lesssim10\%).$$

This formula may be conveniently used to estimate l provided m^*v is known, or vice versa.

For measurements of the size-effect resistivity at a given temperature, the advantage of using a magnetic field, compared with the case of zero field, is that the information on l and m^*v may be obtained from one specimen only. If one measures the conductivity in zero magnetic field, it then becomes necessary to take measurements for a number of specimens of different sizes. However, it is difficult to have the same mean free path in all the specimens. For the purpose of comparison, it may be pointed out that our calculated values of ρ/ρ_0 in Fig. 2, when extrapolated to zero magnetic field, agree with those calculated by Fuchs[10] for thin films in the absence of a magnetic field.

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