APPENDIX

In this appendix we derive the condition for a self-consistent reciprocal bootstrap for the case of several bound and resonant states. In this case the solutions for N_α and D_α can be put into the form

$$N_\alpha(\omega)=\sum_\beta\left(-\frac{A_{\alpha\beta}\lambda_\beta' D_\alpha(-\Omega_\beta)}{\omega+\Omega_\beta}\right) \quad \text{(A1)}$$

$$D_\alpha(\omega)=\Omega_\alpha-\omega \quad \text{(A2)}$$

where only states which are either resonant or bound are considered and Ω_α represents a resonance or a bound-state energy depending upon whether it is above or below threshold. This is now the only distinction made between a resonance and a bound state. The λ_α' are all given by

$$\lambda_\alpha'=N_\alpha(\Omega_\alpha)/D_\alpha'(\Omega_\alpha) \quad \text{(A3)}$$

and the condition for self-consistency arises when Eqs. (A1) and (A2) are substituted into Eq. (A3). The result is

$$\lambda_\alpha'=\sum_\beta A_{\alpha\beta}\lambda_\beta', \quad \text{(A4)}$$

or that the subcrossing matrix connecting all the resonant and bound states have a $+1$ eigenvalue so that

$$\det(A_{\alpha\beta}-\delta_{\alpha\beta})=0. \quad \text{(A5)}$$

The λ_α' which correspond to the coupling constants and reduced widths then form the eigenvector corresponding to the $+1$ eigenvalue. Equation (12) is a special case of Eq. (A5) for a system of one bound and one resonant state.

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Lehmann-Symanzik-Zimmermann Formalism in the Lee Model*†

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The Lehmann-Symanzik-Zimmermann (LSZ) asymptotic condition on the V, N, and θ fields is shown to be equivalent to the usual prescription for constructing scattering states as solutions of the eigenvalue equation. The reduction formula relating the scattering amplitudes to the τ functions is given. Then the coupled equations for the τ functions appropriate to the V-$N\theta$ sector are written down and solved algebraically by transforming to momentum space. This approach is compared with the straightforward solution of the eigenvalue problem and the dispersion treatment found in the literature. It is concluded that LSZ formalism elucidates the basic structure of the model in the most natural way, giving rise to considerable mathematical simplification.

I. INTRODUCTION

IN the formal investigation of field theory, the axiomatic approach of Lehmann, Symanzik, and Zimmermann[1] (LSZ) has played a major role. The LSZ asymptotic condition, expressing the weak convergence of the Heisenberg fields to the "in" and "out" fields, allows an interpretation of field theory in terms of physical particles. Using this condition, a reduction formula can be obtained relating the S-matrix elements for processes involving the physical particles to the "τ functions.[2]" A coupled set of integrodifferential equations can then be derived for these functions from the field equations and commutation relations.[3]

In this paper, the LSZ formalism is investigated as a method of calculation in a solvable field theory, the Lee model.[4] It will be shown that the asymptotic condition can be understood in terms of the usual prescription for constructing scattering states. Then, a calculation of the lowest sector of the model, using the τ functions, will demonstrate that the structure of the theory is expressed in a most basic form.

The Lee model describes the S-wave interaction between two fermions, the V and N particles, and relativistic, spin-zero bosons, the θ particles. Spin and recoil are neglected for the fermions. In Sec. II, we begin by showing that the asymptotic conditions on the three fields (V, N, and θ) are equivalent to formal solutions

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[1] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955).

[2] The τ functions are the vacuum expectation values of time-ordered products of the Heisenberg fields.

[3] P. T. Matthews and A. Salam, Proc. Roy. Soc. (London) **A221**, 128 (1953).

[4] T. D. Lee, Phys. Rev. **95**, 1329 (1954). G. Källén and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 7 (1955).

of the eigenvalue equations for the scattering states. Then the reduction formula for the general S-matrix element is given. Because there is no interaction between the θ particles, the general τ function contains only one time. The Fourier transform of the τ function is introduced, and it is shown that this function, when evaluated on the energy shell,[5] is just a T-matrix element with an additional second-order pole.

Section III deals with the first nontrivial sector of the Lee model in which there is V-particle renormalization and N-θ scattering. The equations for the appropriate τ functions are written down and solved by transformming into momentum space. By studying the analytic structure of the V propagator, the renormalization constants are obtained. The S-matrix element for N-θ scattering is then computed. In the literature, this sector has been solved both by solving the eigenvalue equations[4] and by using a dispersion treatment.[6] We shall see that the LSZ formalism seems to be the most natural method of solving the model.

II. THE ASYMPTOTIC CONDITION AND REDUCTION FORMULA

We investigate the Lee model in which there is one stable V particle and all integrals are finite. The interaction is $V\leftrightarrow N+\theta$ so that the vacuum, one-particle N state, and n-particle θ state are eigenstates of both H and H_0. The most general state, describing n θ particles scattering from a V or N particle, can be written

$$\begin{aligned}\|\alpha;n\pm\rangle&=\sum_{k_1',\cdots,k_n'} f(\omega_1',\omega_1)\cdots\\&\quad\times f(\omega_n',\omega_n)|m_\alpha;\omega_1',\cdots,\omega_n'\pm\rangle \quad (1)\\&=\sum_{(n')}F(n',n)|\alpha;n\pm\rangle,\end{aligned}$$

where

$$m=\begin{pmatrix}m_v\\m_n\end{pmatrix}\quad\text{when}\quad\alpha=\begin{pmatrix}1\\2\end{pmatrix},$$

$$\omega_j'=(k_j'^2+\mu^2)^{1/2}.$$

$|\alpha;n\pm\rangle$ is a solution of the equation

$$H|\alpha;n\pm\rangle=E(\alpha,n)|\alpha;n\pm\rangle, \quad (2)$$

where

$$E(\alpha,n)=m_\alpha+\sum_{\nu=1}^{n}\omega_\nu',$$

with the boundary condition that the state $|\alpha;n\pm\rangle$ in the Schrödinger picture represents a plane wave plus an incoming or outgoing wave in the limit $t\to\mp\infty$. $f(\omega_j',\omega_j)$ is a good function[7] of ω_j', centered about the point $\omega_j'=\omega_j$, satisfying the condition

$$\begin{aligned}\sum_{k''} f^*(\omega_k'',\omega_k)f(\omega_k'',\omega_k')&=0\quad\text{when}\quad k\neq k', \quad (3)\\&=1\qquad\qquad\quad k=k'.\end{aligned}$$

Let us construct the scattering state,

$$\|\alpha;n,k\pm\rangle=\sum_{k'} f(\omega',\omega)\sum_{(n')}F(n',n)|\alpha;n,k'\pm\rangle, \quad (4)$$

with an additional θ particle having an energy $\omega=(k^2+\mu^2)^{1/2}$. This state is a solution of the equation

$$\begin{aligned}H\|\alpha;n,k\pm\rangle=&\sum_{k'} f(\omega',\omega)\sum_{(n')}F(n',n)\\&\times[E(\alpha,n)+\omega']|\alpha;n,k'\pm\rangle \quad (5)\end{aligned}$$

with the boundary condition stated above. Writing $H=H_0+H_I$, (5) becomes

$$\begin{aligned}&\sum_{k'} f(\omega',\omega)\sum_{(n')}F(n',n)[H_0-E(\alpha,n)-\omega']|\alpha;n,k'\pm\rangle\\&\quad=-\sum_{k'} f(\omega',\omega)\sum_{(n')}F(n',n)H_I|\alpha;n,k'\pm\rangle. \quad (6)\end{aligned}$$

Thus, the state $\|\alpha;n,k\pm\rangle$ must satisfy the integral equation

$$\begin{aligned}\|\alpha;n,k\pm\rangle=&\|\alpha;n,k\rangle_0-\sum_{k'} f(\omega',\omega)\sum_{(n')}F(n',n)\\&\times[H_0-E(\alpha,n)-\omega'\pm i\epsilon]^{-1}H_I|\alpha;n,k'\pm\rangle, \quad (7)\end{aligned}$$

where limit $\epsilon\to0+$ is understood.

$$\|\alpha;n,k\rangle_0\equiv\sum_{k'} f(\omega',\omega)\sum_{(n')}f(n',n)|\alpha;nk'\rangle_0$$

is a solution of the equation

$$\begin{aligned}&\sum_{k'} f(\omega',\omega)\sum_{(n')}F(n',n)\\&\qquad\times(H_0-E(\alpha,n)-\omega')|\alpha;n,k'\rangle_0=0. \quad (8)\end{aligned}$$

Since the state $\|\alpha;n\pm\rangle$ was a wave packet constructed from eigenstates of H, $|\alpha;n,k'\rangle_0$ will not be simply $a_{k'}^\dagger|\alpha;n\rangle$. Using the relation $[a_{k'}^\dagger,H_0]=-\omega' a_{k'}^\dagger$, we can write the solution to (8) as

$$\begin{aligned}\|\alpha;n,k\rangle_0=&\sum_{k'} f(\omega',\omega)\sum_{(n')}F(n',n)\\&\times[a_{k'}^\dagger+(H_0-E(\alpha,n)-\omega'\pm i\epsilon)^{-1}a_{k'}^\dagger H_I]\\&\times|\alpha;n\pm\rangle. \quad (9)\end{aligned}$$

The state $\|\alpha;n,k\rangle_0$ represents a superposition of plane-wave states since the second term on the right-hand side of (9) cancels the incoming or outgoing wave contained in the state $|\alpha;n\pm\rangle$. The solution to the integral equation (7) is

$$\begin{aligned}\|\alpha;n,k\pm\rangle=&\sum_{k'} f(\omega',\omega)\sum_{(n')}F(n',n)\\&\times\{a_{k'}^\dagger-(H-E(\alpha,n)-\omega'\pm i\epsilon)^{-1}\\&\times[H_I,a_{k'}^\dagger]\}|\alpha;n\pm\rangle. \quad (10)\end{aligned}$$

This solution can be verified by substituting (10) and (9) into (7) and obtaining an identity, using the operator relation

$$\frac{1}{A+B}=\frac{1}{A}-\frac{1}{A}B\frac{1}{A+B}. \quad (11)$$

If the state $\|\alpha;n,k\pm\rangle$ is written

$$\|\alpha;n,k\pm\rangle=a^\dagger(k\pm)\|\alpha;n\pm\rangle, \quad (12)$$

[5] Fourier transform variable set equal to the initial or final energy of the system.

[6] M. L. Goldberger and S. B. Trieman, Phys. Rev. **113**, 1663 (1959). P. De Celles and G. Feldman, Nucl. Phys. **14**, 517 (1960).

[7] A. good function is one which is everywhere differentiable any number of times, such that it and all its derivatives are $O(|x|^{-N})$, as $|x|\to\infty$, for all N.

then it is clear that the operator $a^\dagger(k\pm)$ is the in or out field for a θ particle of energy ω, since the state $\|\alpha;n\pm\rangle$ is an arbitrary member of a complete set of states for our system. If we take the product of this equation with an arbitrary, fixed physical state $\|\beta;m\pm\rangle$ on the left, the following expression is obtained for matrix elements of a $a^\dagger(k\pm)$:

$$\begin{aligned}\langle\beta;m\|a^\dagger(k\pm)\|\alpha;n\rangle &= \textstyle\sum_{k'} f(\omega',\omega)\langle\beta;m\|a_{k'}^\dagger\|\alpha;n\rangle \\ &\quad -\textstyle\sum_{k'} f(\omega',\omega)\sum_{(n')} \mathrm{F}(n',n)\sum_{(m')} F^*(m',m) \\ &\quad \times\langle\beta;m|[H_I,a_{k'}^\dagger]|\alpha;n\rangle/(E(\beta')-E(\alpha')-\omega'\pm i\epsilon)\,. \end{aligned}\tag{13}$$

Thus we have a representation for matrix elements of the in or out field in terms of matrix elements of the Heisenberg field taken at time $t=0$. The above argument proceeds in the same manner for the fermion fields. If the state $\|\alpha;n\pm\rangle$ is taken to be the vacuum, then (10) becomes the familiar Lippman-Schwinger equation. If $\|\alpha;n\pm\rangle$ is taken to be the one-nucleon state, then (10) becomes the Wick equation for meson-nucleon scattering.

Now let us consider

$$\lim_{t\to\mp\infty}\textstyle\sum_{k'} f(\omega',\omega)e^{-i\omega' t}\langle\beta;m\|a_{k'}^\dagger(t)\|\alpha;n\rangle\,,\tag{14}$$

where $\|\alpha;n\rangle$ and $\|\beta;m\rangle$ are physical scattering states and $a_{k'}^\dagger(t)$ is the Heisenberg field. We know that

$$i\frac{d}{dt}a_{k'}^\dagger(t)=[a_{k'}^\dagger(t),H(t)] =-\omega' a_{k'}^\dagger+[a_{k'}^\dagger(t),H_I(t)]\tag{15}$$

which has the formal solution

$$a_{k'}^\dagger(t)=e^{i\omega' t}\left\{a_{k'}^\dagger+\frac{1}{i}\int_0^t dt' e^{-i\omega' t'}[a_{k'}^\dagger(t'),H_I(t')]\right\}\,.\tag{16}$$

Taking the matrix element of (16) with the state $\|\alpha;n\rangle$ on the right and $\|\beta;m\rangle$ on the left, multiplying by $f(\omega',\omega)e^{-i\omega' t}$, and summing over k', we obtain the expression

$$\begin{aligned}\textstyle\sum_{k'} f(\omega',\omega)e^{-i\omega' t}\langle\beta;m\|a_{k'}^\dagger(t)\|\alpha;n\rangle &=\textstyle\sum_{k'} f(\omega';\omega)\langle\beta;m\|a_{k'}^\dagger\|\alpha;n\rangle \\ &\quad+\textstyle\sum_{k'} f(\omega',\omega)\frac{1}{i}\int_0^t dt' e^{-i\omega' t'} \\ &\quad\times\langle\beta;m\|[a_{k'}^\dagger(t'),H_I(t')]\|\alpha;n\rangle\,.\end{aligned}\tag{17}$$

Using a time translation,

$$\langle\beta;m\|[a_{k'}^\dagger(t'),H_I(t')]\|\alpha;n\rangle =e^{-i[\omega'+E(\alpha)-E(\beta)]t'}\langle\beta;m\|[a_{k'}^\dagger,H_I]\|\alpha;n\rangle\,.\tag{18}$$

Since

$$\int_0^{\pm\infty} dt' e^{-iat'}=-\frac{i}{a\mp i\epsilon}\,,\tag{19}$$

we find by comparing with (13) that

$$\lim_{t\to\mp\infty}\textstyle\sum_{k'} f(\omega',\omega)e^{-i\omega' t}\langle\beta;m\|a_{k'}^\dagger(t)\|\alpha;n\rangle =\langle\beta;m\|a^\dagger(k\pm)\|\alpha;n\rangle\,.\tag{20}$$

Equation (20) is the statement of the LSZ asymptotic condition satisfied by the field $a_k^\dagger(t)$.[8] The same argument holds for the fermion fields. Thus, we find that if solutions to Eq. (5) with the indicated boundary conditions exist, and can be expressed as (10), then the Heisenberg fields $\psi_V(t)$, $\psi_N(t)$, and $a_k(t)$ converge weakly to the V, N, and θ "in" and "out" fields:

$$\begin{aligned}&\lim_{t\to\mp\infty}\langle\beta;m\|e^{-im_\alpha t}\psi_\alpha^\dagger(t)\|\alpha;n\rangle =\langle\beta;m\|\psi_\alpha^\dagger(\pm)\|\alpha;n\rangle\,,\\ &\lim_{t\to\mp\infty}\textstyle\sum_{k'} f(\omega',\omega)e^{-i\omega' t}\langle\beta;m\|a_{k'}^\dagger(t)\|\alpha;n\rangle =\langle\beta;m\|a^\dagger(k\pm)\|\alpha;n\rangle\,.\end{aligned}\tag{21}$$

We have assumed that a complete set of states can be obtained by successively applying the appropriate in or out operators on the vacuum $|0\rangle$, defined by the three relations:

$$\begin{aligned}\psi_V(\pm)|0\rangle&=0\,,\\ \psi_N(\pm)|0\rangle&=0\,,\\ a(k\pm)|0\rangle&=0\,.\end{aligned}\tag{22}$$

Let us now derive the reduction formula using the LSZ asymptotic conditions (21). Dropping the wave-packet notation, we obtain[9]

$$S_{\alpha\beta}=\langle\beta;m-|\alpha;n+\rangle\,.\tag{23}$$

Reducing first on the fermion "from the right," we obtain

$$S_{\alpha\beta}=\lim_{t\to-\infty}\langle\beta;m-|e^{-im_\alpha t}\psi_\alpha^\dagger(t)|0;n+\rangle\,.\tag{24}$$

Since $\|0;n+\rangle$ is a state of n θ particles, it is the same as the suitably normalized bare state:

$$|0;n\pm\rangle=\frac{1}{\sqrt{n!}}\prod_1^n a_{k_\nu}^\dagger(0)|0\rangle =\frac{1}{\sqrt{n!}}\prod_1^n e^{-i\omega_\nu t}a_{k_\nu}^\dagger(t)|0\rangle\,.\tag{25}$$

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[8] Note that the lim $_{t\to\mp\infty}$ has been interchanged with $\sum_{k'}$. For this to be valid, the integrand as a function of ω' and t must satisfy certain properties. These properties are very similar to the conditions under which the expression (9) is a valid solution to (8). Recognizing that one would have to prove these properties in order to make the above argument rigorous, we assume that matrix elements of the Heisenberg fields are sufficiently well behaved that the above steps are valid.

[9] Henceforth, we denote energies in the state $|\alpha;n\rangle$ as ω_ν and energies in the state $|\beta;m\rangle$ as ω_μ'.

We have used the fact that

$$a_{k_\nu}{}^\dagger(t)e^{i\omega_\nu t}=a_{k_\nu}{}^\dagger(0)+D(t)\,,$$

where

$$D(t)|0\rangle=0$$

and

$$[a_{k_\nu}{}^\dagger(t),D(t)]=0.$$

Thus, (24) becomes

$$S_{\alpha\beta}=\lim_{t\to-\infty}\langle\beta;m-|\alpha^\dagger(t)|0\rangle\,, \tag{26}$$

where

$$\alpha^\dagger(t)=\exp[-i(m_\alpha+\sum_1^n\omega_\nu)t]\psi_\alpha{}^\dagger(t)\prod_1^n\frac{a_{k\nu}{}^\dagger(t)}{\sqrt{n!}}\,.$$

In the usual manner, this leads to the result

$$S_{\alpha\beta}=\delta_{\alpha\beta}'+\frac{i}{\sqrt{n!}}\int_{-\infty}^{\infty}dt\bigg[\langle\beta;m-|\psi_\alpha{}^\dagger(t)\prod_1^n a_{k_\nu}{}^\dagger(t)|0\rangle$$
$$\times\bigg(i\frac{\overleftarrow{d}}{dt}+m_\alpha+\sum_1^n\omega_\nu\bigg)\bigg]\exp[-i(m_\alpha+\sum_1^n\omega_\nu)t]\,, \tag{27}$$

where

$$\delta_{\alpha\beta}'=\delta_{nm}\delta_{\alpha\beta}.$$

Reducing "from the left," we obtain the full reduction formula

$$S_{\alpha\beta}=\delta_{\alpha\beta}'-\frac{1}{(n!m!)^{1/2}}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt\,\exp[i(m_\beta+\sum_1^n\omega_\mu')t']$$
$$\times\overrightarrow{D}_\beta(t')\tau_{\alpha\beta}(t',t)\overleftarrow{D}_\alpha{}^*(t)\exp[-i(m_\alpha+\sum_1^n\omega_\nu)t]\,, \tag{28}$$

where

$$D_\alpha(t)=\bigg(i\frac{d}{dt}-m_\alpha-\sum_1^n\omega_\nu\bigg)$$

and

$$\tau_{\alpha\beta}(t',t)=\langle0|T[\psi_\beta(t')\prod_1^m a_{k_\mu}{}'(t')\psi_\alpha{}^\dagger(t)\prod_1^n a_{k_\nu}{}^\dagger(t)]|0\rangle \tag{29}$$

is the so-called τ function. By a time translation,

$$\tau_{\alpha\beta(t',t)}=\langle0|T[\psi_\beta(s)\prod_1^m a_{k_{\mu'}}(s)\psi_\alpha{}^\dagger(0)\prod_1^n a_{k_\nu}{}^\dagger(0)]|0\rangle$$
$$=\tau_{\alpha\beta}(s)\,, \tag{30}$$

where $s=t'-t$. Thus, the most general τ function in the Lee model is a function of $n+m$ energies and one time.

It is convenient to introduce the Fourier transforms of the τ functions,

$$\hat{\tau}_{\alpha\beta}(W)=\frac{1}{i}\int_{-\infty}^{\infty}dse^{iWs}\tau_{\alpha\beta}(s)$$

so that

$$\tau_{\alpha\beta}(s)=\frac{i}{2\pi}\int_{-\infty}^{\infty}dWe^{-iWs}\hat{\tau}_{\alpha\beta}(W)\,. \tag{31}$$

Substituting (31) into (28), the reduction formula becomes

$$S_{\alpha\beta}=\delta_{\alpha\beta}'+2\pi i\delta(m_\alpha+\sum_1^n\omega_\nu-m_\beta-\sum_1^m\omega_\nu')$$
$$\times(n!m!)^{-1/2}(m_\alpha+\sum_1^n\omega_\nu-m_\beta-\sum_1^m\omega_\nu')^2$$
$$\times[\tau_{\alpha\beta}(W)]_{W=m_\alpha+\sum_1^m\omega_\nu}. \tag{32}$$

Therefore, the Fourier transforms of the τ functions, when evaluated on the energy shell, are just the T-matrix elements with an additional second-order pole. It can also be shown that the coefficients in the expansion of the state vector can be obtained from the τ functions. Thus, the problem of finding all the possible scattering amplitudes reduces to solving the Matthews-Salam equations[3] for the τ functions. Now, we give a brief review of the Lee model.

The renormalized Hamiltonian describing the system of V, N, and θ particles is

$$H=m_VZ\psi_V{}^\dagger\psi_V+m_N\psi_N{}^\dagger\psi_N+\sum_k\omega a_k{}^\dagger a_k+\delta m_VZ\psi_V{}^\dagger\psi_V$$
$$+g\bigg[\psi_V{}^\dagger\psi_N\sum_k\frac{u(\omega)}{(2\omega)^{1/2}}a_k+\psi_N{}^\dagger\psi_V\sum_k\frac{u(\omega)}{(2\omega)^{1/2}}a_k{}^\dagger\bigg]\,, \tag{33}$$

where ω is $(k^2+\mu^2)^{1/2}$ and μ is the mass of a particle; m_V, m_N are the masses of the V and N particles; δm_V, Z are the mass and wave function renormalization constants, and g is the renormalized coupling constant. The summation is used for convenience. $u(\omega)$ is the cutoff function and is chosen such that all integrals in the theory will be finite. In order to have a stable V particle, we insist that $m_V<m_N+\mu$. The interaction allows a V particle to emit a θ and become and N, or the N particle can absorb a θ and become a V.

The commutation relations are

$$\{Z^{1/2}\psi_V,Z^{1/2}\psi_V{}^\dagger\}=\{\psi_N,\psi_N{}^\dagger\}=1\,,\quad [a_k,a_{k'}{}^\dagger]=\delta_{kk'}\,. \tag{34}$$

From (33), the field equations are

$$Z(i(d/dt)-m_0)\psi_V(t)=g\sum_k(u(\omega)/(2\omega)^{1/2})\psi_N(t)a_k(t)\,,\quad(\text{where}\quad m_0=m_V+\delta m_V)\,, \tag{35a}$$

$$(i(d/dt)-m_N)\psi_N(t)=g\sum_k(u(\omega)/(2\omega)^{1/2})\psi_V(t)a_k{}^\dagger(t)\,, \tag{35b}$$

$$(i(d/dt)-\omega)a_k(t)=g(u(\omega)/(2\omega)^{1/2})\psi_N{}^\dagger(t)\psi_V(t)\,. \tag{35c}$$

There are two operators which commute with H:
$Q_1=Z\psi_V^\dagger\psi_V+\psi_N^\dagger\psi_N=$ total number of fermions,

$$Q_2=Z\psi_V^\dagger\psi_V+\sum_k a_k^\dagger a_k$$
$$= \text{total number of } V \text{ particles} + \theta \text{ particles}. \quad (36)$$

Thus the model breaks up into sectors designated by the integer eigenvalues of Q_1 and Q_2. The first three sectors are trivial since the physical vacuum, one-particle N state, and one-particle θ states are the same as the corresponding bare states. The first nontrivial sector is spanned by the physical V-particle state and the N-θ scattering states. One proceeds to each of the higher sectors by adding θ particles, one at a time.

III. THE V-$N\theta$ SECTOR ($q_1=1$; $q_2=1$)

The physical V-particle state and the N-θ scattering states form a complete set for this sector. There are four τ functions approprate to this sector. In momentum space, these correspond to the V propagator, the V-$N\theta$ vertex, and the Green's function for N-θ scattering. The τ functions are

$$\tau^1(s)=\langle 0|T[\psi_V(s)\psi_V^\dagger(0)]|0\rangle\,, \quad (37a)$$

$$\tau^2(s,\omega)=((2\omega)^{1/2}/u(\omega))\langle 0|T[\psi_N(s)a_k(s)\psi_V^\dagger]|0\rangle\,, \quad (37b)$$

$$\tau^3(s;\omega)=((2\omega)^{1/2}/u(\omega))\langle 0|T[\psi_V(s)\psi_N^\dagger a_k^\dagger]|0\rangle\,, \quad (37c)$$

$$\tau^4(s;\omega,\omega')=((4\omega\omega')^{1/2}/u(\omega)u(\omega'))\times\langle 0|T[\psi_N(s)a_k(s)\psi_N^\dagger a_{k'}^\dagger]|0\rangle\,, \quad (37d)$$

and their Fourier transforms are

$$\hat{\tau}^\alpha(W;\omega,\omega')=\frac{1}{i}\int_{-\infty}^{\infty}ds\, e^{iWs}\tau^\alpha(s;\omega,\omega')\,. \quad (37e)$$

From (34) and (35), the Matthews-Salam equations are

$$\text{(A)}\begin{cases}\left(i\frac{d}{ds}-m_0\right)\tau^1(s)=\frac{i}{Z}\delta(s)+\frac{g}{Z}\sum_k\frac{u^2(\omega)}{2\omega}\tau^2(s;\omega)\,, & (38a)\\[2ex] \left(i\frac{d}{ds}-m_N-\omega\right)\tau^2(s;\omega)=g\tau^1(s)\,, & (38b)\end{cases}$$

$$\left(i\frac{d}{ds}-m_N-\omega\right)\tau^3(s;\omega)=g\tau^1(s)\,, \quad (38c)$$

$$\left(i\frac{d}{ds}-m_N-\omega\right)\tau^4(s;\omega,\omega')=i\frac{2\omega}{u^2(\omega)}\delta_{kk'}+g\tau^3(s;\omega')\,. \quad (38d)$$

Since $\tau^2(0;\omega)=\tau^3(0;\omega)=0$, (38b) and (38c) give

$$\tau^2(s;\omega)=\tau^3(s;\omega)\,. \quad (39)$$

An alternate equation for $\tau^4(s;\omega,\omega')$ is

$$(i(d/ds)-m_N-\omega')\tau^4(s;\omega,\omega')=i(2\omega/u^2(\omega))\delta_{kk'}+g\tau^2(s;\omega)\,.$$

Interchanging ω with ω' and using (39),

$$\tau^4(s;\omega,\omega')=\tau^4(s;\omega',\omega) \quad (40)$$

since

$$\tau^4(0;\omega,\omega')=\tau^4(0;\omega',\omega)=(2\omega/u^2(\omega))\delta_{kk'}\,.$$

We see that (38a) and (38b) form a coupled set of equations (A) for $\tau^1(s)$ and $\tau^2(s;\omega)$. Since $\tau^3(s;\omega)$ was expressed in terms of $\tau^2(s;\omega)$, and $\tau^4(s;\omega,\omega')$ was expressed in terms of $\tau^3(s;\omega')$, the entire sector is solved when we find a solution to the system (A). This property of the Matthews-Salam equations will persist in the higher sectors.

Transforming (38) into momentum space,

$$(W-m_0)\hat{\tau}^1(W)=\frac{1}{Z}+\frac{g}{Z}\sum_k\frac{u^2(\omega)}{2\omega}\hat{\tau}^2(W;\omega)\,, \quad (41a)$$

$$(W-m_N-\omega)\hat{\tau}^2(W;\omega)=g\hat{\tau}^1(W)\,, \quad (41b)$$

$$(W-m_N-\omega)\hat{\tau}^4(W;\omega,\omega')=\frac{2\omega}{u^2(\omega)}\delta_{kk'}+g\hat{\tau}^2(W;\omega')\,. \quad (41c)$$

From (41b),

$$\hat{\tau}^2(W;\omega)=g\frac{\hat{\tau}^1(W)}{(W-m_N-\omega+i\epsilon)}\,, \quad (42)$$

where $\lim_{\epsilon\to 0+}$ is understood. Substituting (42) into (41a), we obtain

$$\hat{\tau}^1(W)=Z^{-1}[W-m_0-f(W)]^{-1}\,, \quad (43)$$

where

$$f(W)=\frac{g^2}{Z}\sum_k\frac{u^2(\omega)}{2\omega}\frac{1}{(W-m_N-\omega+i\epsilon)}\,, \quad (44)$$

and the sector has been solved.

To obtain the renormalization constants Z and δm_V, the analytic structure of the V propagator, $\hat{\tau}^1(W)$, is studied. Using the definition (37e), and inserting a complete set of intermediate states, we obtain

$$\hat{\tau}^1(W)=\frac{1}{i}\int_0^{\infty}ds e^{iWs}[|\langle 0|\psi_V(s)|V\rangle|^2+\sum_k|\langle 0|\psi_V(s)|N\theta\omega\rangle|^2]\,. \quad (45)$$

Using a time translation in the integral and the asymptotic condition on the V field, (45) can be written in the form

$$\hat{\tau}^1(W)=\frac{1}{(W-m_V+i\epsilon)}+\sum_k\frac{|\langle 0|\psi_V|N\theta\omega\rangle|^2}{(W-m_N-\omega+i\epsilon)}\,. \quad (46)$$

Thus, $\hat{\tau}^1(W)$ has a simple pole at the physical mass of the V particle, with a residue equal to $+1$. Insisting that the denominator in (43) vanish at $W=m_V$,

$$f(m_V)=-\delta m_V. \tag{47}$$

This gives the mass renormalization constant

$$\delta m_V=-\frac{g^2}{Z}\sum_k \frac{u^2(\omega)}{2\omega}\frac{1}{(m_V-m_N-\omega)}. \tag{48}$$

Setting the residue equal to 1, we obtain

$$Z=1-g^2\sum_k \frac{u^2(\omega)}{2\omega}\frac{1}{(m_V-m_N-\omega)^2}. \tag{49}$$

Substituting (48) and (49) into (43), we obtain

$$\hat{\tau}^1(W)=(W-m_V+i\epsilon)^{-1}[1-\beta(W)]^{-1}, \tag{50}$$

where

$$\beta(W)=g^2(W-m_V)\sum_k \frac{u^2(\omega)}{2\omega} \times\frac{1}{(m_V-m_N-\omega)^2(W-m_N-\omega+i\epsilon)}. \tag{51}$$

Z is restricted to lie between 0 and 1, so the coupling constant g will lie between 0 and some critical value.

N-θ Scattering

The reduction formula for N-θ scattering is, from (32),

$$S_{kk'}=\delta_{kk'}+2\pi i\delta(\omega-\omega')(u(\omega)u(\omega')/(4\omega\omega')^{1/2}) \times(\omega-\omega')^2\hat{\tau}^4(W;\omega,\omega')\,|_{W=m_N+\omega}. \tag{52}$$

From (41c), (42), (50), and (52)

$$S_{kk'}=\delta_{kk'}+2\pi i\delta(\omega-\omega')\frac{u(\omega)u(\omega')}{(4\omega\omega')^{1/2}} \times\frac{g^2}{(m_V-m_N-\omega)[1-\beta(\omega+m_N)]}. \tag{53}$$

The four τ functions are

$$\hat{\tau}^1(W)=(W-m_V)^{-1}[1-\beta(W)]^{-1}, \tag{54a}$$

$$\hat{\tau}^2(W;\omega)=\hat{\tau}^3(W;\omega)=g(W-m_N-\omega)^{-1} \times(W-m_V)^{-1}[1-\beta(W)]^{-1}, \tag{54b}$$

$$\hat{\tau}^4(W;\omega,\omega')=(2\omega/u^2(\omega))\delta_{kk'}(W-m_V-\omega)^{-1} +g^2(W-m_N-\omega)^{-1}(W-m_N-\omega')^{-1} \times(W-m_V)^{-1}[1-\beta(W)]^{-1}, \tag{54c}$$

where

$$\beta(W+m_N)=\frac{g^2}{4\pi^2}(m_V-m_N-W) \times\int_\mu^\infty \frac{u^2(\omega)(\omega^2-\mu^2)^{1/2}d\omega}{(m_V-m_N-\omega)^2(\omega-W-i\epsilon)}, \tag{55}$$

transforming to a continuous space.

In the literature, this sector was first solved by finding solutions to the eigenvalue equations for the V-particle state and the N-θ scattering states.[4] In this approach, an integral equation arises in the N-θ problem, but the kernel is separable and the equation reduces to an algebraic relation. More recently, a dispersion treatment has been given for the N-θ amplitude and the problem of finding the renormalization constants.[6] From this approach one solves a Low or Omnes type singular integral equation.

IV. CONCLUDING REMARKS

We have shown in Sec. II that the LSZ asymptotic conditions on the V, N, and θ fields are equivalent to the statement that scattering states can be constructed from the eigenvalue equation in the usual manner. The reduction formula was derived giving the relation between the Fourier transform of the τ functions, when evaluated on the energy shell, and the conventional T-matrix elements. Since there is no interaction between θ particles, the most general τ function contains only one time variable.

The solution to the V-$N\theta$ sector, given in Sec. III, clearly demonstrates the advantage in using the LSZ formalism as a calculational approach. The structure of the Matthews-Salam equations is such that the solution to one algebraic equation (or integral equation in the higher sectors) solves the entire sector. Also, the relations between τ functions and the symmetry properties of these functions under interchange of variables follow immediately from the equations. Thus, it appears that the τ-function approach elucidates the basic structure of the model in the most natural way, thereby simplyfying the mathematics considerably. This will become obvious in a forthcoming article when the solution to the V-θ sector is presented.

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