PHYSICAL REVIEW VOLUME 137, NUMBER 4B 22 FEBRUARY 1965

Four-Pion Models for η and X^0 Mesons

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(Received 5 October 1964)

A four-pion formalism is set up through a semirelativistic Schrödinger equation with pairwise $\pi-\pi$ interactions in the p state. Two kinds of interaction with factorable kernels, both of which reproduce the mass and width of the ρ meson, are considered: (a) an intrinsically attractive interaction which has a very short range and (b) a long-range interaction which is repulsive at low energies but becomes attractive at high enough energies. The four-pion states of $T=0$ are analyzed in terms of spatial and isospin functions of appropriate symmetries, and these are used to examine the spin-parity states of 0^{-+} and 1^{++}, which are the only 4π states of $T=0$ depending on π-π interation in odd-l states, via Bose statistics. By the assumption of factorable kernels, these equations are exactly reducible to "equivalent three-body equations." The solutions of these equations, which can be obtained under certain reasonable approximations (discussed in the text), show that the results are incompatible with a bound pseudoscalar state of $T=0$, so that the η meson cannot be understood on this model. The model also rules out an axial-vector state of $T=0$, bound or resonant. However, the model predicts a resonant-pseudoscalar state at a mass very close to that of X^0 (960 MeV), and a second one at about 1.4 BeV, only when the $\pi-\pi$ interaction is of type b, but not if it is of type a. Certain other implications of a type b interaction are discussed briefly.

1. INTRODUCTION

If the η meson is looked upon as a composite state made up entirely of pions, one would need at least four of them to build up the quantum numbers 0^{-+}, $T=0$ appropriate to its structure. Though the relatively low mass of η ($m_\eta \approx 550$ MeV) makes it energetically impossible for it to decay into four pions ($4m_\pi \approx 560$ MeV), it is interesting to study the possibilities of understanding this particle as, say, a bound state of four pions, in much the same way as ω was first looked upon as a resonant state of three pions.[1] One feature common to both ω and η concerns the effect of Bose statistics which prevents π–π interaction in even-l states from producing their respective quantum numbers. As for odd-l states, the dominant contribution is of course expected from $l=1$. Thus if, ignoring the effects of other channels, ω and η mesons are looked upon as 3π and 4π states, respectively, they would provide rather clean examples of the effects of p-wave π–π interaction. More specifically, the quantum numbers of these particles, so to say, "saturate" the p-wave π–π interaction in 3π and 4π systems.

Analogous arguments may be extended to at least two other particles. The ϕ meson at 1020 MeV, with its quantum numbers 1^{--}, $T=0$, should admit of a 3π analysis in terms of p-wave interaction alone, so far as the $\rho+\pi$ channel is relevant to its structure.[2,3] A more interesting case is provided by the recently discovered[4] X^0 resonance at 960 MeV, whose quantum numbers are believed to be either $T=0$ with 0^{-+} or $T=1$ with 1^{++}.[5] While the decay analysis of this resonance suggests the most important decay channel as $\eta+2\pi$ [4], it is likely that the 4π channel (being less massive) could play an important role in its formation, and to that extent a 4π analysis with only p-wave π–π interaction, analogous to the η case, should apply to this resonance as well.

Some time ago one of us had proposed a scheme of p-wave interaction through a relativistic Tamm-Dancoff type equation in which the kernel was assumed separable in momentum space.[6] This interaction, with its parameters adjusted to fit the mass and width of the ρ meson, was used to study the ω meson as a 3π system of $T=0$ through a three-body Tamm-Dancoff type equation.[6] We propose here an extension of this scheme to study certain 4π states of $T=0$ which can be generated in terms of p-wave π–π interaction.

The main disadvantage of any potential model, however relativistic, is its failure to take account of channels with different numbers or types of particles from what are present in the original channel. While this limitation can be quite serious at energies high enough for the *production* of particles, a potential model through a Schrödinger type equation still offers certain compensating advantages not yet available through dispersion-theoretic techniques. For, while dispersion theory has reached a high degree of perfection in handling multichannel two-body problems, it has not yet been able to provide adequate methematical facilities for the treatment of systems involving three or more particles. This limitation has necessitated suitable two-body approximations on more complicated systems before its techniques can be applied. However, such approximations which must regard certain composite particles as stable ones of *fixed* masses, cannot

[1] G. F. Chew, Phys. Rev. Letters 4, 142 (1960).
[2] The present experimental limits on the $\rho+\pi$ decay mode of ϕ are 35±20% (Ref. 3).
[3] P. L. Connolly, E. L. Hart, K. W. Lai, G. London, G. C. Moneti *et al.*, Phys. Rev. Letters **10**, 371 (1963); see also, Y. Nikitin; Proceedings of the Dubna Conference on High Energy Physics, 1964 (to be published).
[4] G. R. Kalbfleisch, L. W. Alvarez, A. Barbaro-Galtieri, O. I. Dahl, P. Eberhard *et al.*, Phys. Rev. Letters **12**, 527 (1964).
[5] M. Goldberg, M. Gundzik, J. Leitner, M. Primer, P. L. Conolly *et al.*, Phys. Rev. Letters **13**, 249 (1964).
[6] A. N. Mitra, Phys. Rev. **127**, 1342 (1962); referred to as A.

simulate the effect of the internal momentum distributions of the composite particles, or the so-called "polarization effects" due to the possibility of interactions of the individual components of a composite system with external particles. Moreover, if the composite particle is a resonance, rather than a bound state, one encounters additional difficulties by way of certain overlapping cuts in the physical region (for example, as in the "Peierls mechanism").[7] Since on the other hand, a Schrödinger model at least offers a mathematically consistent scheme for the treatment of more than two particles, it is in principle free from the necessity of making two-body approximations to many-body problems, so that the detailed momentum distributions of all the "elementary" particles are allowed to play their full role on the problem at hand. We believe that these aspects of the detailed momentum distributions in a composite system are no less important than the role of "other channels" in the study of particle resonance.[8] As for the limitation of particle conservation, the possibility of identifying certain particles as composite states of "more elementary particles" (in a spirit converse to dispersion theory)[9] affords enough flexibility of approach in a practical problem even within a potential framework.

The importance of treating multiparticle states *as such*, and not merely approximated by certain structureless composites in a practical calculation, has been emphasized in the context of the three-body problem in a very recent paper by Lovelace.[10] Using the three-particle formalism of Faddeev[11] as the basis of his approach, Lovelace has shown, among other things, that in a three-body problem, the separable kernel approximation to two-body scattering is indeed a valid description provided each of the two-particle subsystems is dominated by a limited number of bound states and resonances. In this respect, the separable kernel approach of Ref. 6 which used a p-wave π–π interaction seems to have some formal justification, because of the dominance of the ρ meson in the π–π system. In the present calculation of a 4π, $T=0$ system with quantum numbers 0^{-+} or 1^{++}, which also requires only p-wave interaction between π–π pairs, we hope that a similar justification will be available, and as such a separable π–π potential in the p state, adjusted to fit the parameters of ρ, will form the basis of this paper.

In Sec. 2, the model of p-wave π–π interaction for the 4π problem is discussed on the basis of results obtained from earlier calculations on 3π systems. Two distinct types of interaction are suggested, (a) an intrinsically attractive interaction[6] and (b) an interaction which is repulsive at low energies but becomes attractive at high enough energies.[12] Sec. 3 is concerned with the application of symmetry principles to the four-pion wave function following the analysis of Grynberg and Koba,[13] and the derivation of a coupled set of equations for the spatial functions after elimination of the isospin dependence. In Sec. 4, certain equivalent "three-body" equations are deduced for various combinations of spin and parity for the 4π system, on lines closely analogous to Ref. 6, which is referred to as A. An approximation procedure for the solution of these equations for the total energy (real part) of the 4π system looked upon as an "eigenvalue problem," is outlined in Appendices I and II. In Sec. 5 is given a simple variational treatment for the determination of the total energy, to be compared with the result of more direct evaluation in Sec. 4. Section 6 is devoted to a discussion of the numerical results obtained with the two types of π–π force considered, the main conclusion being that the energy of the η meson is too low to be explained by such a mechanism, but that the X^0 meson admits of an explanation on the basis of the type b force.

2. π-π INTERACTION FOR THE 4π PROBLEM

Let the momenta of the four pions be taken as $\mathbf{P}_i$ ($i=1, 2, 3, 4$), so that in the over-all c.m. system

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3+\mathbf{P}_4=0. \tag{2.1}$$

The pairwise interaction is normalized by

$$\langle \mathbf{P}_i\mathbf{P}_j | V_{ij} | \mathbf{P}_i'\mathbf{P}_j' \rangle = \delta(\mathbf{P}_{ij}-\mathbf{P}_{ij}')\langle \mathbf{p}_{ij} | V_{ij} | \mathbf{p}_{ij}' \rangle , \tag{2.2}$$

where

$$\mathbf{P}_{ij}=\mathbf{P}_i+\mathbf{P}_j, \quad 2\mathbf{p}_{ij}=\mathbf{P}_i-\mathbf{P}_j \tag{2.3}$$

and the p-wave π–π interaction is given by

$$\langle \mathbf{p} | V | \mathbf{p}' \rangle = 3\lambda\rho(1)(\mathbf{p}\cdot\mathbf{p}')v(p)v(p'). \tag{2.4}$$

$\rho(1)$ is the isospin projection operator for $T=1$ and $v(p)$ is a shape factor. This interaction must be incorporated into a relativistic Schrödinger type equation which was taken in A as

$$(\omega_1+\omega_2-E)\Psi = -V_{12}\Psi \tag{2.5}$$

and

$$(\omega_1+\omega_2+\omega_3-E)\Psi = -(V_{12}+V_{23}+V_{31})\Psi \tag{2.6}$$

for the 2π and 3π systems, respectively. Here ω_i is the relativistic energy of the ith pion and

$$V_{ij}\Psi = \int\int d\mathbf{P}_i' d\mathbf{P}_j' \langle \mathbf{P}_i\mathbf{P}_j | V_{ij} | \mathbf{P}_i'\mathbf{P}_j' \rangle \Psi(\mathbf{P}_i'\mathbf{P}_j'). \tag{2.7}$$

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[7] See for a detailed discussion on this question, as well as earlier references, C. Goebels, Phys. Rev. Letters **12**, 134 (1964).

[8] Indeed, if a resonance is reasonably sharp one might expect that even a single-channel approach to it would at least locate its position (if not the width) in a reliable way.

[9] For example, a nucleon could be regarded as a bound state of a pion and nucleon in order to conserve the number of particles, namely three, in the reaction $\pi+N \rightarrow 2\pi+N$.

[10] C. Lovelace, Phys. Rev. **135**, B1224 (1964).

[11] L. D. Faddeev, Dokl. Akad. Nauk. SSSR **145**, 301 (1962) [English transl.: Soviet Phys.—Doklady **7**, 600 (1963)]; for a more complete set of references to Faddeev's papers, as well as to other many-body approaches, see Ref. 10.

[12] A. N. Mitra, Nuovo Cimento **33**, 1220 (1964).

[13] M. Grynberg and Z. Koba, Phys. Letters **1**, 130 (1962).

While this form has the advantage of incorporating relativistic kinematics, its extension to the 4π problem through the equation

$$(\omega_1+\omega_2+\omega_3+\omega_4-E)\Psi=-\sum_{i<j}\sum V_{ij}\Psi \qquad (2.8)$$

leads to enormous computational difficulties. A simpler scheme which, however, ignores the retardation effects between individual pion pairs was proposed by Schiff.[14] This replaces Eq. (28) by[15]

$$(P_1^2+P_2^2+P_3^2+P_4^2-E')\Psi=-\sum_{i<j}\sum V_{ij}\Psi, \qquad (2.9)$$

where E' is related to the total relativistic energy E by

$$4E'=E^2-16\mu^2. \qquad (2.10)$$

We shall naturally find it more convenient to use Eq. (2.9) for the present problem and our apology for this simplified approach is that the details of the difference between (2.8) and (2.9) are less important than the basic assumption of a pair wise π–π interaction within the Schrödinger framework for a four-body problem.

Now the fits to the ρ-meson parameters can be obtained in either of two ways, (a) through an intrinsically attractive interaction ($\lambda<0$) or (b) with the help of a "repulsive looking" potential ($\lambda>0$).[16] Type a needs a very short-ranged interaction to make the rather large mass of the ρ meson (≈760 MeV) compatible with its moderately large width of ≈100 MeV. An interaction of type b on the other hand, provides the resonance mechanism through a long-ranged force, the longer the range the sharper the high-energy resonance. Since, in the language of dispersion theory, the ρ-meson problem is a multichannel one, our characterization of the resultant situation by an *effective* potential of type a or b, is linked with the question as to which one of the two types represents a closer approximation to the actual situation. So far we have examined this question in the context of the isoscalar 3π problem for the quantum numbers of the ω and ϕ mesons. It has been found that whereas type a leads to a (tightly bound) $\rho+\pi$ bound state with a mass too low for ω,[17] type b leads to a sharp $\pi+\rho$ resonance lying somewhat higher than the mass of ϕ.[12] Presumably both types of forces are needed in appropriate combinations to account for ω and ϕ simultaneously. However, the present study of isoscalar 4π systems with quantum numbers 0^{-+} or 1^{++} so as to involve only p-wave π–π interactions, is expected to throw further light on the question of discrimination between types a and b. We have therefore chosen to investigate *separately* the effects of these two types of π–π forces on the 4π problem. Taking the shape factor $v(p)$ as

$$v(p)=\exp(-\tfrac{1}{2}p^2\beta^{-2}), \qquad (2.11)$$

the respective parameters which fit the ρ-meson mass and width are[12,16,17]

$$\text{(a)}\ \beta=28.5,\quad \sigma=\tfrac{1}{2}\pi^{3/2}\lambda\beta^3=-0.99 \qquad (2.12)$$

$$\text{(b)}\ \beta=\ 0.9,\quad \sigma=3.3; \qquad (2.13a)$$

$$\beta=\ 1.0,\quad \sigma=2.3. \qquad (2.13b)$$

3. 4π PROBLEM: SYMMETRY CONSIDERATIONS

A considerable reduction of the 4π problem is achieved through the use of symmetry requirements on a system of $T=0$, which case is particularly simple because of certain formal similarities of the corresponding representation matrices to those for a three-body problem.[18] Indeed, the only $T=0$ isospin functions are, in the notation of Grynberg and Koba,[13]

$$\eta[2,2]_{\rm I}\equiv\chi''=\tfrac{2}{3}\chi_0-\tfrac{1}{3}(5)^{1/2}\chi_2, \qquad (3.1)$$

$$\eta[2,2]_{\rm II}\equiv\chi'=\chi_1, \qquad (3.2)$$

$$\eta[4]\equiv\chi_s=\tfrac{2}{3}\chi_2+\tfrac{1}{3}(5)^{1/2}\chi_0, \qquad (3.3)$$

where χ_I is the four-pion isospin function of $T=0$ obtained by combining two pairs of each of total isospin I ($I=0.1.2$). Of these functions, the completely symmetric one, viz., $\eta[4]$ can be generated only through π–π pairs in mutual s waves, and is therefore ruled out for the problem at hand. The permutation matrices (ij) in this case admit of the rather compact 2×2 representations[13]

$$(12)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix};$$
$$(23),(13)=\begin{pmatrix}\tfrac{1}{2} & \pm\sqrt{3}/2\\ \pm\sqrt{3}/2 & -\tfrac{1}{2}\end{pmatrix}. \qquad (3.4)$$

Also, if i, j, k represent 1, 2, 3 in cyclic order, the permutations (ij) and $(k4)$ have *identical* representations. In this respect we notice a strong similarity with the representation matrices for a three-nucleon problem,[19,20] the mixed symmetries (2,2) in the present case corresponding to the mixed symmetries (2,1) for the three-body problem. To make the analogy with the three-body problem closer, we define the basic per-

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[14] L. I. Schiff, Phys. Rev. **125**, 777 (1962).

[15] The dimensional disparity between the V_{ij}'s in Eqs. (2.8) and (2:9) can of course be absorbed in the definition of the coupling constant λ in Eq. (2.4).

[16] The interpretation of such potentials as repulsive at lower energies and attractive at higher ones, has been discussed in the context of s-wave forces by A. N. Mitra, Nuovo Cimento (L) **32**, 506 (1964).

[17] N. Panchapakesan and A. N. Mitra (to be published); the widths at the observed position of the ρ meson come out as 100, 80, and 120 MeV in terms of (2.12), (2.13a), and (2.13b), respectively.

[18] Cf., M. Verde, in *Hanbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 170.

[19] We follow here the representation of Ref. 18 rather than Ref. 13, for the permutation matrices.

[20] A. N. Mitra and V. S. Bhasin, Phys. Rev. **131**, 1265 (1963); referred to as B.

mutation operators, T_1, T_2, T_3, for the present case as in Ref. 19, viz.,

$$T_k=\tfrac{1}{2}[(ij)+(k4)],\quad (i,j,k=1,2,3), \tag{3.5}$$

so that these operators have the same representations as (3.4). The operators corresponding to T_s, T', and T'' of the 3-nucleon problem[18] are, in the basis (3.4),

$$\begin{aligned}T_s&=T_1+T_2+T_3,\\ T'&=\tfrac{1}{2}(3)^{1/2}(T_1-T_2),\\ T''&=T_3-\tfrac{1}{2}(T_1+T_2),\end{aligned} \tag{3.6}$$

so that the isospin functions χ' and χ'' are expressible as

$$\chi''=T''\chi_0,\quad \chi'=T'\chi_0, \tag{3.7}$$

where

$$\chi_0=(\mathbf{u}_1\cdot\mathbf{u}_2)(\mathbf{u}_3\cdot\mathbf{u}_4), \tag{3.8}$$

and $\mathbf{u}_i$ is the (vector) isospin function for the ith pion. If ψ' and ψ'' are the corresponding (2,2) spatial functions for the 4π system, the complete wave function Ψ which is totally symmetric is given by[19]

$$\Psi=2^{-1/2}(\psi'\chi'+\psi''\chi''). \tag{3.9}$$

Elimination of the isospin functions χ', χ'' from the Schrödinger equation (2.9) is now effected by noting that the isospin projection operators $\rho_{ij}(1)$ in the potentials V_{ij} of Eq. (2.4) are expressible as

$$\rho_{ij}=\tfrac{1}{2}[1-(ij)]. \tag{3.10}$$

This leads to the coupled spatial equations

$$D_{E'}(\mathbf{P}_i)\psi'=-(\Lambda^s+\Lambda'')\psi'+\Lambda'\psi'', \tag{3.11}$$

$$D_{E'}(\mathbf{P}_i)\psi''=-(\Lambda^s-\Lambda'')\psi''+\Lambda'\psi', \tag{3.12}$$

where

$$D_{E'}(\mathbf{P}_i)=P_1^2+P_2^2+P_3^2+P_4^2-E' \tag{3.13}$$

$$=2p_{ij}^2+2p_{k4}^2+Q_k^2-E', \tag{3.14}$$

$$\mathbf{Q}_k\equiv\tfrac{1}{2}(\mathbf{P}_i+\mathbf{P}_j-\mathbf{P}_k-\mathbf{P}_4)=\mathbf{P}_i+\mathbf{P}_j, \tag{3.15}$$

and the operators Λ^s etc., are defined analogously to B as

$$\Lambda^s=\tfrac{1}{2}(\Lambda_{12}+\Lambda_{34}+\Lambda_{31}+\Lambda_{24}+\Lambda_{23}+\Lambda_{14}), \tag{3.16}$$

$$\Lambda'=\tfrac{1}{4}(3)^{1/2}(\Lambda_{23}+\Lambda_{14}-\Lambda_{31}-\Lambda_{24}), \tag{3.17}$$

$$\Lambda''=\tfrac{1}{2}(\Lambda_{12}+\Lambda_{34})-\tfrac{1}{4}(\Lambda_{31}+\Lambda_{24}+\Lambda_{23}+\Lambda_{14}). \tag{3.18}$$

Here Λ_{ij} is related to V_{ij} through the equation

$$V_{ij}=\rho_{ij}(1)\Lambda_{ij}. \tag{3.19}$$

The operators on the right-hand sides of Eqs. (3.11) and (3.12) must be understood in the sense explained in Eq. (2.12) of B.

4. REDUCTION TO EQUIVALENT THREE-BODY EQUATIONS

Our next task is to deduce the algebraic structures for ψ' and ψ'' for various spin-parity assignments, using the separable potential (2.4) in the coupled equations (3.11) and (3.12) for ψ' and ψ''. In this connection, the symmetry requirements on ψ' and ψ'' for various permutations of the momentum coordinates are as given in Table I.[21] Following the techniques of A and B, the only possible structures of ψ' and ψ'' are

TABLE I. Symmetry requirements on ψ' and ψ''.

Operation	$1\rightleftarrows 2$	$3\rightleftarrows 4$	$1\rightleftarrows 2$, $3\rightleftarrows 4$
ψ'	A	A	A
ψ''	S	S	S

$$D_{E'}(\mathbf{P}_i)\psi'=A_3-\tfrac{1}{2}(A_1+A_2), \tag{4.1}$$

$$D_{E'}(\mathbf{P}_i)\psi''=\tfrac{1}{2}(3)^{1/2}(A_1-A_2), \tag{4.2}$$

where the functions A_i must be chosen appropriately for the various spin-parity assignments under consideration, subject to the requirements imposed by (i) the separable potential (2.4) in conjunction with Eqs. (3.11)–(3.14), and (ii) the symmetry requirements given in Table I. Thus for the assignments 0^{-+} and 1^{++}, the only structures of A_i compatible with the above requirements are

$$A_i(0^-)=(\mathbf{p}_{jk}\times\mathbf{p}_{i4})\cdot\mathbf{Q}_i[v(p_{jk})\mathfrak{F}^P(\mathbf{Q}_i,\mathbf{p}_{i4})+v(p_{i4})\mathfrak{F}^P(\mathbf{Q}_i,\mathbf{p}_{jk})] \tag{4.3}$$

and

$$\mathbf{A}_i(1^+)=(\mathbf{p}_{jk}\times\mathbf{p}_{i4})[v(p_{jk})\mathfrak{F}^A(\mathbf{Q}_i,\mathbf{p}_{i4})-v(p_{i4})\mathfrak{F}^A(\mathbf{Q}_i,\mathbf{p}_{jk})], \tag{4.4}$$

where the letters i, j, k are cyclic permutations of 1, 2, 3, and $\mathfrak{F}^P$ and $\mathfrak{F}^A$ are by themselves scalar and even functions each of *two* vectors.[22] We emphasize as in A and B, that the maintenance of this cyclic symmetry is extremely important in the vectors $\mathbf{p}_{ij}$ (which arise out of p-wave interactions) in order to bring out the full symmetries demanded by Table I. For the same reason, the definition of $\mathbf{Q}_i$ as

$$\mathbf{Q}_i=\tfrac{1}{2}(\mathbf{P}_j+\mathbf{P}_k-\mathbf{P}_i-\mathbf{P}_4), \tag{4.5}$$

rather than its simplified form $\mathbf{P}_j+\mathbf{P}_k$ through Eq. (2.1), must be maintained, until the symmetry requirements have been checked. Similar constructions are possible for other spin-parity assignments as well. For example in the case 0^{++}, the factor $(\mathbf{p}_{jk}\times\mathbf{p}_{i4})\cdot\mathbf{Q}_i$, in the right-hand side of Eq. (4.3), should be replaced by the two independent combinations

$$\mathbf{p}_{jk}\cdot\mathbf{p}_{i4}, \tag{4.6}$$

[21] The notation $1\rightleftarrows 2$ stands for the interchange of the momenta $\mathbf{P}_1$ and $\mathbf{P}_2$. The letters S and A denote, respectively, symmetric and antisymmetric behavior with respect to the operation concerned.

[22] Note the plus sign in Eq. (4.3) and the minus sign in Eq. (4.4) which meet the respective symmetry requirements for the two cases.

and

$$3(\mathbf{p}_{jk}\cdot\mathbf{Q}_i)(\mathbf{p}_{i4}\cdot\mathbf{Q}_i)-Q_i^2(\mathbf{p}_{i4}\cdot\mathbf{p}_{jk}), \tag{4.7}$$

each combination getting multiplied by an independent function of the form

$$v(p_{jk})\mathfrak{F}(\mathbf{Q}_i,\mathbf{p}_{i4})+v(p_{i4})\mathfrak{F}(\mathbf{Q}_i,\mathbf{p}_{jk}). \tag{4.8}$$

Similarly, for the case 2^{++}, possible angular functions are[23]

$$p_{jk}^{\mu}p_{i4}^{\nu}+p_{jk}^{\nu}p_{i4}^{\mu}-\tfrac{2}{3}\delta^{\mu\nu}(\mathbf{p}_{jk}\cdot\mathbf{p}_{i4}), \tag{4.9}$$

$$(\mathbf{p}_{jk}\cdot\mathbf{p}_{i4})(Q_i^{\mu}Q_i^{\nu}-\tfrac{1}{3}Q_i^2\delta^{\mu\nu}),\ \text{etc.}, \tag{4.10}$$

where each such function gets multiplied by an independent expression of the form (4.8). However, since the analysis of the cases 0^+, 2^+, etc. depends *also* on π–π interaction in even partial waves (a situation not covered by this work), the present p-wave formalism is inadequate for their description, and as such these spin-parity cases will not be pursued further in this paper.

Following the general procedure outlined in A and B, the function $\mathfrak{F}^P$ for the case 0^{-+} is found, on substitutinos of (4.1)–(4.3) in (3.11) and (3.12), to satisfy the following integral equations:

$$(\mathbf{p}_{12}\times\mathbf{p}_{34})\cdot\mathbf{Q}_3\mathfrak{F}^P(\mathbf{Q}_3,\mathbf{p}_{34})[1+\lambda h(p_{34}^2+\tfrac{1}{2}Q_3^2)]=-3\lambda v(p_{34})\int d\mathbf{p}_{12}'v(p_{12}')(\mathbf{p}_{12}\cdot\mathbf{p}_{12}')[(\mathbf{p}_{12}'\times\mathbf{p}_{34})\cdot\mathbf{Q}_3]$$
$$\times(2p_{12}'^2+2p_{34}^2+Q_3^2-E')^{-1}\mathfrak{F}^P(\mathbf{Q}_3,\mathbf{p}_{12}')+3\lambda\int d\mathbf{p}_{12}'v(p_{12}')(\mathbf{p}_{12}\cdot\mathbf{p}_{12}')[(\mathbf{p}_{31}'\times\mathbf{p}_{24}')\cdot\mathbf{Q}_2']$$
$$\times(2p_{12}'^2+2p_{34}^2+Q_3^2-E')^{-1}[v(p_{31}')\mathfrak{F}^P(\mathbf{Q}_2',\mathbf{p}_{24}')+v(p_{24}')\mathfrak{F}^P(\mathbf{Q}_2',\mathbf{p}_{31}')], \tag{4.11}$$

where

$$h(p^2)=2\pi\int_0^\infty q^4dq\,v^2(q)(q^2+p^2-\tfrac{1}{2}E')^{-1}, \tag{4.12}$$

and the various "mixed momenta" are given by

$$\mathbf{p}_{31}',\mathbf{p}_{24}'=\mp\tfrac{1}{2}\mathbf{Q}_3-\tfrac{1}{2}(\mathbf{p}_{12}'-\mathbf{p}_{34}), \tag{4.13}$$

$$\mathbf{p}_{23}',\mathbf{p}_{14}'=\tfrac{1}{2}\mathbf{Q}_3\mp\tfrac{1}{2}(\mathbf{p}_{12}'+\mathbf{p}_{34}), \tag{4.14}$$

$$\mathbf{Q}_1',\mathbf{Q}_2'=\mp\mathbf{p}_{12}'+\mathbf{p}_{34}. \tag{4.15}$$

A few words on the interpretation of Eq. (4.11) may be in order. The vectors $\mathbf{p}_{ij}$ and $\mathbf{p}_{k4}$ which represent the relative momenta of the π–π pairs, give a realization of the internal structure of the ρ meson in this composite picture. The function $\mathfrak{F}(\mathbf{Q}_3,\mathbf{p}_{34})$ represents essentially the wave function of an equivalent three-body system, viz., the pions 3 and 4 together with the composite of 1 and 2.[24] The resonance structure in 1 and 2 manifests itself in the function $\mathfrak{F}(\mathbf{Q}_3,\mathbf{Q}_{34})$ through its dependence on the factor

$$[1+\lambda h(p_{34}^2+\tfrac{1}{2}Q_3^2)]^{-1}, \tag{4.16}$$

as can be seen from the left-hand side of Eq. (4.11). Again, the first term on the right-hand side of Eq. (4.11) represents the possibility of resonant interaction between the pairs 3 and 4, when 1 and 2 are already in resonance. The last term in this equation stands for the resultant effect of "crosswise interactions" between the four other pion pairs in a representation in which 1 and 2 are resonantly associated. Thus in this picture the "true" four-particle effects which give rise to connected diagrams are essentially contained in the *last term* of Eq. (4.11).

We list here the corresponding integral equation for the function $\mathfrak{F}^A$ of the axial vector case, viz.,

$$(\mathbf{p}_{12}\times\mathbf{p}_{34})\mathfrak{F}^A(\mathbf{Q}_3,\mathbf{p}_{34})[1+\lambda h(p_{34}^2+\tfrac{1}{2}Q_3^2)]=-3\lambda v(p_{34})\int d\mathbf{p}_{12}'v(p_{12}')(\mathbf{p}_{12}\cdot\mathbf{p}_{12}')(\mathbf{p}_{12}'\times\mathbf{p}_{34})$$
$$\times(2p_{12}'^2+2p_{34}^2+Q_3^2-E')^{-1}\mathfrak{F}^A(\mathbf{Q}_3,\mathbf{p}_{34})+3\lambda\int d\mathbf{p}_{12}'v(p_{12}')(\mathbf{p}_{12}\cdot\mathbf{p}_{12}')(\mathbf{p}_{31}'\times\mathbf{p}_{24}')$$
$$\times(2p_{12}'^2+2p_{34}^2+Q_3^2-E')^{-1}[v(p_{31}')\mathfrak{F}^A(\mathbf{Q}_2',\mathbf{p}_{24}')-v(p_{24}')\mathfrak{F}^A(\mathbf{Q}_2'\,\mathbf{p}_{31}')], \tag{4.17}$$

with identical interpretations for the various terms. We note, however, that the last term in (4.17) has a minus sign between the two $\mathfrak{F}$ functions unlike Eq. (4.11), in which the corresponding sign is positive. Since the functions $\mathfrak{F}^{P,A}$ are *even* in their respective arguments this means that while the "last term" may be quite important for the 0^- case, it has a much smaller

[23] Here the greek indices μ, ν, etc., are three-dimensional *tensor* indices as distinct from the letters i, j, k which label the momenta.

[24] The first argument $\mathbf{Q}_3$ of $\mathfrak{F}^P$ is the relative momentum between the pairs (1,2) and (3,4). The second argument $\mathbf{p}_{34}$ represents the momentum distribution within the pair (3,4).

magnitude for the 1^+ case. Therefore, according to our interpretation of the "last term" as giving rise to the true four-particle effects, we expect these to be much less important for the axial vector than for the pseudoscalar case.

So far our treatment is exact.[25] Further manipulations of Eqs. (4.11) or (4.17) must depend on suitable approximations which can best be made on the basis of the physical interpretations given in the last two paragraphs. Thus the resonant π–π structure can be extracted by the ansatz that $\mathfrak{F}^{P,A}$ are both proportional to (4.16). Again, for the Gaussian potential (2.11), it is easy to see from the structures on the right-hand sides of (4.11) or (4.17) that $\mathfrak{F}(\mathbf{Q}_3,\mathbf{p}_{34})$ is also proportional to

$$\exp[-\tfrac{1}{2}p_{34}^2\beta^{-2}-\tfrac{1}{4}Q_3^2\beta^{-2}]. \quad (4.18)$$

Thus taking account of (4.15) and (4.17), we write

$$\mathfrak{F}(\mathbf{Q}_3,\mathbf{p}_{34})=\exp[-\tfrac{1}{2}(p_{34}^2+\tfrac{1}{2}Q_3^2)\beta^{-2}]f(\mathbf{Q}_3,\mathbf{p}_{34}) \times[1+\lambda h(p_{34}^2+\tfrac{1}{2}Q_3^2)]^{-1}, \quad (4.19)$$

where we expect $f(\mathbf{Q}_3,\mathbf{p}_{34})$ to depend less strongly on its arguments, so that further approximations on this function are warranted. Thus according to our interpretation of the first terms on the right-hand sides of Eq. (4.11) or Eq. (4.17) the dependence of $f(\mathbf{Q}_3,\mathbf{p}_{34})$ on its second argument is expected to arise mostly from the former which is primarily responsible for producing a composite structure in the pair (3,4). These and other details concerning the effects of angular correlations in the last terms of (4.11) and (4.17) are discussed in Appendix II. An approximate formula for the evaluation of certain Gaussian integrals which appear repeatedly in these calculations is obtained in Appendix I for both types of interaction defined by (2.12) and (2.13).

The results of evaluation for the pseudoscalar case are summarized in Table II. For the axial vector case on the other hand, it is shown in Appendix II that the true four-particle effects represented by the last term of Eq. (4.17) are almost negligible, so that no 4π resonance can develop in such a quantum state.

5. VARIATIONAL CALCULATION OF ENERGY

For comparison with the direct evaluation of the energy in Sec. 4, we outline here a variational calculation of the 4π resonance energy for the pseudoscalar case on lines similar to Schiff's treatment for the energy of the ω meson.[14] The basis is provided by writing the 4π Schrödinger equation (2.9) in the form

$$(T+V)\Psi=E'\Psi, \quad (5.1)$$

where T is the total kinetic energy symmetric with respect to cyclic permutations of the momenta, viz.,

$$T=P_1^2+P_2^2+P_3^2+P_4^2=2p_{ij}^2+2p_{k4}^2+Q_3^2, \quad (5.2)$$

and

$$V=\sum_{i<j}\sum V_{ij} \quad (5.3)$$

is the total potential energy. The variational estimate $\bar{E}'$ of E' is then given by

$$E'\leqslant\bar{E}'=\langle T\rangle+\langle V\rangle, \quad (5.4)$$

where for any operator A

$$\langle A\rangle=(\Psi,A\Psi)/(\Psi,\Psi). \quad (5.5)$$

Using (3.9), we have

$$(\Psi,\Psi)=\tfrac{1}{2}\int(|\psi'|^2+|\psi''|^2)d\tau, \quad (5.6)$$

$$(\Psi,T\Psi)=\tfrac{1}{2}\int(2p_{34}^2+2p_{12}^2+Q_3^2) \times(|\psi'|^2+|\psi''|^2)d\tau, \quad (5.7)$$

where

$$d\tau=d\mathbf{p}_{12}d\mathbf{p}_{34}d\mathbf{Q}_3. \quad (5.8)$$

In deriving these expressions certain obvious δ-function integrations have been carried out, so that only the nontrivial momenta in the c.m. system are represented by $d\tau$, which, incidentally, is invariant under cyclic permutations of the indices (1, 2, 3).

For ψ' and ψ'' we now write analogously to (4.1) and (4.2),

$$\psi'=B_3-\tfrac{1}{2}(B_1+B_2), \quad (5.9)$$

$$\psi''=\tfrac{1}{2}(3)^{1/2}(B_1-B_2), \quad (5.10)$$

where the B_i are certain functions similar to the A_i functions of Sec. 4. Since, however, the B_i have to be determined variationally, computational convenience should also be an important criterion in the choice of their algebraic structures. Assuming that the B functions are real,[26] the quantities relevant to the deter-

TABLE II. Calculated energies of the 0^- state (in units of the pion mass).

β/μ	σ	E_l	E_u	$E_{\rm I}$	$E_{\rm II}$
1	2.33	6.8–6.9	10.0–10.05	11.0	10.5
0.9	3.3	6.3–6.4	9.7–9.8	10.7	9.9
28.8	−0.99	86.1			
28.8	−3.30	4.0			

[25] The word "exact" is used in the sense that, apart from the basic assumptions of a single-channel four-pion state with p-wave π–π interactions through a Schrödinger equation, no approximations have so far been made in the deduction of Eqs. (4.11) or (4.17) from Eqs. (3.11) and (3.12).

[26] The reality of the B functions is assumed primarily on grounds of simplicity, analogously to variational functions generally assumed for bound-state energies. Of course in the present context of investigation of a resonance, such assumptions might well need to be relaxed, especially if a program of elaborate variational calculation were undertaken. However, since our variational method is intended mainly as a guide for comparison with the results of Sec. 4, we have chosen very simple forms for the functions [see Eqs. (5.19)–(5.21)]. With such a limited scope, it seems to us that not much is likely to be gained by relaxing the reality condition on these functions.

mination of E' are

$$(\Psi,\Psi)=\tfrac{3}{2}\int d\tau(B_3^2-\tfrac{1}{2}B_3B_1-\tfrac{1}{2}B_3B_2)\,, \qquad (5.11)$$

$$(\Psi,T\Psi)=\tfrac{3}{2}\int(2p_{34}^2+2p_{12}^2+Q_3^2)\times d\tau(B_3^2-\tfrac{1}{2}B_3B_1-\tfrac{1}{2}B_3B_2)\,, \qquad (5.12)$$

$$(\Psi,V\Psi)=\tfrac{3}{2}\sum_{i=1}^{3}(B_i,V_{34}B_i)-\tfrac{3}{4}\sum^{3}\sum^{3}_{i\neq j}(B_i,V_{34}B_j)\,, \qquad (5.13)$$

$$(B_i,V_{34}B_j)=3\lambda\int d\tau' B_i(\mathbf{p}_{jk},\mathbf{p}_{i4},\mathbf{Q}_i)v(p_{34})\times(\mathbf{p}_{34}\cdot\mathbf{p}_{34}')v(p_{34}')B_j(\mathbf{p}_{ki}',\mathbf{p}_{j4}',\mathbf{Q}_j')\,, \qquad (5.14)$$

$$d\tau'=d\mathbf{p}_{12}d\mathbf{p}_{34}d\mathbf{Q}_3d\mathbf{p}_{34}'\,. \qquad (5.15)$$

In writing these matrix elements the results of certain obvious permutation symmetries have been freely used, e.g.,

$$(B_i,B_j)=(B_2,B_3)=(B_3,B_1)\,,\quad i\neq j\,, \qquad (5.16)$$

$$(B_i,B_i)=(B_3,B_3)\,, \qquad (5.17)$$

$$(B_i,V_{j4}B_k)=(B_j,V_{k4}B_i)\,,\ \text{etc.} \qquad (5.18)$$

Further, the arguments of the B functions that appear on the right-hand side of (5.14) are the ones appropriate for bringing out the cyclic symmetries relevant to their structure. Also, in the {(12), (34)} representation, the values of the "mixed momenta" $\mathbf{p}_{ij}$, $\mathbf{p}_{ij}'$, $\cdots$, etc., in terms of the integration variables (5.15) are as given in Eqs. (4.13)–(4.15).

For the pseudoscalar case, the following simplest variational functions have been considered:

$$\text{I: } B_i=(\mathbf{p}_{jk}\cdot\mathbf{p}_{i4})F\,, \qquad (5.19)$$

$$\text{II: } B_i=F[c_1Q_i^2(\mathbf{p}_{jk}\cdot\mathbf{p}_{i4})+c_2(\mathbf{p}_{jk}\cdot\mathbf{Q}_i)(\mathbf{p}_{i4}\cdot\mathbf{Q}_i)]\,, \qquad (5.20)$$

where

$$F=(\mathbf{p}_{jk}\times\mathbf{p}_{i4})\cdot\mathbf{Q}_i\exp[-\tfrac{1}{2}\alpha(p_{jk}^2+p_{i4}^2+\tfrac{1}{2}Q_i^2)] \qquad (5.21)$$

is an *invariant* function under cyclic permutations of the various arguments. Such forms are largely suggested by the analysis of Sec. 4, and have the strong advantage that the exponential function, being *invariant* under cyclic permutations of the indices, can be cast in any representation without the introduction of angular correlations between the momentum vectors. This simplification largely facilitates the evaluation of the integrals in (5.11)–(5.12), since the angular functions are contained entirely in the functions multiplying F in (5.19) and (5.20). The integrals which are thus all elementary, though laborious, lead to the following[27]:

$$\bar{E}_I'=19\alpha^{-1}+16\sigma\alpha^{5/2}\beta^{-3}(\alpha+\beta^{-2})^{-2}\times(3\alpha^2+3\beta^{-4}+2\alpha\beta^{-2}) \qquad (5.22)$$

$$\bar{E}_{II}'=23\beta^2+(9/7)\sigma\beta^2(36c_1^2+34c_1c_2+9c_2^2)/(30c_1^2+12c_1c_2+5c_2^2) \qquad (5.23)$$

for the trial functions I and II given by (5.19) and (5.20), respectively. It is clear from these expressions that the possibility of a bound state can arise only for $\lambda<0$ (type a interaction), since this is the only case for which the total energy is less than the kinetic energy. To see the possibility for a *just-bound* state, viz., $E'=0$, with type a interaction, we have considered Eq. (5.22) derived from the trial function I, and determined α, σ variationally from the equations

$$\bar{E}'=0\,,\quad \partial\bar{E}'/\partial\alpha=0\,, \qquad (5.24)$$

as was also done by Schiff for the 3π problem.[14] The values of α and σ which make $\partial^2\bar{E}'/\partial\alpha^2>0$, are

$$\alpha=2.75/\beta^2\,,\quad \sigma=-3.0\,. \qquad (5.25)$$

A comparison of (5.25) with (2.12) shows that one needs an enormously stronger attraction to bind four pions at about the mass of η than is provided by the strength of the two-body interaction. This conclusion is just the opposite of Schiff's for the 3π case, which showed too much binding in ω for the π–π interaction appropriate to the ρ meson.

A similar conclusion is reached with trial function II. Thus, type a interaction can at most give rise to a 4π resonance but certainly not a bound state.

For type b interaction ($\lambda>0$), there is of course no question of a bound state since now both the kinetic- and potential-energy terms are positive in (5.22) or (5.23). As for the possibilities of resonance, variational estimates of the energy have been made by taking the parameter α in the exponent (5.21), the *same* as was obtained in Eq. (4.19), from the structure of the integral equation, viz.,

$$\alpha=\beta^{-2}\,. \qquad (5.26)$$

This assumption makes the most rapidly varying part of the variational wave function conform to the shape deduced from the exact structure of the 4π integral equation.[28] Using this value of α in (5.19)–(5.21), variational estimates of the resonance energy E can be made with the parameters of type b interaction that are listed in Eq. (2.13). For the trial function I there are now no free parameters. For function II, on the other hand, the ratio c_1/c_2 can be variationally estimated, and the value of this ratio which makes $\bar{E}_{II}'$ a

[27] In writing (5.23), the parameter α of (5.21) has been taken as β^{-2}; see Eq. (5.26).

[28] Note that $\mathfrak{F}^P$ enters the A_i function with the multiplicative factor $v(p_{12})$ so that it is the structure of $v(p_{12})\mathfrak{F}^P$ that should be compared with the B_i functions (5.19)–(5.21).

minimum is

$$c_1/c_2 = -0.512. \tag{5.27}$$

The results for E are collected in Table II along with the results of direct evaluation described in Sec. 4 and Appendix II, for the pseudoscalar case.

6. CONCLUSION

One definite result of this investigation is that the 4π formalism through pair-wise p-wave interaction rules out an axial vector (1^{++}) resonance of $T=0$, since the corresponding state has no properly connected four-particle graphs. This conclusion is valid for both types of interaction considered in this paper, and may therefore be regarded as a largely model-independent result.

The only other 4π state of $T=0$ which depends entirely on p-wave π–π interaction is the pseudoscalar state (0^{-+}). According to our investigation, such a state can of course develop appreciable four-body effects of a *connected* type, but the energy of the composite system depends rather markedly on the nature of the π–π interaction that is assumed. Thus with type a interaction whose parameters are given by Eq. (2.12) and which is characterized by an intrinsically attractive short-range force, Table II shows that the predicted energy is about 86 μ, far too high for any known meson resonance. On the other hand, to obtain an energy $E\approx 4\mu$, which is almost the mass of η, the required value of the strength parameter σ is -3.3, as against its two-pion value of -0.99, given in (2.12). The large difference in the scales of σ and E can be partly understood in terms of the approximate formula

$$E' \approx 2.9\beta^2 \tag{6.1}$$

which is derivable from Eq. (II.11) for the parameters of (2.12). This direct estimate of $\sigma=-3.3$ needed for $E=4\mu$ is rather close to the corresponding "variational estimate" in Table II of σ needed to give zero binding energy, viz., $\sigma=-3.0$.[29] This is a fortunate circumstance since our approximation treatment outlined in Appendices I and II is formally not so much justified for type a interaction as for type b interaction, and may therefore be taken to provide some support for the qualitative validity of the formula (6.1) which is a clue to the understanding of the extremely large value of E with type a interaction. Without further refinement, therefore, it is possible to say that, far from producing a bound 4π structure like the η, the attraction provided by type a interaction in the state of 0^{-+} is far too little to produce any resonance *even in the GeV region*. This conclusion is just the opposite of those of Ref. 14 or Ref. 16 for the three-pion energy in the $T=0$ state of 1^{--}, where the problem was one of too strong attraction. It is not easy to say whether this difference between the 3π and 4π results, using the same π–π interaction has something to do with any intrinsic difference between the three-body and four-body problems. In our view, the "lack of sufficient attraction" with the short-range interaction of type a, might well be a peculiar characteristic of the "pseudoscalar state."[30]

The situation is entirely different with type b interaction, which, though incapable of producing bound states (see Sec. 5), can yet give rise to 4π resonances. Indeed, the prospects of resonance formation at moderate energies are much brighter with such interaction. For, as an inspection of Eq. (4.11) shows, the last term on the right-hand side which represents the true four-body effects is *attractive* in this case ($\lambda>0$), thus helping to bring the pions together. Moreover, the method of treatment outlined in Appendices I and II is much more justified with long-range type b forces than with short-range type a. The figures in Table II already suggest that the resonance energy with type b forces is only of the order of a few pion masses [rather than several baryon masses with type a forces]. Table II lists two solutions, E_l and E_u of Eq. (4.11),[31] the higher one E_u lying a few pion masses above the lower one E_l. As a check on the validity of these solutions, the results of the variational estimates of Sec. 5 with trial functions I and II are also listed for comparison. The values of the variational energies E_{I} and E_{II}, being larger than the higher energy solutions E_u, seem to support our claim on the authenticity of the results E_l and E_u of direct evaluation.

We are as yet unable to comment on the significance of the solution E_u in the context of the present experimental data, though we are tempted to point out its possible relevance to a heavy pseudoscalar particle of mass around 1.5 BeV predicted by Schwinger[32] on the basis of his W_3 model. Another possibility might be to associate it with a reported peak in the $\bar{K}K\pi$ system at 1410 MeV,[33] *provided* the latter could be regarded as a sharp resonance of $T=0$, realizable through a 4π channel as well.[34] However, such assignments are extremely speculative at the present stage of experimental knowledge about the $\bar{K}K\pi$ peak.

[29] Such rapid variation of E with σ is in agreement with the conclusion of Schiff (Ref. 14) for the 3π problem, and seems to be a characteristic of short-range forces.

[30] In this respect it may be in order to point out that the analysis in A, of the integral equation for the 0^{--} state of the 3π system had also indicated a repulsive kernel for that state.

[31] The quantities E_l and E_u are shown as lying within small ranges like 6.8–6.9, etc. The limits of these ranges correspond respectively to the assumptions $p_{34}^2\approx 0$ and $p_{34}^2\approx\beta^2$ in Eq. (II.10) of Appendix II. The closeness of these limits in all the cases justifies the approximation made after Eq. (II.10).

[32] J. Schwinger, Phys. Rev. Letters **12**, 237 (1964).

[33] R. Armenteros *et al.*, Proceedings of the Siena Conference on Elementary Particles, Siena, Italy, 1963 (unpublished).

[34] A calculation by Oakes [R. J. Oakes, Phys. Rev. Letters **12**, 134 (1964)] using the Peierls mechanism [R. F. Peierls, Phys. Rev. Letters **6**, 641 (1961)] suggests, on the other hand, that such a peak could be understood through the exchange of a pion in $\bar{K}-K^*$ scattering, so that it need not have a definite isospin or angular momentum. However, since the statistical ratio of $T=0$ and 1 mixtures is 9:1, it may be regarded as an almost pure $T=0$ state.

As for the solution E_l, the most obvious candidate is the X^0 meson[4] whose existence was one of the main motivations for the present approach. Indeed, all the figures for E_l in Table II are rather close to the experimental value $6.86\,\mu$ (≈ 960 MeV) of the X mass. (There is some variation with the range parameter β, a higher mass corresponding to a higher β.) We therefore feel rather strongly that our 4π model with type b interaction predicts the X^0 resonance as a pseudoscalar particle rather than axial vector, which seems to be in agreement with the latest findings of Kalbfleisch *et al.*[35]

In summary, our four-pion model of pair-wise π–π interaction through a Schrödinger equation, does not give a bound 0^- state, or a 1^+ state of any form (bound or resonant). However, it predicts two 0^- resonances of moderate energies, the lower one of which is identifiable with the X^0 meson, provided the π–π interaction is of type b, namely, repulsive at low energies but changing to attraction at high energies. In this connection one of us (ANM) has discussed elsewhere[36] certain other advantages of type b interaction. Among the more important advantages, one might mention (i) easier fits to the ρ meson, (ii) compatibility with SU_3 generalization for K–π interaction, (iii) qualitative understanding of the ϕ meson as a $\rho+\pi$ resonance, and of the more recent $K\pi\pi$ resonance at 1175 MeV.[37] The relevance of the present investigation to the X^0 meson may be taken to provide additional support in favor of type b interaction as a useful form of parametrization of the effective π–π force.

ACKNOWLEDGMENTS

We are thankful to Professor R. G. Majumdar for his interest in this investigation. One of us (SR) acknowledges financial assistance from the University Grants Commission.

APPENDIX I

The function that appears most frequently in the integral equations reduced from (4.11) or (4.17), after the substitution of (4.19), is

$$\lambda h(x)=2\pi\lambda\int_0^\infty dp\; p^4e^{-p^2/\beta^2}(p^2+x)^{-1}. \tag{I.1}$$

The exact values of this function for $x>0$ and $x<0$ are, respectively,

$$\sigma\left[1-2x\beta^{-2}+2\pi^{1/2}(x/\beta^2)^{3/2}e^{x/\beta^2}\times\left\{1-\frac{2}{\pi^{1/2}}\int_0^{(\surd x)/\beta}dy e^{-y^2}\right\}\right] \tag{I.2}$$

and

$$\sigma\left[1-2x\beta^{-2}-4(-x/\beta^2)^{3/2}e^{x/\beta^2}\int_0^{(\surd x)/\beta}dy e^{+y^2}\right], \tag{I.3}$$

where

$$\sigma=\tfrac{1}{2}\pi^{3/2}\beta^3\lambda, \tag{I.4}$$

and (I.3) is evaluated in the sense of a principal value integral for (I.1).

As these expressions are rather unwieldy it is desirable to approximate them by more handy formulas which are reasonably accurate. A possible approximation which is certainly valid for small β (appropriate to the case $\lambda>0$) is based on the observation that the numerator function $p^4e^{-p^2/\beta^2}$ in (I.1) has a sharp maximum and hence varies much more rapidly than the denominator, so that the latter may be replaced by a constant of the form p_m^2+x, where p_m^2 corresponds to the maximum of the numerator function. More precisely, $p_m^2\approx\frac{5}{2}\beta^2$, which yields

$$\lambda h(x)\approx 3\sigma\beta^2(5\beta^2+2x)^{-1}. \tag{I.5}$$

This approximation, which is extremely good for x/β^2 large and positive, works quite well also for smaller values of $x(>0)$. It is tolerable even for negative values of x/β^2, except in the neighborhood of the pole of the function (I.5). Fortunately, however, negative values of x are much less pertinent to the solution of our integral equation than its positive values, so that (I.5) is a good approximation for the integrals encountered in Appendix II with type b interaction.

For type a interaction, viz., large β, the corresponding approximation is

$$\lambda h(x)\approx 3\sigma\beta^2/(3\beta^2+2x), \tag{I.6}$$

which again works very satisfactorily for $x>0$.[38]

The formulas (I.5) and (I.6) can be combined into a single unified formula

$$\lambda h(x)\approx\tfrac{3}{2}\beta^2\sigma(\gamma\beta^2+x)^{-1}, \tag{I.7}$$

where

$$\gamma=\tfrac{5}{2}\quad\text{or}\quad\tfrac{3}{2}, \tag{I.8}$$

for type b or type a interaction, respectively. The approximation (I.5) for small β, is valid also for integrals of the form

$$\mathcal{G}=\int_0^\infty dq\; q^4e^{-q^2/\beta^2}(q^2+x)^{-1}g(q^2), \tag{I.9}$$

where $g(q^2)$ is a *slowly varying* function of q^2 and $x>0$. Thus (I.9) may be replaced by

$$\mathcal{G}\approx(\beta^2\gamma+x)^{-1}\int_0^\infty dq\; q^4e^{-q^2/\beta^2}g(q^2), \tag{I.10}$$

where $\gamma=\frac{5}{2}$.

[35] G. R. Kalbfleisch, O. I. Dahl, and A. Rittenburg, Phys. Rev. Letters **13**, 349 (1964).
[36] A. N. Mitra, Phys. Letters **12**, 61 (1964).
[37] T. Wangler *et al.*, Phys. Letters **9**, 71 (1964).

[38] The difference between the denominators in (I.5) and (I.6) can be understood through the first two terms in the expansion (for $x>0$) of $(p^2+x)^{-1}$ in powers of (i) x/p^2 for large β and (ii) p^2/x for small β.

The corresponding formula for large β cannot of course be taken to be necessarily valid without a more detailed knowledge of the structure of $g(q^2)$. However, as our results will eventually show, the case of large β (type a interaction) leads in several ways to the common conclusion of an extremely high mass resonance, so that any inacccuracies in a formula like (I.10) with $\gamma\approx\frac{3}{2}$ will not make any qualitative difference to this basic result. With this understanding we have, in Appendix II, formally used the approximation (I.10) for (I.9) for both cases, though its numerical accuracy can be guaranteed only for the case of small β.

APPENDIX II

We remark at the outset that the treatment in this section will apply with much better accuracy for type b interaction than for type a interaction (see Appendix I).

Substitution of (4.19) in (4.11) leads to an integral equation for $f^P(\mathbf{Q}_3,\mathbf{p}_{34})$ of the form

$$(\mathbf{p}_{12}\times\mathbf{p}_{34})\cdot\mathbf{Q}_3 f^P(\mathbf{Q}_3,\mathbf{p}_{34})=-4\pi\lambda(\mathbf{p}_{12}\times\mathbf{p}_{34})\cdot\mathbf{Q}_3\int_0^\infty dq\, e^{-q^2/\beta^2}f^P(\mathbf{Q}_3,\mathbf{q})[1+\lambda h(q^2+\tfrac{1}{2}Q_3^2)]^{-1}$$
$$\times(2q^2+2p_{34}^2+Q_3^2-E')^{-1}+3\lambda\int d\mathbf{p}_{12}'(\mathbf{p}_{12}\cdot\mathbf{p}_{12}')[(\mathbf{p}_{12}'\times\mathbf{p}_{34})\cdot\mathbf{Q}_3]$$
$$\times\exp(-p_{12}'^2\beta^{-2})\left[\frac{f^P(\mathbf{p}_{12}'+\mathbf{p}_{34},\,\tfrac{1}{2}\mathbf{Q}_3+\tfrac{1}{2}\mathbf{p}_{34}-\tfrac{1}{2}\mathbf{p}_{12}')}{(2p_{12}'^2+2p_{34}^2+Q_3^2-E')\{1+\lambda h(p_{24}'^2+\tfrac{1}{2}Q_2'^2)\}}+\{\mathbf{Q}_3\rightleftarrows-\mathbf{Q}_3\}\right],\qquad\text{(II.1)}$$

where the mixed momenta are given by (4.13)–(4.15), and $\mathbf{Q}_3\rightleftarrows-\mathbf{Q}_3$ signifies a term obtained from its predecessor by the replacement indicated. According to the arguments given at the end of Sec. 4, the dependence of f^P on its second argument $\mathbf{p}_{34}$, comes mainly from the first term of (II.1), which already shows an integral structure of the form (I.9). Thus, according to (I.10) the dependence on $\mathbf{p}_{34}$ is simply described by the factor

$$(p_{34}^2+\tfrac{1}{2}Q_3^2+\beta^2\gamma-\tfrac{1}{2}E')^{-1}.$$

Therefore we use the ansatz:

$$f^P(\mathbf{Q}_3,\mathbf{p}_{34})\approx H(\mathbf{Q}_3)(p_{34}^2+\tfrac{1}{2}Q_3^2+\beta^2\gamma-\tfrac{1}{2}E')^{-1},\qquad\text{(II.2)}$$

so that (II.1) reduces to

$$\frac{H(\mathbf{Q}_3)[(\mathbf{p}_{12}\times\mathbf{p}_{34})\cdot\mathbf{Q}_3]}{(p_{34}^2+\tfrac{1}{2}Q_3^2+\beta^2\gamma-\tfrac{1}{2}E')}\left\{1+2\pi\lambda\int_0^\infty\frac{dq\, q^4e^{-q^2/\beta^2}}{q^2+\gamma\beta^2+\tfrac{1}{2}Q_3^2-\tfrac{1}{2}E'+\tfrac{3}{2}\sigma\beta^2}\right\}$$
$$=3\lambda\int d\mathbf{p}_{12}'(\mathbf{p}_{12}\cdot\mathbf{p}_{12}')[(\mathbf{p}_{12}'\times\mathbf{p}_{34})\cdot\mathbf{Q}_3](2p_{12}'^2+2p_{34}^2+Q_3^2-E')^{-1}H(\mathbf{p}_{12}'+\mathbf{p}_{34})$$
$$\times[2x/(x^2-\tfrac{1}{4}\{\mathbf{Q}_3\cdot(\mathbf{p}_{34}-\mathbf{p}_{12}')\}^2)],\qquad\text{(II.3)}$$

where

$$x=x_0+\tfrac{1}{2}\mathbf{p}_{34}\cdot\mathbf{p}_{12}'\qquad\text{(II.4)}$$

and

$$x_0=\beta^2\gamma+\tfrac{1}{4}Q_3^2+\tfrac{3}{4}(p_{34}^2+p_{12}'^2)+\tfrac{3}{2}\sigma\beta^2-\tfrac{1}{2}E'.\qquad\text{(II.5)}$$

Now, the last factor on the right-hand side of (II.3) may be approximated simply by $(2/x)$, provided $x^2\gg\frac{1}{4}[\mathbf{Q}_3\cdot(\mathbf{p}_{34}-\mathbf{p}_{12}')]^2$, a condition satisfied to within less than 2% error near the maximum of the Gaussian function, namely for $p_{12}'^2\approx\gamma\beta^2$. This facilitates the integration over the azimuthal angle ϕ of $\mathbf{p}_{12}'$, which is necessary for extracting the "pseudoscalar factor" from both sides. Then (II.3) reduces to

$$\frac{H(Q_3)}{(p_{34}^2+\gamma\beta^2+\tfrac{1}{2}Q_3^2-\tfrac{1}{2}E')}\left[1+\frac{\tfrac{3}{2}\sigma\beta^2}{2\gamma\beta^2+\tfrac{3}{2}\sigma\beta^2+\tfrac{1}{2}Q_3^2-\tfrac{1}{2}E'}\right]=6\pi\lambda\int_0^\infty dq\int_0^\pi d\theta\, q^4e^{-q^2/\beta^2}\sin^3\theta H(\mathbf{q}+\mathbf{p}_{34})x^{-1}$$
$$\times(2q^2+2p_{34}^2+Q_3^2-E')^{-1},\qquad\text{(II.6)}$$

where $\cos\theta=\hat{q}\cdot\hat{p}_{34}$, and (I.1) has once again been used to simplify the left-hand-side integral in Eq. (II.3).

To "match" the p_{34} dependence on both sides of (II.6) we must first carry out the angular integration on $\cos\theta$.

For this purpose we ignore the angular correlations in $H(\hat{q}+\mathbf{p}_{34})$ and x, viz.,

$$H(\mathbf{q}+\mathbf{p}_{34})\approx H((q^2+p_{34}^2)^{1/2}), \tag{II.7}$$

$$x\approx x_0. \tag{II.8}$$

This approximation can of course be tested for (II.8). Indeed, an expansion of x^{-1} in powers of $(\mathbf{p}_{34}\cdot\mathbf{q})/2x_0$ shows that the first "correction term" is less than 6% in the "worst case" which occurs when

$$q^2\sim p_{34}^2\gg 2\sigma\beta^2. \tag{II.9}$$

Since now (II.8) is a good approximation for x, we hope (II.7) to have also a similar validity, as the functional form of H is essentially governed by that of x^{-1}. Using (II.7), (II.8) and (I.10) in (II.6) gives

$$H(Q_3)[1+\tfrac{3}{2}\sigma\beta^2(2\gamma\beta^2+\tfrac{3}{2}\sigma\beta^2+\tfrac{1}{2}Q_3^2-\tfrac{1}{2}E')^{-1}]=4\pi\lambda\int_0^\infty dq\, q^4e^{-q^2/\beta^2}H((q^2+p_{34}^2)^{1/2})/x_0. \tag{II.10}$$

The left-hand side is now independent of p_{34} but the right-hand side still shows some dependence on this momentum. This is due to our ansatz (II.2) which neglected the p_{34} dependence of $f(\mathbf{Q}_3,\mathbf{p}_{34})$ coming from the last term of (II.1). The only obvious approximation which now suggests itself is to regard the right-hand side as independent of p_{34} and evaluate it by setting $p_{34}=0$. A crude estimate of the error which may be made by evaluating the first correction term in the expansion of x_0^{-1} in powers of p_{34}, yields a correction of 10–12% for $p_{34}^2\sim\beta^2$ and $q^2\sim\gamma\beta^2$. A similar estimate may be assumed valid also for the function $H((q^2+p_{34}^2)^{1/2})$. Ignoring these corrections and setting $p_{34}=0$ on the right-hand side of (II.10) now enables us to reduce this equation, through repeated application of (I.10), to the form

$$1=4\pi\lambda\int_0^\infty dq\frac{e^{-q^2/\beta^2}q^4(2\gamma\beta^2+\frac{1}{2}q^2-\frac{1}{2}E'+\frac{3}{2}\sigma\beta^2)}{(2\gamma\beta^2+\frac{1}{2}q^2-\frac{1}{2}E'+3\sigma\beta^2)[\frac{1}{4}q^2+\frac{3}{2}\sigma\beta^2-\frac{1}{2}E'+(7/4)\gamma\beta^2]} \tag{II.11}$$

whose numerical solutions for E according to (2.10) are summarized in Table II of the text.

For a closer estimate of the order of error in neglecting p_{34} from the right-hand side of (II.10), an equation similar to (II.11) may be derived by assuming p_{34}^2 as constant. Taking $p_{34}^2\approx\beta^2$, the numerical solutions for type b interaction which is also listed in Table II, seem more than to bear out our assertion that the correction is indeed small due to the neglect of p_{34}^2 from the right-hand side of Eq. (II.10).

For the axial vector case, the presence of the minus sign between the two terms of the "mixed integral" in Eq. (4.17) leads, with an analogous treatment, to an equation like (II.3), but with the replacement

$$\frac{2x}{x^2-\frac{1}{4}[\mathbf{Q}_3\cdot(\mathbf{p}_{34}-\mathbf{p}_{12}')]^2}\rightarrow\frac{\mathbf{Q}_3\cdot(\mathbf{p}_{34}-\mathbf{p}_{12}')}{x^2-\frac{1}{4}[\mathbf{Q}_3\cdot(\mathbf{p}_{34}-\mathbf{p}_{12}')]^2}.$$

Therefore according to the argument given after (II.3), this integral is negligible compared to the corresponding term of the pseudoscalar case. Since, on the other hand, the "mixed integral" is the only one which can give rise to a properly *connected* graph for the 4π system, it is clear that the axial vector state of the 4π system consists of two essentially uncorrelated pairs, viz., (1,2) and (3.4) in our representation, and is therefore incapable of developing any resonance.