Approximate Solution to Partial-Wave Dispersion Relations

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The partial-wave dispersion relations of the scattering amplitude are considered by means of the inverse amplitude technique. The sense in which the N/D and inverse amplitude methods of solution are equivalent is discussed. A method of solution that is symmetric and free from subtraction parameters is given. The procedure is illustrated for the $\pi\pi\rho$ bootstrap problem for the single-channel case. A self-consistent solution is obtained.

I. INTRODUCTION

In the study of partial-wave dispersion relations there have generally been given two methods of solution: the N/D method[1] and the inverse A method.[2] A good deal of interesting work has been done on the N/D approach, where a given input force $L(s)$ is assumed. It has been shown that for a given symmetric input force $L(s)$, the partial-wave amplitude is symmetric either for the unsubtracted dispersion relations[3] or for the subtracted one.[4] It is further shown that formally the scattering amplitude is independent of the choice of the subtraction point.[4]

The determinantal method[5] has often been used to solve approximately the N/D equations for the single-channel problem[6] as well as for the coupled-channel problem,[7] because its application requires simply the evaluation of integrals rather than the solving of integral equations, and probably because it is not at all clear that an exact solution to the N/D equations is a better approximation to A than just the first-order iteration when the input force $L(s)$ is merely some approximation to the left-hand cut in $A(s)$.

However, the determinantal method gives an approximate solution which is not only dependent on the subtraction point s_0 but also nonsymmetric, thus violating time-reversal invariance. The solutions obtained by the determinantal method are sensitive to the choice of subtraction point, though it is computationally simple. An alternative formulation of the approximate N/D method which is symmetric and independent of a subtraction point has already been presented.[8] However, we shall present a more general formulation of the problem which yields the results of the above-mentioned work as a special case.

It has been claimed that virtually all statements that can be made for the N/D method can also be applied to the inverse A method.[9] This, however, has not been discussed for the approximate solutions. The purpose of this paper is to discuss some aspects of this claim, and to present an approximate solution which is symmetric, independent of subtraction parameters, and yet can be applied to the interesting case of bootstrap technique. Although our formulation is essentially a many-channel one, we present here an application to a single-channel case of $\rho\pi\pi$ as an example.

In Sec. II the inverse amplitude method is discussed along lines similar to those of the customary N/D method. Section III presents the derivation of an approximate solution for the partial-wave dispersion relations that is symmetric and free from subtraction parameters. The procedure is applied to the single-channel bootstrap problem in Sec. IV. Section V contains concluding remarks.

II. THE INVERSE AMPLITUDE METHOD

The purpose of the formalism is to solve for the scattering amplitude $A(s)$ for a particular partial wave as a function of s, in case of n-coupled two-body channels. The partial-wave dispersion relation takes the form

$$A(s)=L(s)+\frac{1}{\pi}\int_{C_R}\frac{\mathrm{Im}A(s')}{s'-s-i\epsilon}ds', \qquad (1)$$

where C_R denotes the right-hand cut in the s plane coming from unitarity, and $L(s)$ is the symmetric "input" potential which is the left-hand cut contribution. The unitarity condition reads

$$\frac{1}{2i}[A_{ij}(s)-A_{ij}^*(s)]=\sum_k A_{ik}^*(s)\rho_k(s)\theta(s-s_k)A_{kj}(s), \qquad (2)$$

where $\rho_k(s)$ is the kinematical factor, s_k is the threshold

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[1] J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960); G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960). In the text, references are made mostly to the once-subtracted N/D formalism developed by J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).
[2] P. T. Mathews and A. Salam, Nuovo Cimento **13**, 381 (1959); J. W. Moffat, Phys. Rev. **121**, 926 (1961).
[3] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).
[4] A. W. Martin, Phys. Rev. **135**, B967 (1964).
[5] M. Baker, Ann. Phys. (N.Y.) 4, 271 (1958).
[6] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961), and Erratum p. 268.
[7] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[8] G. Shaw, Phys. Rev. Letters **12**, 345 (1964).
[9] M. Sugawara and A. Kanazawa, Phys. Rev. **126**, 2251 (1962).

for the state k, and $\theta(s-s_k)$ is the usual step function which insures that the right-hand cut starts at s_k. If (2) is multiplied by $A^{-1}(s)$ on the right and by $[A^*(s)]^{-1}$ on the left, the unitarity condition becomes

$$\mathrm{Im}(A^{-1}(s))_{ij}=-\delta_{ij}\rho_j(s)\theta(s-s_j). \tag{3}$$

The dispersion relation (1) implies that $A(s)$ is itself a boundary value of an analytic function.

Analyticity and unitarity are satisfied if we consider

$$F(s)=A^{-1}(s)L(s) \tag{4}$$

assuming $A(s)$ is nonsingular, and write[10]

$$A^{-1}(s)L(s)=1-\frac{s-s_0}{\pi}\int_{C_R}\frac{\rho(s')L(s')}{(s'-s)(s'-s_0)}ds' +\frac{s-s_0}{\pi}\int_{C_L}\frac{\mathrm{Im}(A^{-1}(s')L(s'))}{(s'-s)(s'-s_0)}ds', \tag{5}$$

where the subtraction point s_0 is taken at the position below the physical threshold where $L(s)$ becomes infinite, such as the beginning of the left-hand cut C_L when $L(s)$ includes a one-particle exchange force. In writing (5), it is understood that $L(s)$ is infinite in general at some point s_0 below the threshold, so that $L(s)$ will dominate the amplitude $A(s)$ in the vicinity of s_0. The existence of such s_0 may be justified by viewing all the exchange forces between strongly interacting particles as governed by the Yukawa mechanism. If, in addition, there are pole terms closer to the physical region than the beginning of the left-hand cut C_L of such a kind, it will then be assumed that they will dominate the scattering amplitude. If $L(s)$ does not become infinite at any finite point below the threshold, one may take s_0 at $-\infty$ in (5). For the left-hand cut contribution may dominate the scattering amplitude on the distant left-hand cut in this case; one may have the situation in which $A(s)\to L(s)$ as $s\to-\infty$. In writing the once-subtracted dispersion relation (5), it is also kept in mind that the formalism resulting from (5) will proceed parallel to the once-subtracted N/D formalism. Then in the physical region where the input potential $L(s)$ is regular, we have from (5)

$$A(s)=\left[L^{-1}(s)-\frac{s-s_0}{\pi}\left(P\int_{C_R}\frac{\rho(s')L(s')}{(s'-s)(s'-s_0)}ds' -\int_{C_L}\frac{\mathrm{Im}(A^{-1}(s')L(s'))}{(s'-s)(s'-s_0)}ds'\right)L^{-1}(s')-i\rho(s)\right]^{-1}. \tag{6}$$

The dispersion relation (5) tacitly assumes that $\mathrm{Im}(A^{-1}(s')L(s'))$ on C_L is bounded by s' for the effective s' which contribute significantly to the integral over C_L. Furthermore, near threshold s, if $|s'-s|\gg|s-s_0|$ and $\mathrm{Im}(A^{-1}(s')L(s'))$ is small for the effective s', one may neglect the third term in (6) to obtain the approximate solution

$$A_{\mathrm{det}}(s)=\left[L^{-1}(s) -\left(\frac{s-s_0}{\pi}P\int_{C_R}\frac{\rho(s')L(s')}{(s'-s)(s'-s_0)}ds'\right)L^{-1}(s)-i\rho(s)\right]^{-1}. \tag{7}$$

We observe that (7) is the approximate solution one gets in the N/D formalism by the determinantal method. It is interesting to remark that the determinantal method in the N/D formalism is consistent with neglecting the left-hand cut integral other than $L(s)$ in this formalism.

As was mentioned before, $A_{\mathrm{det}}(s)$ does contain a subtraction parameter s_0 and is not symmetric. We can eliminate the dependence on s_0 in our equations by using the following relations:

$$L(s)=\frac{1}{\pi}\int_{C_L}\frac{\mathrm{Im}L(s')}{s'-s}ds'\equiv\frac{1}{\pi}\int_{C_L}\frac{\mathrm{Im}A(s')}{s'-s}ds', \tag{8}$$

where it is recalled that

$$\mathrm{Im}L(s)\equiv\mathrm{Im}A(s)\quad\text{for}\quad s\quad\text{on}\quad C_L \tag{9}$$

from (1). On the other hand, we get from (4)

$$\mathrm{Im}L(s)=\mathrm{Im}(A(s)F(s))\quad\text{for}\quad s\quad\text{on}\quad C_L, \tag{10}$$

and thus from (5)

$$\mathrm{Im}L(s)=[\mathrm{Im}A(s)]\left(1-\frac{s-s_0}{\pi}\int_{C_R}\frac{\rho(s')L(s')}{(s'-s)(s'-s_0)}ds' +\frac{s-s_0}{\pi}P\int_{C_L}\frac{\mathrm{Im}(A^{-1}(s')L(s'))}{(s'-s)(s'-s_0)}ds'\right)+\mathrm{Re}A(s)\mathrm{Im}(A^{-1}(s)L(s)). \tag{11}$$

The second term in (5) is real for s on C_L while the third term in (5) is complex there. Inserting (11) into (8),

[10] A similar form is used by G. Feldman, P. T. Mathews, and A. Salam [Nuovo Cimento **16**, 549 (1960)] for the single-channel case when the input force $L(s)$ is a pole term. It is noted that $A^{-1}(s)L(s)$ and $A(s)$ are two different functions with different high-energy behaviors. The fact that one writes a nonsubtraction relation for $A(s)$ does not necessarily mean another nonsubtraction relation for $A^{-1}(s)L(s)$. If $A(s)$ and $L(s)$ went to zero as $1/s$ for large s, then $A^{-1}(s)L(s)$ would approach a constant for large s.

one obtains

$$L(s)=\frac{1}{\pi}\int_{C_L}\frac{ds'}{s'-s}[\operatorname{Im}A(s')]\left(1-\frac{s'-s_0}{\pi}\int_{C_R}dx\frac{\rho(x)L(x)}{(x-s')(x-s_0)}+\frac{s'-s_0}{\pi}P\int_{C_L}dx\frac{\operatorname{Im}(A^{-1}(x)L(x))}{(x-s')(x-s_0)}\right)+\frac{1}{\pi}\int_{C_L}\frac{ds'}{s'-s}\operatorname{Re}A(s')\operatorname{Im}(A^{-1}(s')L(s')).\quad(12)$$

For s on C_R, where $L(s)$ is real, (12) becomes, by making use of (8)

$$L(s)=L(s)+\frac{1}{\pi}\int_{C_R}\frac{ds'}{s'-s}\left(L(s')-\frac{s'-s_0}{s'-s_0}L(s)\right)\rho(s')L(s')-\frac{1}{\pi}\int_{C_L}\frac{ds'}{s'-s}\left(\operatorname{Re}L(s')-\frac{s-s_0}{s'-s_0}L(s)\right)\operatorname{Im}(A^{-1}(s')L(s'))+\frac{1}{\pi}\int_{C_L}\frac{ds'}{s'-s}\operatorname{Re}A(s')\operatorname{Im}(A^{-1}(s')L(s'))\quad(13a).$$

The integral over C_R on the right of (13a) is not a principal value integral because $L(s')-L(s)(s-s_0)/(s'-s_0)$ vanishes at $s'=s$. The integrals over C_L are real for s on C_R.

Similarly, for s on C_L where $L(s)$ becomes complex, (12) reduces to

$$L(s)=L(s)+\frac{1}{\pi}\int_{C_R}\frac{ds'}{s'-s}\left(L(s')-\frac{s-s_0}{s'-s_0}L(s)\right)\rho(s')L(s')-\frac{1}{\pi}\int_{C_L}\frac{ds'}{s'-s}\left(\operatorname{Re}L(s')-\frac{s-s_0}{s'-s_0}\operatorname{Re}L(s)\right)\operatorname{Im}(A^{-1}(s')L(s'))+\frac{1}{\pi}\int_{C_L}\frac{ds'}{s'-s}\operatorname{Re}A(s')\operatorname{Im}(A^{-1}(s')L(s'))+i\frac{s-s_0}{\pi}\int_{C_L}ds'\frac{\operatorname{Im}L(s')}{(s'-s)(s'-s_0)}\operatorname{Im}(A^{-1}(s')L(s')).\quad(13b)$$

Here, in interchanging the order of repeated principal value integrations, we have used the Poincaré-Bertrand transformation formula[11]

$$P\int_{C_L}\frac{ds'}{s'-s}P\int_{C_L}\frac{\varphi(s',x)}{x-s'}dx=-\pi^2\varphi(s,s)+\int_{C_L}dxP\int_{C_L}\frac{\varphi(s',x)}{(s'-s)(x-s')}ds',\quad(14)$$

where s is a fixed point on C_L not coinciding with one of its ends. The integral over x on the right of (14) exists in the ordinary Riemann sense because of cancellation of one pole by the other.

Therefore, it follows from (13a) and (13b) that

$$-\frac{(s-s_0)}{\pi}\int_{C_R}ds'\frac{\rho(s')L(s')}{(s'-s)(s'-s_0)}+\frac{s-s_0}{\pi}\int_{C_L}ds'\frac{\operatorname{Im}(A^{-1}(s')L(s'))}{(s'-s)(s'-s_0)}=L^{-1}(s)\left[-\frac{1}{\pi}\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}+\frac{1}{\pi}\int_{C_L}ds'\frac{(\operatorname{Re}L(s')-\operatorname{Re}A(s'))}{s'-s}\operatorname{Im}(A^{-1}(s')L(s'))\right]\quad(15)$$

for s on C_R, and

$$-\frac{(s-s_0)}{\pi}\int_{C_R}ds'\frac{\rho(s')L(s')}{(s'-s)(s'-s_0)}+\frac{s-s_0}{\pi}\int_{C_L}ds'\frac{\operatorname{Im}(A^{-1}(s')L(s'))}{(s'-s)(s'-s_0)}=[\operatorname{Re}L(s)+i\operatorname{Im}L(s)]^{-1}\left[-\frac{1}{\pi}\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}+\frac{1}{\pi}\int_{C_L}ds'\frac{\operatorname{Re}L(s')-\operatorname{Re}A(s')}{s'-s}\operatorname{Im}(A^{-1}(s')L(s'))\right]\quad(16)$$

for s on C_L. Inserting (15) into (5), we get

$$A^{-1}(s)L(s)=1-L^{-1}(s)\left[\frac{1}{\pi}P\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}-\frac{1}{\pi}\int_{C_L}ds'\frac{\operatorname{Re}L(s')-\operatorname{Re}A(s')}{s'-s}\operatorname{Im}(A^{-1}(s')L(s'))\right]-i\rho(s)L(s)\quad(17)$$

[11] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953) p. 57; G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

for s on C_R while we obtain from (16) and (5)

$$A^{-1}(s)L(s)=1-L^{-1}(s)\left[\frac{1}{\pi}\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}-\frac{1}{\pi}\int_{C_L}ds'\frac{\mathrm{Re}L(s')-\mathrm{Re}A(s')}{s'-s}\mathrm{Im}(A^{-1}(s')L(s'))\right] \tag{18}$$

for s on C_L. Thus, it is noted that (17) and (18) are free from the subtraction parameter s_0. It is not surprising that s_0 is eliminated since $A(s)$ in principle should not depend on any subtraction point.[4] We prefer (17) or (18) to (5) since they will give the correct residue at all singularities, in the unphysical region, where the input force $L(s)$ becomes infinite. However, this is true for (5) only when there is one such singularity present, and when s_0 is taken at the point where $L(s)$ becomes infinite.

In principle, one can solve for $A(s)$ from (17) by calculating $\mathrm{Re}A(s)$ and $\mathrm{Im}(A^{-1}(s)L(s))$ from (18) by iterations and by inserting into (17). However, it is clear that the process brings in computational difficulties, and in practice one is using only some approximations to the left-hand cut input force $L(s)$. Thus we might as well use only the approximated form for (17) or (18). In making approximations one has to be sure that the scattering amplitude should manifest time reversal invariance; that is, we should preserve the symmetry of the expressions.

III. APPROXIMATE SOLUTIONS

In this section we give a symmetric, approximate solution to the scattering amplitude obtained from (17) and (18), which is applicable to the many-channel problems.

The simplest approximation is to neglect the left-hand cut contribution beyond $L^{-1}(s)$ in (17) or (18):

$$A_c(s)=\left[L^{-1}(s)-L^{-1}(s)\left(\frac{1}{\pi}P\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}\right)L^{-1}(s)-i\rho(s)\right]^{-1} \tag{19}$$

for s on C_R.

This is exactly the same as the approximate solution to the N/D equations obtained by Shaw.[12] It is observed that Shaw's procedure of arriving at (19) is less transparent and does not provide a clear way of obtaining higher order corrections of the left-hand cut contributions beyond $L^{-1}(s)$. However, the present procedure provides a straightforward way of obtaining such higher order corrections, if necessary.

We want to get an approximate form for the integral over C_L in (17), by calculating (18) near the starting point of the left-hand cut where $L(s)$ becomes infinite. We start with an approximate form of (18)

$$A^{-1}(s)L(s)=1-L^{-1}(s)\left(\frac{1}{\pi}\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}\right) \tag{20}$$

for s in the unphysical region where $A(s)\approx L(s)$. The form (20) is equivalent to $A_c(s)$ of (19) evaluated in the unphysical region. It is noted that the integral over C_R in (20) is real for s below the physical threshold and is denoted there

$$C(s)\equiv\frac{1}{\pi}\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}. \tag{21}$$

One gets from (20) and (21)

$$L^{-1}(s)A(s)=1+\left(1+\frac{1}{1-L^{-1}(s)C(s)}L^{-1}(s)C(s)\right)L^{-1}(s)C(s). \tag{22}$$

By neglecting terms in (22) beyond $O(L^{-1}(s))$, it reduces to

$$\mathrm{Re}L(s)-\mathrm{Re}A(s)\simeq -C(s) \tag{23}$$

and

$$\mathrm{Im}(A^{-1}(s)L(s))\simeq-[\mathrm{Im}L^{-1}(s)]C(s) \tag{24}$$

in the unphysical region. Thus, it follows that

$$\frac{1}{\pi}\int_{C_L}ds'\frac{\mathrm{Re}L(s')-\mathrm{Re}A(s')}{s'-s}\mathrm{Im}(A^{-1}(s')L(s'))\simeq\frac{1}{\pi}\int_{C_L}ds'\frac{C(s')\mathrm{Im}L^{-1}(s')C(s')}{s'-s}, \tag{25}$$

and finally from (17) and (25) we obtain

$$A(s)=\left[L^{-1}(s)-L^{-1}(s)\left(\frac{1}{\pi}P\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}-\frac{1}{\pi}\int_{C_L}ds'\frac{C(s')\mathrm{Im}L^{-1}(s')C(s')}{s'-s}\right)L^{-1}(s)-i\rho(s)\right]^{-1} \tag{26}$$

in the physical region. It is apparent that $A(s)$ like $A_c(s)$ does not contain any subtraction parameters and is symmetric. Equation (26) is applied here to the bootstrap problem. It is realized that its applicability is not limited to such problems. One may use it to simply calculate the scattering amplitude. Higher order corrections beyond the last term within the inner bracket of (26) may be obtained by including more terms in the expansion (22). Near the starting point of the left-hand cut, (22) converges absolutely, so that the higher order

[12] The notation $A_c(s)$ is after Ref. 8.

terms will not change appreciably the integral over C_L in (17) from (25). In the vicinity of the region where $L(s)$ dominates $A(s)$, the integral over C_L in (17), and thus the estimated form (25), will be very small when compared to the integral over C_R. Hence (26) becomes very similar to (19). This small term which plays a role as correction beyond $L^{-1}(s)$ in this formalism, however, may become important when a *derivative condition* is involved.

It should be mentioned that our procedure is not exactly the same iteration scheme as the one we face in the N/D equations. There, under each iteration, the right-hand cut contribution in the D equation should be re-evaluated. It is also well-known that the $L(s)$ from exchange considered in Sec. IV necessitates a cutoff in order to obtain a fully iterated solution to the partial-wave dispersion relations. Hence instead of iterating (18), we start with an approximation in the expansion (22), while keeping the same right-hand cut contribution, to estimate the integral over C_L in (17) rather than simply neglecting it. By including more terms in the expansion (22) the integral over C_L in (26) only changes and it still converges without introducing a cutoff at least for the $L(s)$ considered in the following section. For the example discussed in Sec. IV, when we included up to $C^2(s)$ term in the expansion (22), it turned out that the integral over C_L in (17) was not very different from (25) and the final result did not change appreciably from that of (26). Thus for our example in Sec. IV, it is felt that the form (26) is good enough.

It might seem that the approximation (26) involves a lot more work than the approximation (19). But since one is iterating only one integral $C(s)$ of (21) in (26), instead of whole $A(s)$, the whole procedure is not so much involved. Actually the integral $C(s)$ of (21) is determined completely by the knowledge of $L(s)$ in the physical region and it is merely an analytic continuation of the right-hand cut integral below the physical threshold. The estimated form (25) seems reasonable especially in the region where $L(s)$ dominates $A(s)$. The point is that one may estimate the integral over C_L in (17) starting with the first-order approximation via the customary procedure of the inverse amplitude technique, for this may be important as far as its derivatives are concerned.

One may try to compare numerically the various approximations (19), (26), and similar forms with different integrals over C_L, and even the determinantal method (7) for a given input $L(s)$, if one so desired. However, when the input $L(s)$ is just some approximation to the left-hand cut contribution in $A(s)$, it is not at all clear that an exact solution to (7) is necessarily a better approximation to $A(s)$ than (26) or (19) or even (7) with a properly chosen s_0. Hence it is felt that the comparison may not be a very conclusive one. For this reason, in Sec. IV, we apply the approximations (19) and (26) to the ρ-exchange bootstrap problem for which there exist calculations by the determinantal method,[6,7] and compare their results instead of comparing numerically the various approximations with the fully iterated solution of (17) for the approximate $L(s)$.

It is seen that when $L(s)$ is given by a single-pole term, both (19) and (26) give the same amplitude as the one from (17). The determinantal method (7) gives also the same answer provided s_0 is taken at the pole position.

IV. APPLICATION TO A SINGLE-CHANNEL BOOTSTRAP PROBLEM

In this section we want to apply $A(s)$ of (26) as well as $A_c(s)$ to the bootstrap of the ρ meson in $\pi\pi$ scattering. The input force $L(s)$ is calculated from the single dispersion relation

$$A^I(s,t,u)=\frac{1}{\pi}\int_4^\infty dt'\frac{A_t{}^I(s,t',u)}{t'-t}+\frac{1}{\pi}\int_4^\infty du'\frac{A_u{}^I(s,t,u')}{u'-u}, \quad (27)$$

where the pion mass is taken to be unity. The absorptive parts are shown[13] as

$$A_t{}^I(s,t,u)=\sum_{I'}\chi_{II'}A_t{}^{I'}(t,s,u)=\sum_{I'}\chi_{II'}\,\mathrm{Im}A^{I'}(t,\cos\theta_t), \quad (28)$$

$$A_u{}^I(s,t,u)=(-1)^I\sum_{I'}\chi_{II'}A_u{}^{I'}(u,s,t)=(-1)^I\sum_{I'}\chi_{II'}\,\mathrm{Im}A^{I'}(u,\cos\theta_u), \quad (29)$$

where $(\chi_{II'})$ is the 3×3 crossing matrix. The lth partial wave is defined by

$$A_l{}^I(s)=\frac{1}{2}\int_{-1}^1 d(\cos\theta_s)P_l(\cos\theta_s)A^I(s,\cos\theta_s). \quad (30)$$

Thus, we get from (27), (28), (29), and (30) that

$$A_l{}^I(s)=\frac{1}{\pi}\frac{4}{s-4}\int_4^\infty dt\sum_{I'}\chi_{II'}\times\mathrm{Im}A^I(t,\cos\theta_t)Q_l\left(1+\frac{2t}{s-4}\right), \quad (31)$$

and

$$\mathrm{Im}A_l{}^I(s)=\frac{2}{s-4}\int_4^{-s+4}dt\,P_l\left(1+\frac{2t}{s-4}\right)\times\sum\chi_{II'}\,\mathrm{Im}A^I(t,\cos\theta_t) \quad (32)$$

in the unphysical region. The one-channel approxi-

[13] K. Kang, Phys. Rev. **134**, B1324 (1964).

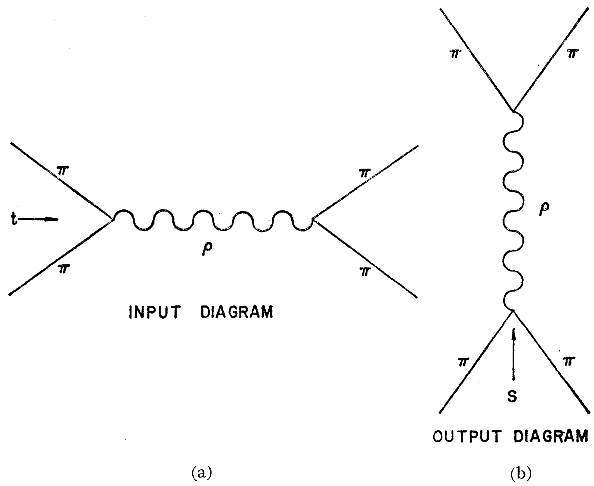

FIG. 1. (a) The input diagram, describing the exchange force for the process $\pi\pi\rightarrow\pi\pi$. (b) The output diagram, describing the appearance of the ρ as a resonance in the $\pi\pi$ state.

mation considers only the $I=1$, $l=1$ partial wave and ignores all others. In fact there is good evidence of an $I=0$, $l=2$ $\pi\pi$ resonance at around 1.25 BeV.[14] In some calculations[13] the s waves are shown to be important. Nevertheless, they will not be included, for the present calculation is merely an illustration of the material of Sec. III. It will be shown that our procedure provides a self-consistent solution; namely, the same values which we put into (31) and (32) for the position and the width of ρ are produced in (26), for this oversimplified model. [Compare Figs. 1(a) and 1(b).]

In order to have correct threshold behavior, the scattering amplitude is redefined

$$A_l(s)\rightarrow\left(\frac{4}{s-4}\right)^l A_l(s). \tag{33}$$

We also assume a Breit-Wigner form for the ρ resonance

$$\frac{4}{t-4}A_{l=1}{}^{I=1}(t)=\frac{4\Gamma}{m_\rho{}^2-t-i\Gamma(t-4)^{3/2}/t^{1/2}}. \tag{34}$$

If we further make the narrow-width approximation, we get from (34)

$$\operatorname{Im}A_{l=1}{}^{I=1}(t)=\pi\Gamma(t-4)\delta(m_\rho{}^2-t). \tag{35}$$

Thus, the approximation amounts to

$$\sum_{I'}\chi_{II'}\sum_{l}(2l+1)P_l\left(1+\frac{2s}{t-4}\right)\operatorname{Im}A_l{}^{I'}(t)$$
$$=\tfrac{3}{2}\pi\Gamma(t-4+25)\delta(m_\rho{}^2-t). \tag{36}$$

[14] W. Selov, V. Hagopian, H. Brody, A. Baker, and E. Leboy, Phys. Rev. Letters 9, 272 (1962); Y. Y. Lee, B. P. Roe, D. Sinclair, and J. C. Vander Velde, Phys. Rev. Letters 12, 342 (1964).

Then we get from (31) and (32)[15]

$$B(s)\equiv\frac{4}{s-4}A_1{}^1(s)=12\Gamma\frac{m_\rho{}^2-4+2s}{(s-4)^2}$$
$$\times\left[\left(1+\frac{2m_\rho{}^2}{s-4}\right)\ln\left(1+\frac{s-4}{m_\rho{}^2}\right)-2\right] \tag{37}$$

and

$$\operatorname{Im}B(s)\equiv\operatorname{Im}\left(\frac{4}{s-4}A_1{}^1(s)\right)$$
$$=12\Gamma\frac{m_\rho{}^2-4+2s}{(s-4)^2}\left(1+\frac{2m_\rho{}^2}{s-4}\right) \tag{38}$$

in the unphysical region. It should be noted that the width Γ corresponds to $\frac{1}{3}(\gamma_\rho{}^2{}_{\pi\pi}/4\pi)$ of Ref. 7 where they obtained the input force from calculating the exchange graph [Fig. 1(a)]. From (26) $A(s)$ is given by

$$A(s)=\left[B^{-1}(s)-B^{-1}(s)\left(\frac{1}{\pi}P\int_4^\infty ds'\frac{B(s')\rho(s')B(s')}{s'-s}\right.\right.$$
$$\left.\left.-\frac{1}{\pi}\int_{-\infty}^{-m_\rho{}^2+4}ds'\frac{C(s')\operatorname{Im}B^{-1}(s')C(s')}{s'-s}\right)B^{-1}(s)-i\rho(s)\right]^{-1}, \tag{39}$$

where $B^{-1}(s)=1/B(s)$ in the single-channel case,

$$\rho(s)=(\tfrac{1}{4})(s-4)^{3/2}/s^{1/2}, \tag{40}$$

and

$$\operatorname{Im}B^{-1}(s)=-\operatorname{Im}B(s)/|B(s)|^2. \tag{41}$$

Then $A(s)$ will depend on $x=s/4$, $r=m^2/4$, and Γ;

$$A(x)=B(x)/D(x), \tag{42}$$

where

$$D(x)=1-B^{-1}(x)\left(\frac{1}{\pi}P\int_1^\infty dx'\frac{B(x')\rho(x')B(x')}{x'-x}\right.$$
$$\left.-\frac{1}{\pi}\int_{-\infty}^{-r+1}dx'\frac{C(x')\operatorname{Im}B^{-1}(x')C(x')}{x'-x}\right)-i\rho(x)B(x). \tag{43}$$

We want r and Γ such that

$$\operatorname{Re}D(r)=0, \tag{44}$$

and

$$\Gamma=-B(r)\bigg/\left(\frac{d\operatorname{Re}D(x)}{dx}\right)_{x=r}. \tag{45}$$

The inverse of the scattering amplitude A^{-1} is calculated for a set of values of m_ρ and Γ which specify the

[15] It should be observed that (37) is actually an approximation to the whole amplitude $A(s)$ in the unphysical region rather than just to the left-hand cut contribution. However, we further assume that this is an approximation to $L(s)$. In this respect, the determinantal method can satisfy crossing symmetry roughly by taking the subtraction point s_0 at the beginning of the left-hand cut since $A_{\text{det}}(s_0)=L(s_0)$ there. See Ref. 7. $B(s)$ denotes the expression approximating $L(s)$ in the main text for the $\rho\pi\pi$ example.

input mass and width of the resonance, respectively. For the given input mass and width, the output mass m_ρ obtained from the zero of $\mathrm{Re}(A^{-1})$ or $\mathrm{Re}D(x)$ of (43), as well as the output width given by (45), are readily obtained. A self-consistent mass occurs if the input m_ρ value agrees with the output m_ρ value. Similarly a self-consistent solution in the width occurs if the input Γ value agrees with the output Γ value. A completely self-consistent resonance is obtained when both m_ρ (input) equals m_ρ (output) and Γ (input) equals Γ (output). It turns out that there is such a self-consistent solution, and the position and width of the ρ resonance are found to be $m_\rho=322$ MeV, $\Gamma=0.112$ (≈ 45 MeV). The resulting parameters are determined clearly by insisting that $\mathrm{Re}(A^{-1})$ should not have ghost zeros near the physical threshold. It should be noted that if $B(s)$ contains a dynamical zero, it may produce a spurious pole in the approximation (26) like that in (19). One may avoid a ghost pole below threshold by including this dynamical zero in $\rho(s)$.[8] Indeed the $B(s)$ of (37) contains such a zero at $s=2-\frac{1}{2}m_\rho^2$ and thus such zeros in $\mathrm{Re}(A^{-1})$ may still exist in the unphysical region. Our final $\mathrm{Re}(A^{-1})$ with the resulting parameters is examined and found to have no spurious zeros for $2-\frac{1}{2}m_\rho^2<s<4$ in the unphysical region. The behavior of $\mathrm{Re}D(x)$ for large x is such as to reach a minimum negative point beyond which it approaches the real x axis asymptotically from below. However, we do not attribute any physical significance to this behavior, since our approximation does not hold for large x.

Similar calculations are carried out using the expression $A_c(s)$. Although a self-consistent solution in the mass is obtainable at about $m_\rho=350$ MeV, input $\Gamma=0.15$, the output width is found to be of the order of one half the input width. Thus it is seen that the additional term in our expression $A(s)$ plays a role in the output width value to bring about a completely self-consistent solution. This may not mean that the approximation (26) is necessarily better than the approximation (19), in the sense that the output width calculated from a plot of the phase shift instead of (45) may give different results for the two approximations. Insofar as the determination of the parameters is concerned, using (45) instead of the phase-shift plot reduces the work greatly. It should be noticed that both (19) and (26) give a similar value of the self-consistent mass.

The calculation was performed with the aid of the University of Michigan IBM 7090 computer.

V. CONCLUDING REMARKS

Despite the fact that our results in the example of Sec. IV are considerably off the measured values of the ρ resonance, we feel that the present calculation gives good results when compared to previously published results. Further, our procedure is free from subtraction parameters and is symmetric.

It should be kept in mind that the calculation in the previous section makes the following approximations which should be improved for a more realistic calculation:

First of all, the input force $L(s)$ is taken from the exchange force in the $I=1$, $l=1$ state, neglecting all other partial waves, and is assumed to be valid all over the left-hand cut. Secondly, the contributions from other inelastic channels to the scattering amplitude have been ignored.

The procedure developed in Sec. III is basically a many-channel one. It is easily applicable to the many-channel problem. Other forms of $L(s)$ that can render the dispersion relations algebraically soluble can be discussed within the present framework. Also the procedure can be applied to the Reggeized bootstrap problems. Any improvement on the calculation should be preceded by an improvement on the calculation of the input force $L(s)$.

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