the reaction $^{27}Al(n,\alpha)^{24}Na$, the cross section for which is 115 mb at a neutron energy of 14.6 MeV.[19] The disintegration of ^{24}Na is always accompanied by the emission of a γ ray of energy 2.76 MeV and a relative measure of the source strength of ^{78}As and ^{24}Na could be obtained from measurements of the γ spectra. For these measurements, an Intertechnique 400-channel analyzer was used. The ratio between the cross sections of the (n,α) reactions in ^{81}Br and ^{27}Al was determined by this method and was found to be 0.07 ± 0.03. This confirms the results given above.

Cross sections of (n,p) and (n,α) reactions may be more accurately determined if the different activities are separated by chemical methods. An experiment of this kind was carried out parallel to our own experiments by E. Steinnes[20] from the Institute of Nuclear Chemistry, University of Oslo. The SAMES J accelerator was used in this case too. Steinnes obtained a very low value for the cross section of the reaction $^{81}Br(n,\alpha)^{78}As$.

Our cross section values for bromine are listed in Table II, together with values obtained by other groups.[5–8 19,20]

[20] E. Steinnes (private communication, 1964).

CONCLUSION

There is very good agreement between the results obtained by the annihilation method and the extrapolation method for phosphorus and copper. The agreement is not so good for bromine. Examination of this element is more complicated since so many activities are present, but there certainly is a discrepancy here. Unfortunately the measurements could not be repeated, since the accelerator had to be used for other purposes.

For the (n,p) and (n,α) reactions in phosphorus and for all $(n,2n)$ reactions our limits of error are relatively small. The reason for this probably is that a large number of measurements were always made. We have no special reason to expect any systematic errors to be present, but obviously the possibility cannot be excluded.

As mentioned above, there is a large disagreement between earlier values and our value for the cross section of the (n,α) reaction in ^{81}Br. Otherwise the agreement between our cross-section values and those obtained by other groups seems to be satisfactory.

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Analytic Distorted-Wave Approximation for High-Energy Electron Scattering Calculations*

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A three-dimensional WKB approximation for high-energy electron scattering on nuclei is derived and approximated analytically in the vicinity of the nucleus. This approximation contains five parameters which can be calculated from the charge distribution and are not adjustable. It is applied here to calculations of elastic scattering and monopole and quadrupole excitation; the method employed involves an asymptotic expansion in inverse powers of qR, where q is the momentum transfer and R is the nuclear radius. In spite of the several approximations, the calculations reproduce qualitatively all the features of the exact results, such as position and filling in of diffraction minima, and they are quantitatively very good as well. Although the charge and excitation distributions treated here are somewhat idealized, it should be straightforward to extend the results to more realistic situations.

I. INTRODUCTION

THE examination of nuclear shapes by measurement of high-energy electron elastic scattering has the great theoretical attraction that the interaction with individual nucleons is electromagnetic, and well understood. The assumption that for elastic scattering the nucleus can be represented by a static charge distribu-

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tion reduces the theoretical problem to that of solving a Dirac equation for the electron in which the effect of the nucleus is contained entirely in the static Coulomb potential. The salient physical features of the observed differential cross sections may then be associated with properties of the charge distribution.[1] For the lightest nuclei, where the Born approximation may safely be employed, this association, involving only the Fourier transform of the charge distribution, the form factor, is very direct and useful. For heavier nuclei (and for light nuclei beyond the region of the first diffraction dip) it is necessary to make an exact, perforce numerical, solution of the Dirac equation. The procedure is straightforward but lengthy, and one loses the direct connection between the charge distribution and the structure of the resulting differential cross sections. The cross sections obtained, however, turn out to be very similar in character to those of the Born approximation.[2] The purpose of this paper is to develop a physical picture of the scattering process involving high-energy electrons and heavy nuclei, and from it to derive an approximation which reproduces very closely the exact partial-wave results for elastic scattering.

Preliminary announcements of such a paper were made a number of years ago in connection with earlier work on this subject.[3,4] We will try to clarify the relationship of the present method to those earlier studies, and to approximations suggested by others. The problem of analytic high-energy approximations to potential scattering is common to a number of fields, of course. The individual feature of the electron-scattering problem is that the part of the differential cross section useful for investigating nuclear structure occurs at large angles, many times the classical scattering angle. At such large angles the cross section is very small, and makes negligible contribution to the "total" cross section. The many recent attempts to discuss the close-to-forward scattering are thus not directly relevant to our problem, and we shall not refer to them in detail.[5] Although the method we derive is describable as a distorted-wave Born approximation, the extensive calculations of inelastic nucleon scattering which go under this name are different enough in purpose and scope that we have not attempted to make comparison with them.

The necessity for an approximate method was forced on us when scattering experiments with strongly deformed heavy nuclei were made which included inelastic with the elastic scattering.[3] The similarity of the exact partial-wave cross sections to the Born-approximation results had been noted earlier, and empirical scaling rules for the angles of the diffraction structure developed.[6] A regularity in the behavior of the scattering phase shifts and radial wave functions reinforced our intuitive reasoning that the electron wave functions inside the nucleus were approximately plane waves, with modified wave number and amplitude. The phase-shift regularity implies also that they were not exactly plane waves, but were distorted by the smooth Coulomb potential. It was possible to give an analytic form to the three-dimensional wave function, i.e., to analytically sum the Legendre series of partial waves, in the region of the nucleus. The physical assumption was made that this wave function represents the effect of the smooth part of the potential, in other words, the low Fourier components. The scattering at angles larger than the classical maximum angle must be due to the high Fourier components of the potential. The effect of these was calculated in first-order perturbation theory, taking the distorted plane waves as zero-order wave functions. This approximation was called a modified Born approximation (abbreviated M.B.A.). The scattering amplitude as calculated by this approximation thus bore a close resemblance to that of the Born approximation. The method reproduced the angular variation of the partial-wave cross sections, and because of the distortion of the incident and final waves, it also filled in the diffraction zeros, which are a notorious inadequacy of the Born approximation, to about the correct extent. The absolute value of the cross section was not reproduced very well, however, a deficiency which our recent investigations improve on considerably.

At the same time as we were developing this approach, Schiff[7] proposed another approximation for this process. His method was to make a summation of the infinite series of scatterings given by the Born approximation. A result of the stationary-phase method he employed to evaluate the Born integrals was that, because of the long tail of Coulomb field of a heavy nucleus, all but one of the scatters were at very small angles. The change in phase of the wave function due to the small angle scatterings was summed analytically by assuming straight-line trajectories for the electron (i.e., zero-angle scattering). The scattering amplitude then comes from the one large-angle Born collision, with wave functions which have a modified wave number. The approximate evaluation of the scattering amplitude according to this approximation, by Tiemann,[8] gave differential cross sections in which the angular position of the diffraction structure was in good agreement with partial-

[1] A general discussion of electron scattering and nuclear shapes is given in the review article by R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).

[2] See, for example, D. R. Yennie, D. G. Ravenhall, and R. N. Wilson, Phys. Rev. **95**, 500 (1954).

[3] B. W. Downs, D. G. Ravenhall, and D. R. Yennie, Phys. Rev. **106**, 1285 (1957), Ref. 7.

[4] D. G. Ravenhall and D. R. Yennie, Proc. Phys. Soc. (London) **A70**, 857 (1957), reference to Yennie, Ravenhall, and Tiemann.

[5] See, for example, R. J. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, New York, 1959), Vol. 1, p. 315.

[6] See Ref. 2, Sec. 6.

[7] L. I. Schiff, Phys. Rev. **103**, 443 (1956).

[8] J. J. Tiemann, Phys. Rev. **109**, 183 (1958).

wave results, but for which the maxima were too low by 10–50%, and the minima much too deep.

A development of Schiff's approach was suggested by Saxon,[9] who emphasized its connection with the WKB or eikonal approximation. A later version, by Saxon and Schiff,[10] while confining itself to the Schrödinger equation, made a detailed analytical development. In the final form the scattering amplitude appears to be very similar to Schiff's first approximation, since at an early stage the assumption of straight-line trajectories was made. Error estimates were made, but no detailed numerical comparison with an exact treatment was given. A further contribution was made by Tiemann,[11] who developed Schiff's approximation somewhat, and included some effects suggested by the work on the M.B.A. His results were in much closer agreement with the partial-wave cross section, except for the depth of the diffraction dips, and the over-all normalization. Recently Baker[12] has developed from the nonrelativistic treatment of Glauber an approximation for the present problem which closely resembles the results of Schiff.[7] Rather than introducing distortions in both the incident and outgoing waves, he assumes as an axis for the distortion the recoil momentum. It is somewhat unexpected, therefore, that the numerical agreement he obtains is in closer agreement with the partial-wave calculations than was Tiemann's evaluation of Schiff's approximation. The improvement may well result from a more accurate calculation by Baker of his approximation.

The method we present in this paper has the same basis, fundamentally, as the earlier modified Born approximation, but now rests entirely on analytic approximations, rather than on empirical regularities in the partial-wave results. As has been briefly described previously,[4,13] we imagine a division of the potential into a low-frequency part and a high-frequency part. Fortunately, the work we present here does not need an explicit statement of the division. The low-frequency part, a potential which in configuration space approximates the actual potential, but which has all sudden changes in function and derivatives smoothed away, is well-treated by the eikonal approximation. The small amounts of reflected wave arising from sudden changes in a potential, which the eikonal approximation ignores, are in fact not present even in an exact solution because of the removal of the high frequencies. A verification of this can be provided by the exact numerical differential cross section from such a potential. This cross section tends rapidly to zero for angles greater than the classical maximum angle, which for typical cases of current interest would be from 5° to 10°. The scattering of electrons at angles from 30° to 150°, the phenomenon of experimental and computational interest, is a wave-scattering phenomenon involving the high-frequency part of the potential. We treat it in perturbation theory with the low-frequency wave functions (approximated by the eikonal wave functions) as the zero-order states. It is fortunate for us that the first-order expression seems to provide the major part of the contribution to the large-angle scattering. Our use of the eikonal approximation has led us to small changes in the amplitude of the wave function, too subtle to recognize from the empirical study of the partial-wave solutions, which remove to a large extent the discrepancy in absolute magnitude of our earlier approximation to the cross section.

Our present approximation has a resemblance to the Saxon-Schiff approach in its use of the eikonal (WKB) approximation. In our treatment, however, as with Schiff's original development, it is crucial that the large-angle scattering appear as a dominant, lowest order contribution, rather than in successive improvements of the eikonal approximation. The aspirations and general method of the various approximations proposed are quite similar, especially comparing Schiff's with ours. It is perhaps as an explicit demonstration of the validity of our particular method that our work is worth reporting on at this time. Of great importance, practically, is the inclusion in our actual formulas of the several small but very important terms describing the slight distortions of the electron wave. Without them it is not possible to obtain the correct filling in of the Born zeros or the correct absolute magnitude. An accurate approximation such as ours has an obvious application in inelastic scattering, of course. Partial-wave calculations of inelastic scattering, modeled on the corresponding nucleon-nucleus calculations, are very lengthy.[14,15] The results seem to bear out our physical

[9] D. S. Saxon, Phys. Rev. **107**, 871 (1957).

[10] D. S. Saxon and L. I. Schiff, Nuovo Cimento **6**, 614 (1957).

[11] J. J. Tiemann, Ph.D. thesis, Stanford University, 1960 (unpublished).

[12] A. Baker, Phys. Rev. **134**, B240 (1964). It should be noted that the partial-wave cross section given in Fig. 5 of this reference is somewhat misleading. The expected diffraction structure at small angles is absent. We thank Professor K. Ford for a helpful correspondence concerning this work.

[13] A simple physical description of some recent calculations on electron and positron scattering in terms of our approximation is given in R. Herman, B. C. Clark, and D. G. Ravenhall, Phys. Rev. **132**, 414 (1963).

[14] T. A. Griffy, D. S. Onley, J. T. Reynolds, and L. C. Biedenharn, Phys. Rev. **128**, 833 (1962); D. S. Onley, T. A. Griffy, and J. T. Reynolds, *ibid.* **129**, 1689 (1963). These papers discuss electric-quadrupole excitation, the first with uniform charge distributions, and the second with the realistic Fermi distribution to describe the elastically scattered wave.

[15] K. Alder and T. H. Schucan, Nucl. Phys. **42**, 498 (1963). This paper contains calculations of monopole excitation cross sections with a uniform charge distribution for the elastic wave. Later in this paper we present an alternative method for calculating monopole excitations, using only the elastic partial-wave program. In the one case we have checked in detail, $Z=80$ and $kR=7.09$, our results are in significant disagreement with those given in Fig. 3 of the above reference. Their cross section lacks the third diffraction dip expected from the Born approximation, and which does occur in our partial-wave calculation. We understand from Dr. Schucan that more recent calculations of theirs are in much closer agreement with our results [T. H. Schucan, thesis, University of Basel, 1964 (unpublished)].

predictions as regards shifting of the minima, etc. Our method, albeit an approximation, is simpler and more transparent.

II. SEMICLASSICAL APPROXIMATION FOR SCATTERING WAVE FUNCTIONS

The purpose of this section is to develop a simple, useful, and fairly accurate approximation to the electron-scattering wave function in the vicinity of the nucleus. The physical basis of the approximation is that only a very small part of the incident flux which passes through the nucleus scatters into large angles. This suggests that the main part of the wave function in the vicinity of the nucleus is simply a distorted incident wave. Originally, we were led to this approximation by an empirical study of the properties of the numerically-calculated exact wave functions. The main features appearing in this study are the following: (i) Near the origin, the radial wave functions can be approximated well by spherical Bessel functions with the local wave number replacing the asymptotic one, and, equally important, with a normalization which is independent of angular momentum. If this were the only modification, it would immediately suggest a plane wave with a local wave number. (ii) For small angular momentum, the phase shifts can be represented well by the empirical fit

$$\eta_j=\eta_0-b'j(j+1). \tag{2.1}$$

A little study shows that this difference in phase between the different partial waves corresponds to a curvature of the wave fronts in the vicinity of the nucleus. It is worth noting in passing that two charge distributions whose phase shifts are representable in the form (2.1) with nearly equal parameters may yield very different angular distributions at the wider angles; examples are given in Ref. 4. This is a consequence of the often-noted fact that, when wide-angle electron scattering is calculated by means of a partial-wave expansion, there is a very sensitive cancellation of the various terms in the sum and very slight differences in the phase shifts may produce large relative differences in the cross section. This reinforces our view that the wave function near the nucleus [represented by η_0 and b in (2.1)] depends only on very general features of the charge distribution while the scattering amplitude at large angles depends on fine details of the charge distribution.

It is possible to develop an approximation to the wave function by using the empirical phase shift behavior (2.1) together with the properties of the radial functions. In addition to the modified wave number and wave-front curvature, which are already contained in Schiff's work, two new features emerge. The first is the positional dependence of the spin direction, which is always perpendicular to the surfaces of constant phase. The second is the change in magnitude of the wave function across the nucleus, corresponding to the fact that the nucleus acts somewhat like a lens, tending to bring the rays to a focus at a point somewhat beyond the nucleus. These qualitative properties of the wave function strongly suggest that a three-dimensional WKB, or eikonal, approximation may provide a quite accurate estimate to the wave function at small distances. This procedure will now be studied; and, incidentally, the physical significance of (2.1) will be discovered.

Eikonal Approximation

The starting point of the analysis is the two-component Dirac equation which is valid at high energies

$$(\boldsymbol{\sigma}\cdot\mathbf{p}+V)\psi(\mathbf{r})=E\psi(\mathbf{r}). \tag{2.2}$$

We assume a solution in the form

$$\psi(\mathbf{r})=u(\mathbf{r})e^{iS(\mathbf{r})}, \tag{2.3}$$

where S is to be chosen so that the rapid variations of ψ are contained in the exponential and the slow variations are in the spinor function u. It should be remarked that this form of wave function cannot be valid everywhere with these restrictions. For example, in the asymptotic region it would not be possible to write the sum of the incident and scattered waves in this form with u slowly varying. Thus it would not be feasible to solve the scattering problem directly using the form (2.3) together with the restriction that u be slowly varying. However, if we are willing to forego a direct calculation of the outgoing wave, it should be possible to determine the distorted wave at the nucleus.

The result of substituting (2.3) into (2.2) is

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}S+V-E)u=i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}u. \tag{2.4}$$

Since we want u to be a slowly varying function of position, we define the zeroth approximation by

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}S+V-E)u_0=0. \tag{2.5}$$

Multiplying this through by $(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}S-V+E)$, we see that this homogeneous equation for the components of u_0 has solutions only if

$$(\boldsymbol{\nabla}S)^2-(E-V)^2=0. \tag{2.6}$$

We shall come back to the physical interpretation of this equation later on. As they stand, (2.4) and (2.6) together are presumably equivalent to the Dirac equation (2.2). In particular, since it is a differential equation, (2.4) has wave-like solutions. Now, however, we will proceed to solve it by iteration, starting with u_0; the wave-like solutions will accordingly be lost. As has been repeatedly emphasized, this should not adversely affect the validity of the approximation at small distances. The higher approximations are now given by an iterative procedure

$$u=u_0+u_1+u_2+\cdots$$
$$(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}S+V-E)u_{n+1}=i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}u_n. \tag{2.7}$$

It is expected that this will be an asymptotic, rather than a convergent expansion.

Although there appears to be no practical interest in the higher terms of the expansion, it is interesting to see how each one may be computed in principle from the preceding one. For this purpose it is convenient to split each spinor into its positive and negative helicity parts

$$u_n = u_n^{(+)} + u_n^{(-)},$$
$$(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}S)u_n^{(\pm)} = \pm(E-V)u_n^{(\pm)}. \tag{2.8}$$

This may be accomplished most easily with a projection operator

$$u_n^{(\pm)} = [(E-V\pm\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}S)/2(E-V)]u_n. \tag{2.9}$$

It is seen from (2.5) that u_0 has positive helicity [we may of course reverse all helicities by reversing the sign of the $\boldsymbol{\sigma}$ term in the Dirac equation (2.2)]. The negative helicity part of $u_n (n>0)$ is determined immediately from (2.7), which reduces to

$$u_n^{(-)} = -i(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}u_{n-1})/2(E-V). \tag{2.10}$$

The positive helicity part of u_n is determined by a "solvability condition" on (2.7), required because the matrix on the left side is singular. The right side must have a certain ratio between the two spinor components given by

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}S - V + E)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}u_n = 0,$$

which may also be written

$$u_0^{\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}u_n = 0. \tag{2.11}$$

For the lowest order this may be expressed in the form

$$\boldsymbol{\nabla}\cdot(u_0^{\dagger}\boldsymbol{\sigma}u_0) = 0. \tag{2.12}$$

This is just the expression for current conservation in lowest order; current conservation is also easily verified in higher order. The positive helicity part of u_n may be written

$$u_n^{(+)} = \lambda_n u_0, \tag{2.13}$$

where λ_n is a scalar function of position. It is then easily found that

$$\hat{n}\cdot\boldsymbol{\nabla}\lambda_n = -u_0^{\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}u_n^{(-)}/u_0^{\dagger}u_0, \tag{2.14}$$

where $\hat{n}$ is a unit vector in the direction of $\boldsymbol{\nabla}S$:

$$\hat{n} = \boldsymbol{\nabla}S/(E-V). \tag{2.15}$$

The systematic procedure for developing the eikonal expansion is now clear. The function S is to be determined from (2.6) together with boundary conditions. Equation (2.5) then fixes the local spin direction; i.e., it gives the ratio of components of u_0 at each point. Normalization of u_0 is fixed by (2.12). Finally (2.10) and the solution of (2.14) generate the higher orders in terms of the lower ones.

The Semiclassical Approximation

The zeroth approximation is defined by (2.5), (2.6), and (2.12) together with the boundary conditions. Because of the close connection with the classical scattering problem which will soon become apparent, this approximation will be called the "semiclassical approximation." This connection is immediately obvious because (2.6) is just the Hamilton-Jacobi equation for particles of zero rest mass. Since we wish to exploit some properties of the classical scattering problem, the connection will now be traced out in some detail.

Any solution of (2.6) corresponds to a family of trajectories of classical particles; each trajectory is orthogonal to the surfaces of constant S and the classical momentum at any point of the trajectory is given by

$$\mathbf{P}_{cl} = \boldsymbol{\nabla}S. \tag{2.16}$$

Because of the zero rest mass, the speed of the particle is always the speed of light ($=1$ in natural units). Letting s be the distance along the trajectory, the equations of motion of the particle may be written

$$d\mathbf{r}/ds = \mathbf{p}_{cl}/(E-V), \tag{2.17}$$
$$d\mathbf{p}_{cl}/ds = -\boldsymbol{\nabla}V.$$

There are two important constants of the motion for the particle (along a trajectory):

$$\mathbf{p}_{cl} + V = E, \tag{2.18}$$
$$\mathbf{r}\times\mathbf{p}_{cl} = L_{cl};$$

the latter holds only in a central potential.

The next step is to find the family of trajectories corresponding to each type of scattering eigenfunction. For the plus type (incident plane wave plus outgoing spherical wave), the required family of trajectories corresponds to incident particles of momentum $\mathbf{k}_i = k\hat{n}_i$. For the minus type eigenfunction, the final particles all have a definite momentum $\mathbf{k}_f = k\hat{n}_f$. The unit vectors $\hat{n}_i$ and $\hat{n}_f$ refer to the asymptotic limit of (2.15) in the initial or final direction respectively; these vectors completely characterize the scattering eigenfunction. Clearly the two types are related; by reversing the sign of S, the plus type solutions are converted to the minus type ones, with final momentum the negative of $\mathbf{k}_i$. The two types of eigenfunctions are illustrated in Figs. 1(a) and 1(b). For definiteness, the plus type eigenfunction will be discussed in detail; rather than proliferate notation, an index i will be omitted from S, u_0, etc., whenever no confusion will result.

The function S may be determined by integrating along classical trajectories

$$S(\mathbf{r}) = S(\mathbf{r}_0) + \int_{\mathbf{r}_0}^{\mathbf{r}} (E-V)ds. \tag{2.19}$$

Since for finite impact parameters we wish the incident

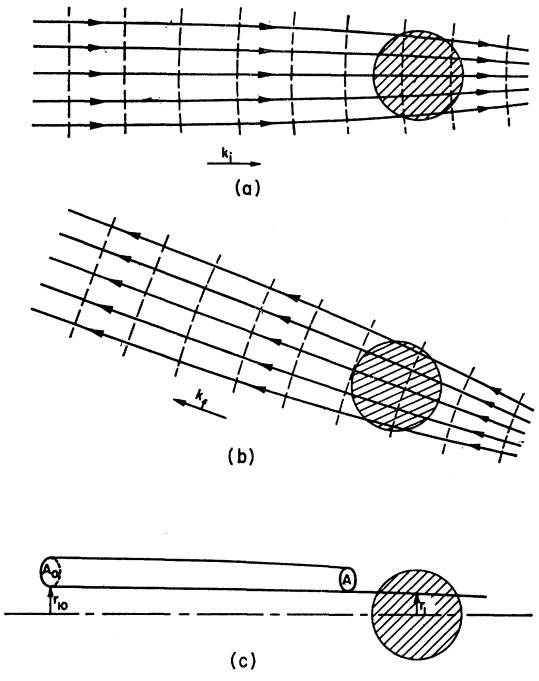

FIG. 1. The + and − type scattering eigenfunctions near the nucleus are illustrated in (a) and (b), respectively. The classical trajectories are represented by solid lines with the direction of ΔS indicated by arrows; the dashed lines represent the surfaces of constant S. A tube of flux is shown in (c), and the area change is indicated.

wave to approach a plane wave at large distances, it is convenient to choose $S(\mathbf{r}_0)$ in the following way:

$$S(\mathbf{r}_0)=\mathbf{k}_i\cdot\mathbf{r}_0-\gamma\ln 2kr_0, \tag{2.20}$$

where $\gamma=Ze^2$. With this choice $S(\mathbf{r})$ is independent of $\mathbf{r}_0$ in the limit $\mathbf{k}_i\cdot\mathbf{r}_0\to-\infty$. Equation (2.20) then amounts to a boundary condition on S, which may now be computed from (2.19) if the classical trajectories are known.

Assuming that S has been determined, we may express the general solution of (2.5) in terms of an arbitrary constant spinor v

$$u_0=N[2(E-V)]^{-1}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}S+E-V)v. \tag{2.21}$$

The normalizing factor N is to be determined from (2.12), which may now be expressed

$$\boldsymbol{\nabla}\cdot(\hat{n}u_0^{\dagger}u_0)=0, \tag{2.22}$$

where $\hat{n}$ is defined in (2.15); a more complete notation for $\hat{n}$ is $\hat{n}_i(\mathbf{r})$, where $\mathbf{r}$ indicates its dependence on position and i its dependence on incident direction ($\hat{n}_i$). This equation has a simple geometrical interpretation. Consider a tube of trajectories [Fig. 1(c)] and let A be the cross-sectional area of the tube. Upon integrating (2.22) over the volume of the tube, it is found that

$$\tfrac{1}{2}N^2A(1+\hat{n}_i\cdot\hat{n})=\text{constant},$$

or

$$N=[\tfrac{1}{2}\beta(1+\hat{n}_i\cdot\hat{n})]^{-1/2}, \tag{2.23}$$

where $\beta=A/A_0$ and where v has been chosen to satisfy

$$v^{\dagger}v=1, \quad \boldsymbol{\sigma}\cdot\hat{n}_iv=v.$$

This corresponds to a uniform flux density for the incident wave. Finally, then, we may write

$$u_0=[2\beta(1+\hat{n}_i\cdot\hat{n})]^{-1/2}(\boldsymbol{\sigma}\cdot\hat{n}+1)v. \tag{2.24}$$

Higher Approximations

We shall confine ourselves here to a few general remarks. Using the results of the semiclassical approximation, (2.14) may be rewritten

$$d\lambda_n/ds=-\beta u_0^{\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}u_n^{(-)}, \tag{2.25}$$

where ds refers to a derivative along the classical trajectory. Now we may discuss why the eikonal approximation can never give wide-angle scattering directly. According to (2.10), the negative helicity part of the spinor in any given order ($u_n^{(-)}$), is just a derivative of the spinor in the preceding order. It will thus not contain an outgoing wave unless the preceding order did. On the other hand, the positive helicity part is given by integrating a quantity along the classical trajectory, and can therefore not give scattering outside the classical region. To sum up, if the classical scattering does not extend beyond a certain angle (and typically, it does not), we cannot hope to determine the scattering in the nonclassical region by an finite number of derivatives of the semiclassical scattering amplitude.

Another feature of the scattering which is not included in the eikonal approximation is the possibility of reflection of the wave from rapid changes of the potential. The condition for this approximation to be valid is that the change in local wave number is small in one wavelength:

$$\lambda\partial V/\partial r\ll k \quad \text{or} \quad \partial V/\partial r\ll k^2.$$

For a nucleus of charge Z, radial extension of order R, this means

$$\gamma/(kR)^2\ll 1.$$

This condition will in fact obtain in those situations in which we are interested. Of course, while the reflection of a tiny fraction of the incident flux at a rapid change in the potential or one of its higher derivatives may be a completely unimportant correction to the wave function at small distances, it could give a relatively important contribution to the scattered wave which has a very small magnitude at large angles.

Another apparent difficulty appears if there exists a focal point in the semiclassical wave function; this may occur without the presence of rapidly varying potentials. In that case β would become very small, and the higher approximations would not converge in the region near the focus. The eikonal approximation would

then break down. Beyond the focal point, where several trajectories pass through each point, the function S becomes multiple valued. Presumably the approximate wave function should then be expressed as a sum of terms of the form (2.3). In the case of our high-energy approximation, a focal point does exist; fortunately it is not in the region for which we wish to determine an approximate function. As will be seen in the following paragraphs, for electron scattering from nuclei the focus lies a distance of order $R(kR/\gamma)$ beyond the nucleus. For positron scattering, the nucleus acts like a diverging lens and there is a virtual focal point on the incoming side of the nucleus.

Estimates of the Eikonal Wave Function

Ideally, one would like to calculate the eikonal wave function and employ it in other calculations without further approximation. However, this would not be a simple matter and we wish to develop another procedure. The difficulty may be understood by considering (2.19). On the surface, this appears to be linear in V and hence linear in the parameter $\gamma=Ze^2$. However, the actual situation is more complicated; each trajectory depends on V and hence $S(\mathbf{r})$ is a very complicated nonlinear function of γ. Let us attempt to find S as a power series in γ by setting

$$S=S(0)+\mathbf{k}\cdot\mathbf{r}+S_1(\mathbf{r})+S_2(\mathbf{r})+\cdots, \tag{2.26}$$

where $S_i(\mathbf{r})$ is proportional to γ^i and each S_i vanishes at the origin. Substituting in (2.6), we find for the first few powers of γ

$$\mathbf{k}\cdot\nabla S_1=-EV, \tag{2.27a}$$

$$2\mathbf{k}\cdot\nabla S_2=V^2-(\nabla S_1)^2, \tag{2.27b}$$

$$\mathbf{k}\cdot\nabla S_3=-\nabla S_1\cdot\nabla S_2. \tag{2.27c}$$

The solution to (2.27a) is easily written down. Let $\mathbf{r}_1\,[=\hat{k}\times(\mathbf{r}\times\hat{k})]$ be the component of $\mathbf{r}$ perpendicular to $\mathbf{k}$ and $\mathbf{z}=z\hat{k}$ be the component parallel to $\mathbf{k}$. Then the solution which satisfies the boundary condition (2.20) is

$$S_1(r_1,z)=\lim_{L\to\infty}\left[\int_0^L V(\lambda)d\lambda - \int_{-z}^{L} V\big((\lambda^2+r_1^2)^{1/2}\big)d\lambda\right]=S_1'+S_1'', \tag{2.28a}$$

where

$$S_1'=\int_0^\infty [V(\lambda)-V\big((\lambda^2+r_1^2)^{1/2}\big)]d\lambda, \tag{2.28b}$$

$$S_1''=-\int_0^z V\big((\lambda^2+r_1^2)^{1/2}\big)d\lambda. \tag{2.28c}$$

The solution to (2.27b) may now be expressed in terms of S_1:

$$S_2=-\frac{1}{2k}\int_{-\infty}^{z}\left[\frac{\partial S_1(r_1,z')}{\partial r_1}\right]^2 dz'. \tag{2.29}$$

Using dimensional arguments it is easy to find the order of magnitude of the various contributions. It is clear that, since S_1 is independent of k, it is of order γ. The additional terms are successively smaller by factors of the order γ/kR, where R is a measure of the spatial extension of the charge distribution.

While it would be desirable and perhaps feasible to use (2.28) and (2.29) directly in further calculations, we shall pursue the simpler procedure of expanding these functions in powers of $\mathbf{r}$. For a typical charge distribution which is flat at the center of the nucleus, the potential may be written

$$V(r)=V(0)+\tfrac{1}{2}ak'^3r^2+\cdots. \tag{2.30}$$

For example, inside a uniformly charged sphere of radius R, one finds $V(0)=-3\gamma/2R$ and $a=\gamma/(k'R)^3$, with no higher powers; k' will be defined shortly. In the remainder of this work, we shall treat only the a terms in (2.30), but higher powers of r^2 could be included if necessary. With (2.30), Eq. (2.28c) is easily integrated

$$S_1''=-V(0)\hat{k}\cdot\mathbf{r}-\tfrac{1}{2}a(k'^3r_1^2z+\tfrac{1}{3}k'^3z^3). \tag{2.31}$$

The $V(0)$ term should clearly be combined with the plane wave term in (2.26) to give $\mathbf{k}'\cdot\mathbf{r}$, where

$$\mathbf{k}'=k'\hat{k},\quad k'=E-V(0). \tag{2.32}$$

We have used already k' in defining a; this is simply a matter of convenience which permits writing later expressions in a more unified form. Next (2.28b) may be expanded in powers of r_1^2

$$\begin{aligned}S_1'&=-\tfrac{1}{2}r_1^2\int_0^\infty\frac{1}{\lambda}\frac{\partial V}{\partial\lambda}d\lambda \\ &\quad -\tfrac{1}{8}r_1^4\int_0^\infty\left(\frac{1}{\lambda^2}\frac{\partial^2V}{\partial\lambda^2}-\frac{1}{\lambda^3}\frac{\partial V}{\partial\lambda}\right)d\lambda\cdots \\ &=-r_1^2\pi\gamma\int_0^\infty\rho(\lambda)d\lambda-\tfrac{1}{8}r_1^4\pi\gamma\int_0^\infty\frac{1}{\lambda}\frac{\partial\rho}{\partial\lambda}d\lambda \\ &\equiv-b_1(r_1k')^2+c_1(r_1k')^4,\end{aligned} \tag{2.33}$$

which serves to define b_1 and c_1:

$$b_1=\frac{\pi\gamma}{k'^2}\int_0^\infty\rho(\lambda)d\lambda=\frac{\gamma}{4k'^2}\left\langle\frac{1}{r^2}\right\rangle, \tag{2.34a}$$

and

$$c_1=-\frac{1}{8}\frac{\pi\gamma}{k'^4}\int_0^\infty\frac{1}{\lambda}\frac{\partial\rho}{\partial\lambda}d\lambda. \tag{2.34b}$$

TABLE I. Comparison of values of b and c from eikonal theory and phase shift analysis. All examples are for electron scattering from gold; for notation see Hahn *et al.*[a] The phase shifts used in cases (a) and (c) are from Ravenhall and Yennie[b]; those for case (b) are from Clark *et al.*[c] The quantities $b_{\min}$, $b_{\max}$, c_1, and c_2 are defined in the text; $b_{\rm ph}$ and $c_{\rm ph}$ are obtained from the phase shifts and $b_{\rm av}$ is the mean of $b_{\min}$ and $b_{\max}$. Entries are in units of 10^{-5}.

Charge distribution	$b_{\min}$	$b_{\rm ph}$	$b_{\rm av}$	$b_{\max}$	$c_{\rm ph}$	c_1	c_2	c_1+c_2
(a) Uniform $kR=8$	550	582±2	580	610	1.04±0.50	0.87	0.27	1.14
(b) Smoothed uniform $kc=7.58$ $kt=2.79$	571	603±1	603	635	0.97±0.07	1.03	···	···
(c) Three-parameter $kc=7.07$ $kt=3.30$ $w=1.20$	490	507±1	516	542	−0.31±0.09	−0.11	···	···

[a] B. Hahn, D. G. Ravenhall, and R. Hofstadter, Phys. Rev. **101**, 1131 (1956).
[b] Reference 4.
[c] Reference 20.

In these expressions, ρ is the charge density normalized to unity:

$$4\pi\int_0^\infty \rho r^2 dr = 1. \tag{2.35}$$

The reduction in (2.33) was accomplished with the help of

$$V(\lambda) = -4\pi\gamma\left\{\frac{1}{\lambda}\int_0^\lambda \rho r^2 dr + \int_\lambda^\infty \rho r dr\right\}. \tag{2.36}$$

It is clear on grounds of symmetry that the higher order contributions to S in the transverse plane will have the same form as (2.33). We may then include some of these higher order effects by modifying b_1 and c_1 to new values b and c. The eikonal function near the origin is then given by

$$S(r) = S(0) + \mathbf{k}'\cdot\mathbf{r} - \tfrac{1}{6}a\mathbf{k}'\cdot\mathbf{r}[3k'^2r^2 - 2(\mathbf{k}'-\mathbf{r})^2] - b(\mathbf{r}\times\mathbf{k}')^2 + c[(\mathbf{r}\times\mathbf{k}')^2]^2 + \cdots. \tag{2.37}$$

It will be shown later that the coefficient b introduced here is to be identified with the b' in (2.1). It will also be shown that the contribution to b of order γ^2 indicates that b is bounded as follows:

$$b_{\max} = \frac{\gamma}{4k'k}\left\langle\frac{1}{r^2}\right\rangle > b > \frac{\gamma}{4k'^2}\left\langle\frac{1}{r^2}\right\rangle = b_{\min}. \tag{2.38}$$

For the simpler charge distributions, b may be calculated quite accurately (correct to relative order γ/kR). Examples of b from phase shifts are compared with these bounds in Table I.

The next step of the calculation is to determine u_0 to the same order of approximation as S. To lowest order in a, b, and c, we find

$$\hat{n} = \nabla S/[E-V(r)] \cong \hat{k} - \frac{1}{k'}[a(\mathbf{r}\cdot\mathbf{k}') + 2b - 4c(\mathbf{k}'\times r)^2] \times [\mathbf{r}k'^2 - \mathbf{k}'(\mathbf{k}'\cdot\mathbf{r})], \tag{2.39}$$

and

$$\nabla\cdot\hat{n} = -[2ak'\mathbf{r}\cdot\mathbf{k}' + 4bk' - 16ck'(\mathbf{k}'\times\mathbf{r})^2]. \tag{2.40}$$

To the present order of approximation, Eq. (2.22) becomes

$$\nabla\cdot(\hat{n}N^2) = 0$$

or

$$\hat{n}\cdot\nabla N^2 = -N^2\nabla\cdot\hat{n}, \tag{2.41a}$$

which yields

$$N^2 = N_0^2[1 + a(\mathbf{k}'\cdot\mathbf{r})^2 + 4b\mathbf{k}'\cdot\mathbf{r} - 16c\mathbf{k}'\cdot\mathbf{r}(\mathbf{k}'\times\mathbf{r})^2], \tag{2.41b}$$

where N_0 is the normalization in the transverse plane through the origin. The latter quantity is easily determined from (2.23) if we know the area change. This may be determined from the conservation of classical angular momentum which may be written

$$kr_1^0 = k'r_1(1 - \tfrac{1}{2}ak'^2r^2)$$

for an incident particle of impact parameter r_1^0 passing through the nucleus at a distance r_1 from the origin; terms of higher order in γ have been neglected. The relation between areas is

$$r_1^0 dr_1^0 = (1/\beta)r_1 dr_1,$$

which yields

$$N_0 \cong \frac{1}{\sqrt{\beta}} = \frac{k'}{k}(1 - ak'^2r_1^2) = \frac{k'}{k}[1 - a(\mathbf{k}'\times\mathbf{r})^2]. \tag{2.42}$$

In preparation for writing our approximation, we also need

$$\boldsymbol{\sigma}\cdot\hat{n}v = [1 + a(\mathbf{k}'\cdot\mathbf{r})^2 + 2b(\mathbf{k}'\cdot\mathbf{r}) - 4c(\mathbf{k}'\cdot\mathbf{r})(\mathbf{k}'\times\mathbf{r})^2]v - k'[a(\mathbf{k}'\cdot\mathbf{r}) + 2b - 4c(\mathbf{k}'\times\mathbf{r})^2]\boldsymbol{\sigma}\cdot\mathbf{r}v. \tag{2.43}$$

Combining all these results, we find for our approximate wave function

$$\phi_k^{(+)} = (k'/k)\{1 - a(\mathbf{k}'\times\mathbf{r})^2 + a(\mathbf{k}'\cdot\mathbf{r})^2 + 3b(\mathbf{k}'\cdot\mathbf{r}) - 10c\mathbf{k}'\cdot\mathbf{r}(\mathbf{k}'\times\mathbf{r})^2 - [\tfrac{1}{2}a\mathbf{k}'\cdot\mathbf{r} + b - 2c(\mathbf{k}'\times\mathbf{r})^2]\boldsymbol{\sigma}\cdot\mathbf{r}k'\}ve^{iS}. \tag{2.44}$$

The (−)-type eigenfunction is obtained from the (+)-type one by reversing the signs of b and c; but not that of a.

For most of our purposes, Eq. (2.37) will be an adequate representation of $S(\mathbf{r})$. However, for completeness, let us consider the next order corrections; as remarked earlier, some of these corrections are included by properly defining b and c so that they give $S(\mathbf{r})$ correctly in the transverse plane through the origin. This amounts to replacing the boundary condition at infinity by one on this plane. In order to determine this new boundary condition, one must of course integrate (2.19) along the correct trajectories up to this plane. The coefficients b and c will be considered in further detail below. Now we write the remainder in (2.36) as δS and find

$$k'(\partial\delta S/\partial z)=-\tfrac{1}{2}a^2r_1^2z^2k'^6-2b^2r_1^2k'^4-8c^2r_1^6k'^8 -2abr_1^2zk'^5+4acr_1^4zk'^7+8bcr_1^4k'^6,$$

or

$$\delta S=-\tfrac{1}{6}a^2(\mathbf{k}'\times\mathbf{r})^2(\mathbf{k}'\cdot\mathbf{r})^3-2b^2(\mathbf{k}'\times\mathbf{r})^2(\mathbf{k}'\cdot\mathbf{r}) -8c^2[(\mathbf{k}'\times\mathbf{r})^2]^3(\mathbf{k}'\cdot\mathbf{r}) -ab(\mathbf{k}'\times\mathbf{r})^2(\mathbf{k}'\cdot\mathbf{r})^2+2ac[(\mathbf{k}'\times\mathbf{r})^2]^2(\mathbf{k}'\cdot\mathbf{r})^2 +8bc[(\mathbf{k}'\times\mathbf{r})^2]^2(\mathbf{k}'\cdot\mathbf{r}). \quad (2.45)$$

In order to obtain b and c to second order in γ we have to evaluate (2.29) for $z=0$. Using

$$\frac{\partial S_1(r_1,z)}{\partial r_1}=-r_1\int_{-z}^{\infty}\frac{1}{\lambda}\frac{\partial V((r_1^2+\lambda^2)^{1/2})}{\partial\lambda}d\lambda$$

and interchanging orders of integration, we find

$$S_2(r_1,0)=-\frac{r_1^2}{2k}\int_0^{\infty}\frac{d\lambda}{\lambda}\frac{\partial}{\partial\lambda}[V((\lambda^2+r_1^2)^{1/2})-V(r_1)]^2 =-\frac{r_1^2}{2k}\int_0^{\infty}\frac{d\lambda}{\lambda^2}[V((\lambda^2+r_1^2)^{1/2})-V(r_1)]^2. \quad (2.46)$$

Expanding this in powers of r_1^2, we find

$$b_2=\frac{1}{2kk'^2}\int_0^{\infty}\frac{d\lambda}{\lambda^2}[V(\lambda)-V(0)]^2>0, \quad (2.47a)$$

$$<\frac{|V(0)|}{2kk'^2}\int_0^{\infty}\frac{d\lambda}{\lambda^2}|V(0)-V(\lambda)| =\frac{|V(0)|}{k}|b_1|=\left(\frac{k'}{k}-1\right)b_1; \quad (2.47b)$$

$$c_2=-\frac{1}{2kk'^4}\int_0^{\infty}\frac{d\lambda}{\lambda^2}[V(\lambda)-V(0)]\left[\frac{1}{\lambda}\frac{\partial V(\lambda)}{\partial\lambda}-ak'^3\right],$$

$$=4\left(\frac{k'}{k}-1\right)c_1-\frac{1}{2kk'^4} \times\int_0^{\infty}\frac{d\lambda}{\lambda^2}V(\lambda)\left[\frac{1}{\lambda}\frac{\partial V(\lambda)}{\partial\lambda}-ak'^3\right]. \quad (2.48)$$

Equation (2.47) leads immediately to the bounds given in (2.38), which are correct to order γ^2.

As illustrations, here are two examples of b and c for simple charge distributions.

Uniformly charged sphere of radius R:

$$b=\frac{3}{4}\frac{\gamma}{(k'R)^2}\left[1+\frac{7}{9}\frac{\gamma}{k'R}\right],$$
$$c=\frac{3}{32}\frac{\gamma}{(k'R)^4}\left[1+\frac{22}{5}\frac{\gamma}{k'R}\right]. \quad (2.49)$$

Charged shell of radius R:

$$b=\frac{1}{4}\frac{\gamma}{(k'R)^2}\left[1+\frac{2}{3}\frac{\gamma}{k'R}\right],$$
$$c=-\frac{1}{32}\frac{\gamma}{(k'R)^4}\left[1+\frac{4}{4}\frac{\gamma}{k'R}\right]. \quad (2.50)$$

Connection Between the Eikonal Approximation and the Phase-Shift Analysis

At the beginning of this chapter, we indicated how we were led to the eikonal approximation by an empirical study of the scattering eigenfunctions computed by the phase-shift analysis. The main feature of the empirical analysis was the recognition that the phase shifts could be approximated rather well by an expansion of the form[16]

$$\eta_j=\eta_{-1/2}-b'(j+\tfrac{1}{2})^2+c'(j+\tfrac{1}{2})^4+\cdots. \quad (2.51)$$

We wish now to connect these empirical constants b' and c' to the properties of the eikonal approximation. The point of connection is provided by consideration of the deflection of a classical particle by the nucleus. The WKB approximation to the phase shifts, which incidentally agrees with the exact phase shifts to within 1.0% in the example cited in footnote 16, are related to the classical scattering angle by the well-known

[16] It must be emphasized that this expression represents the general trend of the "data" from the phase shift analysis. The actual points fluctuate slightly about the smooth curve given by (2.51). However, these fluctuations are very small compared with the over-all change in the magnitude of η_j. For example, for the parameters $\gamma=0.5765$, $kR=8$ (uniformly charged sphere), the over-all change in η_j for the first seven phase shifts is approximately 0.26 radians. Nevertheless, parameters can be found so that (2.51) fits the data to within 0.0005 radians, and accuracy of 0.2%. When viewed from the point of view of the WKB approximation for the partial waves, which should give a fairly smooth dependence on $(j+\frac{1}{2})$, these fluctuations may be thought of as a feature of the wave properties of the partial waves. The fluctuations probably arise from the more or less random phase of the radial wave function at the edge of the charge distribution; these functions would clearly be continuous in any usual WKB approximation.

expression[17]

$$\frac{\partial \eta_j{}^{\mathrm{WKB}}}{\partial j}=-\tfrac{1}{2}\alpha, \tag{2.52a}$$

where α is the scattering angle, reckoned positive for a purely attractive potential, for a classical particle whose angular momentum is $j+\frac{1}{2}$. Taking the derivative of (2.51), and using for $j+\frac{1}{2}$ the classical value at the transverse plane,

$$j+\tfrac{1}{2}=k'r_1(1-\tfrac{1}{2}ak'^2r_1^2), \tag{2.52b}$$

we get the phase-shift prediction for the deflection angle:

$$-\tfrac{1}{2}\alpha=-2b'k'r_1+(4c'+ab')(k'r_1)^3. \tag{2.53a}$$

On the other hand, the deflection angle may be calculated classically to the same order of approximation. Since the momentum is ∇S, the angle is

$$\begin{aligned}-\tfrac{1}{2}\alpha&=\frac{\partial S(r_1,0)/\partial r_1}{\partial S(r_1,z)/\partial z|_{z=0}}\\&=-(2bk'^2r_1+4ck'^4r_1^3)/k'(1-\tfrac{1}{2}ak'^2r_1^2)\\&\cong-2bk'r_1+(4c-ab)(k'r_1)^3.\end{aligned} \tag{2.53b}$$

Comparing the two expressions for α, we conclude that

$$\begin{aligned}b'&=b,\\c'&=c-\tfrac{1}{2}ab.\end{aligned} \tag{2.54}$$

Numerical examples are given in Table I. The extent to which the relationships (2.54) are obeyed is one measure of the applicability of the eikonal approximation.

The eikonal approximation may be compared with the phase-shift analysis also by examining the phase of the wave function at the origin. The eikonal approximation gives it as

$$S(0)=\lim_{L\to\infty}\left\{\int_{-L}^{0}[E-V(\lambda)]d\lambda-kL-\gamma\ln 2kL\right\}. \tag{2.55a}$$

In the phase-shift analysis it is the phase of the $l=0$ partial wave, i.e., the value of (2.51) for $j=\frac{1}{2}$. From (2.51), we thus obtain

$$\eta_{1/2}=\eta_{-1/2}-b=S(0)-b, \tag{2.55b}$$

where $\eta_{-1/2}$ is identified as $S(0)$ by using the WKB approximation for the phase shifts. Unfortunately, there is a clear disagreement between (2.55a) and (2.55b) of a term b. It can only be accounted for by going to the next order in the eikonal approximation. We obtain the next approximation from (2.10) and (2.14). Keeping at each step only terms of lowest order in γ, we get

$$\begin{aligned}\frac{d\lambda_1}{dS}&=\frac{i}{2u_0{}^\dagger u_0}u_0{}^\dagger\boldsymbol{\sigma}\cdot\nabla\frac{1}{E-V}\boldsymbol{\sigma}\cdot\nabla u_0\\&=\frac{+i}{2k}u_0{}^\dagger\nabla^2u_0=\frac{i}{4k}\nabla^2\left\{N\left[1+\frac{\partial S/\partial z}{E-V}\right]\right\},\end{aligned}$$

or

$$d\lambda_1/dz=(i/2k)\nabla^2N. \tag{2.56}$$

The formal integration of this equation gives

$$\lambda_1(r_1,0)=\frac{i}{2k}\frac{\partial N}{\partial z}\bigg|_{z=0}+\frac{i}{2k}\int_{-\infty}^{0}\left(\frac{\partial^2}{\partial r_1{}^2}+\frac{1}{r_1}\frac{\partial}{\partial r_1}\right)N(r_1,z)dz. \tag{2.57}$$

The differential equation for N, obtained from (2.15) and (2.41a), is

$$\frac{\partial N}{\partial z}=-\tfrac{1}{2}\nabla\cdot\left(\frac{\nabla S}{E-V}\right)\cong-\frac{1}{2k}\left(\frac{\partial^2}{\partial r_1{}^2}+\frac{1}{r_1}\frac{\partial}{\partial r_1}\right)S.$$

Integrating this and inserting the result in (2.57), one finds, after some elaborate manipulations,

$$\lambda_1(0,0)=-ib, \tag{2.58}$$

which combines with $S(0)$ to give the proper phase at the origin.

One may also ask whether u_1 gives any important spatial variations in the vicinity of the nucleus which should be taken into account in calculations [an overall phase factor represented by $\lambda_1(0,0)$ is actually immaterial]. The answer is that u_1 gives variations of order $\gamma/(kR)^2$, which should be unimportant in the region of application of the approximation. To see this, we note that $u_1{}^{(-)}$ is given by (2.10), which with (2.44) yields

$$u_1{}^{(-)}\cong+i(k'/k)(\tfrac{3}{4}ak'\boldsymbol{\sigma}\cdot\mathbf{r}-\tfrac{3}{4}a\mathbf{k}'\cdot\mathbf{r})v. \tag{2.59a}$$

Then (2.14) gives

$$\partial\lambda_1/\partial z\cong-\tfrac{3}{2}iak'.$$

Putting this together with (2.58), we find

$$u_1{}^{(+)}\cong i(k'/k)(-b-\tfrac{3}{2}a\mathbf{k}'\cdot\mathbf{r})v, \tag{2.59b}$$

so that

$$u_1\cong i(k'/k)[-b+\tfrac{3}{4}ak'\boldsymbol{\sigma}\cdot\mathbf{r}-(9/4)a\mathbf{k}'\cdot\mathbf{r}]v. \tag{2.59c}$$

Note that in this approximation

$$\boldsymbol{\sigma}\cdot\nabla u_1=0,$$

or

$$(\boldsymbol{\sigma}\cdot\nabla S-V+E)\boldsymbol{\sigma}\cdot\nabla u_1=0$$

which is the solvability condition mentioned after

[17] N. F. Mott and H. S. Massey, *The Theory of Atomic Collisions* (Oxford University Press, New York, 1949), 2nd ed., p. 124.

(2.10). We also see that the last term in (2.59c) could be absorbed in a further change in wave number inside the nucleus.

III. APPROXIMATE CALCULATION OF CROSS SECTIONS

Before proceeding, let us review briefly the nature of our approximate wave functions. To the lowest order in γ, our wave function depends primarily on four parameters: a, which describes the radial dependence of V and is proportional to the charge density at the origin (i.e., inversely proportional to R^3); b, which gives the curvature of the wave front and is proportional to $\langle r^{-2}\rangle$; c, which gives a correction to this curvature; and k', the local wave number which is proportional to $\langle r^{-1}\rangle$. Of these parameters only c is relatively sensitive to the details of the charge distribution; and in fact the same approximate wave function could apply to many different charges. Our approximation scheme is also unable to supply any direct information about outgoing spherical scattered waves which we know are sensitive to the charge. We therefore conclude that the approximate wave function depends on only general features of the charge distribution and in particular on the lower Fourier components of ρ. The approximation is also valid only in the vicinity of the nucleus but as will become evident later, this is the most important region for the purposes of calculation. The plan of this section is as follows. We first develop an asymptotic expansion for the amplitude of the cross section; the derivation is sufficiently general to apply to elastic or inelastic scattering of various multipole orders. The expression is then applied to various cases of interest.

Asymptotic Expansion of the Matrix Elements

Generally we shall be concerned with integrals of the form

$$h=\int e^{i(\mathbf{q}'\cdot\mathbf{r}+\phi_1+\phi_2)}U(r)F(\mathbf{r})d^3r. \tag{3.1}$$

The origin of this type of integral is easy to see. The factor $\exp(i\mathbf{q}'\cdot\mathbf{r})$ comes from the plane-wave parts of the incident and final wave functions, using the modified wave number. The other parts of the phase, ϕ_1 and ϕ_2, arise from the more complicated terms in S, and they are separated into two contributions depending on whether they are odd (ϕ_1) or even (ϕ_2) in $\mathbf{q}'\cdot\mathbf{r}$. The remaining complicated parts of the wave function, including normalization, spinor dependence, etc., are contained in F together with any angular dependence coming from the interaction potential as, for example, in quadrupole excitation.

We are interested particularly in the region of large q' in which case the most rapid variation of the integrand is associated with the exponential factor. This suggests that we try to develop an asymptotic expansion based on this rapid variation. To do this we first introduce polar coordinates

$$\hat{z}\|\mathbf{q}, \quad \hat{x}\perp\mathbf{q} \quad \text{in plane of } \mathbf{k}_i, \mathbf{k}_f.$$

Then

$$F(\mathbf{r})=F(r,\mu,\varphi),$$

where

$$\mu=\cos\theta=\hat{q}'\cdot\hat{r}.$$

Now use the identity

$$e^{i(q'r\mu+\phi_1+\phi_2)}=\frac{1}{i(q'r+\phi_1'+\phi_2')}\frac{\partial}{\partial\mu}e^{i(q'r\mu+\phi_1+\phi_2)} \tag{3.2}$$

with $\phi_i'=\partial\phi_i/\partial\mu$, and integrate (3.1) by parts with respect to μ:

$$\int_{-1}^{1}F(r,\mu,\phi)e^{i(q'r\mu+\phi_1+\phi_2)}d\mu$$

$$=\sum_{\pm}\frac{\pm F(r,\pm1,\varphi)e^{i\Phi_2\pm i(q'r+\Phi_1)}}{i(q'r+\Phi_1'\pm\Phi_2')}-\int_{-1}^{1}e^{i(q'r\mu+\phi_1+\phi_2)}\times\frac{\partial}{\partial\mu}\left\{\frac{F(r,\mu,\phi)}{i(q'r+\phi_1'+\phi_2')}\right\}d\mu, \tag{3.3}$$

where

$$\Phi_i=\phi_i|_{\mu=1}, \quad \Phi_i'=\partial\phi_i/\partial\mu|_{\mu=1}.$$

The integral left over on the right-hand side is of the same form as the original integral and it may be integrated by parts in the same manner as the original integral. In this way an asymptotic series in $(q'r+\Phi_1'+\Phi_2')^{-1}$ is developed. The terms in this new integrand arising from angular derivatives of the wave function are easily seen to be of order $\gamma/(k'R)^2$ relative to the original integrand. We shall now argue that these terms may safely be neglected to the accuracy we desire; other terms associated with angular derivatives of the potential may not be neglected. The integrated terms in (3.3) may be rearranged to the form

$$e^{i\Phi_2}[A_s\sin(q'r+\Phi_1)+iA_c\cos(q'r+\Phi_1)],$$

where A_s and A_c are real and are respectively of order $(1/k'r)$ and $(\gamma/(k'R)^2)$. The terms we wish to ignore then give corrections to these which are imaginary and of order $\gamma/(k'R)^3$. The imaginary correction to A_s is roughly equivalent to a change in phase of the argument of the cosine by $(1/k'R)$, while the imaginary correction to A_c corresponds to a change in phase of the argument of the sine by $\gamma/(k'R)^2$. The latter change is completely negligible. It is easy to see that the effective phase change of the cosine term may also be neglected. Because the cosine term has a small coefficient, it is important only near the zeros of the sine term, where $|\sin(q'r+\Phi_1)|\lesssim\gamma/k'R$. In this region, the error in neglecting the imaginary part of A_s is of relative order

$\gamma/(k'R)^2$. Since $\gamma<1$, and we are interested in large $k'R(k'R>5)$, the error is only a few percent.

Next, we note that in all cases we shall consider

$$\begin{aligned} F(-r,-1,\varphi)&=F(r,1,\varphi),\\ \Phi_1(-r)&=-\Phi_1(r),\\ \Phi_2(-r)&=\Phi_2(r). \end{aligned} \tag{3.4}$$

Hence, if we define the potential for negative values of r by

$$U(-r)\equiv U(r), \tag{3.5}$$

for a spherically symmetric potential, we may neglect the integral on the right side of (3.3), and (3.1) may be written

$$h\cong\frac{2\pi}{iq'}\int_\infty^\infty rdr\,G(r)U(r)e^{i\chi}, \tag{3.6a}$$

where

$$G(r)\equiv\frac{1}{2\pi}\int_0^{2\pi}\frac{F(r,1,\varphi)}{1+(\Phi_1'+\Phi_2')/q'r}d\varphi; \tag{3.6b}$$

$$\chi=q'r+\Phi_1+\Phi_2.$$

This is the desired result: the three-dimensional integral (3.1) has been reduced to a one-dimensional integral with a relative error of order $(\gamma/(k'R)^2)$. The generalization of (3.6) for an asymmetric potential will be illustrated by quadrupole scattering later.

Elastic Scattering

Although we discuss elastic scattering first, it is not the simplest process to formulate in terms of the eikonal approximation. The exact expression for the elastic scattering amplitude is

$$f=\frac{k}{2\pi}\int\chi_{\mathbf{k}_f}{}^\dagger V(r)\psi_{\mathbf{k}_i}{}^{(+)}d^3r, \tag{3.7}$$

where χ is a plane wave and $\psi^{(+)}$ an exact scattering wave function. We desire instead to evaluate f by using our approximate wave functions. While it would appear natural to replace $\psi^{(+)}$ by its eikonal approximation $\phi^{(+)}$ the result would hardly be better than the Born approximation. The trouble is that $\psi^{(+)}$ contains an outgoing spherical wave and $\phi^{(+)}$ does not. Because of the long-range nature of V, this outgoing spherical wave can make a very large contribution to the integral. The distortions contained in $\phi^{(+)}$ correspond in the Born expansion to the succession of small momentum transfers experienced by the *incoming* electron. The contribution mentioned above is the origin, in (3.7) of the corresponding momentum transfers of the *outgoing* electron, and these would be lost with the above replacement. The eikonal wave function $\phi^{(+)}$ does not contain the high Fourier component (i.e., large-angle scattered waves) which produce it. As was seen in the last section, the classical momentum change of the incoming particle is of order γR^{-1}, whereas we wish to consider angles for which the momentum change $q=2k\sin\frac{1}{2}\theta\gg R^{-1}$.

The intuitive picture we have of the process is the following. The incoming wave function is gradually distorted on its way into the nuclear region, but it does not suffer large momentum transfers in this process. While in the vicinity of the nucleus it suffers one large momentum transfer. On the way out further small momentum transfers take place. A similar qualitative picture was obtained by Schiff[7] by summing up the leading terms in the Born expansion. This picture suggests that the proper way to alter (3.7) is to replace *both* $\chi^\dagger$ and $\psi^{(+)}$ by eikonal wave functions

$$f\cong f_1\equiv\frac{k}{2\pi}\int\phi_{\mathbf{k}_f}{}^{(-)\dagger}V(r)\phi_{\mathbf{k}_i}{}^{(+)}d^3r. \tag{3.8}$$

This is the approximation which will be used.

While it may not be possible to give a rigorous mathematical derivation of this expression, we can give a pseudoderivation which helps bring out the physical basis for the approximation and gives an indication of when it should be valid. To obtain our approximation, we assume that there is a comparison potential $\tilde{V}(r)$ which has the following properties: (i) $\tilde{V}(r)$ yields nearly the same eikonal approximations as $V(r)$ near the origin. (ii) $\tilde{V}(r)$ produces very little scattering at large angles compared to that of $V(r)$. The second condition means $\tilde{V}$ is missing high Fourier components that V contains. It is a smoothed out potential corresponding to a charge distribution of approximately the same radius as that giving V. Now let $\tilde{\psi}^{(\pm)}$ be the exact scattering solutions with $\tilde{V}$. Then, the complete scattering amplitude is given by[18]

$$\begin{aligned} f=(k/2\pi)\Big\{&\int\chi_{\mathbf{k}_f}{}^\dagger\tilde{V}\tilde{\psi}_{\mathbf{k}_i}{}^{(+)}d^3r\\ &+\int\tilde{\psi}_{\mathbf{k}_f}{}^{(-)\dagger}(V-\tilde{V})\psi_{\mathbf{k}_i}{}^{(+)}d^3r\Big\}. \end{aligned}$$

For large angles, the first term is negligible. The second term can be simplified by a series of approximations; (i) $V-\tilde{V}$ is small outside the vicinity of the nucleus (note that this argument may fail for non-Coulomb scattering); hence we can approximate both $\tilde{\psi}^{(-)}$ and $\psi^{(+)}$ by the corresponding eikonal wave functions $\phi^{(\pm)}$; (ii) now the $\tilde{V}$ part of the integral is small because it has only soft Fourier components. (With the exact functions, $\tilde{V}$ is not negligible since $\psi^{(+)}$ contains an outgoing spherical wave.) Equation (3.8) follows immediately and the $\tilde{V}$ dependence drops out. We note however that the result depends on the existence of such a $\tilde{V}$. Clearly it is not always possible to find such a com-

[18] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

parison potential. For example, for a Gaussian charge distribution the cross section drops very rapidly with increasing angle. It would be effectively impossible to find a charge distribution giving a more rapid drop yet maintaining a, b, and k'. In this case, double scatterings in the vicinity of the nucleus would presumably be important.

Our discussion indicates that the eikonal appropriate to $\tilde{V}$ rather than V is the one to be used. The significance of this remark is that if S has a higher order discontinuity due to one in V, this discontinuity should be smoothed out in (3.8). In practice this will be done by using the first terms of the power series derived in Sec. II, rather than the exact S. Presumably one could improve our calculational method concerning this point, but probably at the expense of great additional labor. As the present work is exploratory and aimed at obtaining qualitative understanding of elastic scattering rather than accuracy, further refinements will not be attempted. The uniform charge distribution seems to be a case where the foregoing reasoning is most reliable, and the specific results reported later consider only this example.

The preceding remarks are less likely to apply to the details of the wave function associated with the parameter c, which depends more sensitively on the details of the charge distribution. Nevertheless, the c dependence may indicate the accuracy of the method. For this reason, results will be presented both with and without the c terms.

Equation (3.8) is in the form of (3.1) with

$$U(r)=V(r)$$

and

$$(k'/k)^2 v_f^\dagger F v_i \equiv u_0^\dagger(\mathbf{k}_f) u_0(\mathbf{k}_i). \tag{3.9}$$

Instead of proceeding directly it is more convenient to reduce the integral to one involving the charge distribution. The connection between potential and charge density is given in (2.36), which is valid even for negative r if we define

$$\rho(-r)\equiv\rho(r). \tag{3.10}$$

After inserting (2.36) into (3.6a) and interchanging the orders of integration, we find

$$f_1=-\frac{\gamma k}{iq'}e^{2iS(0)}\left(\frac{k'}{k}\right)^2 v_f^\dagger\left\{4\pi\int_{-\infty}^{\infty}r'dr'\rho(r')\times\left[r'\int_{r'}^{\infty}e^{-\lambda|r|}e^{ix}G(r)dr+\int_{-\infty}^{r'}e^{-\lambda|r|}e^{ix}G(r)rdr\right]-\int_{-\infty}^{\infty}e^{-\lambda|r|}e^{ix}G(r)dr\right\}v_i, \tag{3.11}$$

where the Coulomb potential has been screened to make the integral well defined. The contribution of the last term in (3.11) is difficult to calculate and we want simply to ignore it. Some justification for doing this can be made. Consider what happens if we expand it in an asymptotic series in $(q'R)^{-1}$ by repeated partial integrations. Except for discontinuities in the integrands, each integrated term will vanish at the limits of integration. If V has an nth order discontinuity, we see from (2.6) that S and hence G will have an $(n+1)$th order discontinuity; it may also have an nth-order discontinuity, but that will be associated with N and $\boldsymbol{\sigma}\cdot\mathbf{n}$ and will have an extra factor of $(1/k'R)$. Previously we argued that S corresponding to the reference potential $\tilde{V}$ should be used in (3.6). S and $\tilde{S}$ are approximately the same for small r, but $\tilde{S}$ should be smoother near discontinuities in derivatives of V. In any case, it appears that the contribution due to the discontinuity in derivatives of G will be at least of order $(\gamma/k'R)$ relative to those arising from discontinuities in V. As it will turn out, we shall evaluate some easily calculated terms of the same order and obtain some quite good results. The fact that they are not better may perhaps be attributed to the neglect of this term.

Having neglected the last term of (3.11), we now make an asymptotic expansion of the r integration by repeated partial integrations. Terms of relative order $\gamma/(kR)^2$ are neglected by the same argument as that given after Eq. (3.3). The result is

$$f_1\cong+\frac{ik\gamma}{(q')^3}e^{2iS(0)}\left(\frac{k'}{k}\right)^2 v_f^\dagger 4\pi\int_{-\infty}^{\infty}\frac{\rho(r)G(r)}{C(r)^2}e^{ix}rdrv_i, \tag{3.12}$$

where

$$C(r)=1+\frac{1}{q'}\frac{d}{dr}(\Phi_1+\Phi_2). \tag{3.13}$$

This is the desired result. In order to apply it, it will be convenient to use the approximation (2.44) for $\phi^{(\pm)}$ given in the preceding section. With this approximation, we find

$$\begin{aligned}(k'/k)^2 v_f^\dagger F v_i=(k'/k)^2 v_f^\dagger\\ \times\{1+3b\mathbf{q}'\cdot\mathbf{r}+2a[(\mathbf{r}\cdot\mathbf{k}_i')^2+(\mathbf{r}\cdot\mathbf{k}_f')^2-r^2k'^2]\\ -10c[\mathbf{k}_i'\cdot\mathbf{r}(\mathbf{k}_i'\times\mathbf{r})^2-\mathbf{k}_f'\cdot\mathbf{r}(\mathbf{k}_f'\times\mathbf{r})^2]-[\tfrac{1}{2}a\mathbf{K}'\cdot\mathbf{r}\\ -2c(\mathbf{k}_i'\times\mathbf{r})^2+2c(\mathbf{k}_f'\times\mathbf{r})^2]\boldsymbol{\sigma}\cdot\mathbf{r}k'\}v_i,\end{aligned} \tag{3.14a}$$

where

$$\mathbf{K}'=\mathbf{k}_i'+\mathbf{k}_f'.$$

This is to be evaluated for $\mu=1$, which is the same as taking $\mathbf{r}$ parallel to $\mathbf{q}'(\mathbf{r}=\hat{q}r)$; it then simplifies to

$$\begin{aligned}F(r,1,\varphi)=[1+3bq'r\\ +a(q'^2-2k'^2)r^2-\tfrac{5}{2}cq'K'^2r^3]\end{aligned} \tag{3.14b}$$

which is independent of φ.

Before evaluating the various other quantities occurring in G and χ, it will be convenient to estimate orders of magnitude so that we do not unnecessarily carry along contributions smaller than others which have been neglected. So far, we have neglected terms of relative

order $\gamma/(k'R)^2$ in the amplitude. The meaning of this should be made precise. At most angles, the leading term of the amplitude will turn out to be real (with $e^{i\phi_2+2iS(0)}$ factored out). Its correction is complex and of relative order $\gamma/k'R$; terms of relative order $\gamma/(k'R)^2$ are neglected. At certain angles the leading term passes through zero. In this region the imaginary terms originally of relative order $\gamma/k'R$ become dominant and fill in the zeros in the cross section; but, according to the discussion after Eq. (3.3), the error of the terms neglected is still only $\gamma/(k'R)^2$ compared to the terms which actually contribute at these minima. Now we want to make a further simplification which will lead ultimately to errors of order $\gamma/k'R$ at these minima. The simplification will consist simply of neglecting all the contributions which arise from δS_2, given in (2.45). The point is that the odd powers of $\mathbf{r}$ in δS_2 always go along with $\mathbf{q}'\cdot\mathbf{r}$ and hence are of relative order $\gamma^2/(k'R)^2$ for all scattering angles. On the other hand, the even powers of $\mathbf{r}$ are to be compared with the b and c terms in (2.36). At the angles where the real part of the cross section becomes small, it is just the b and c contributions which give rise to the imaginary terms which fill in the zeros in the cross section. To be consistent at these points, the ab and ac terms of (2.45), which are of relative order $(\gamma/k'R)$, should really be retained. It may be noted that the error introduced by this simplification is comparable to that arising from the uncertainty in b indicated by (2.38) With these remarks, we may simply state the results.

$$\chi(r)=q'r-a[\tfrac{1}{2}q'k'^2-(1/12)q'^3]r^3 -\tfrac{1}{2}bK'^2r^2+\tfrac{1}{8}cK'^4r^4 =q'r+\Phi_1+\Phi_2, \tag{3.15}$$

$$C(r)=1-a(\tfrac{3}{2}k'^2-\tfrac{1}{4}q'^2)r'^2 -b(K'^2r/q')+\tfrac{1}{2}c(K'^4r^3/q'), \tag{3.16}$$

$$\begin{aligned}1+(\phi_1'+\phi_2')/q'r &=1-a[\tfrac{1}{2}k'^2-\tfrac{1}{4}q'^2+\tfrac{1}{2}K'^2\cos^2\varphi]r^2 \\ &\quad+b[q'^2-K'^2\cos^2\varphi]r/q' \\ &\quad+c[-\tfrac{1}{2}q'^2+(2k'^2-\tfrac{3}{2}q'^2)\cos^2\varphi]K'^2r^3/q' \\ &=1-a[\tfrac{3}{2}k'^2-\tfrac{1}{2}q'^2]r^2 \\ &\quad+b[-2k'^2+\tfrac{3}{2}q'^2]\frac{r}{q'}+c[k'^2-(5/4)q'^2]K'^2r^3/q',\end{aligned} \tag{3.17}$$

where the second form results from doing the angular integration in (3.6b) and approximating as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{1+u+v\cos^2\varphi} =\frac{1}{[(1+u)(1+u+v)]^{1/2}}\cong\frac{1}{1+u+\frac{1}{2}v}.$$

This is as much as may be conveniently worked out by general analysis.

Now let us take as an example the simple one of a uniformly charged sphere where the charge density is given by

$$\rho(r)=\frac{3}{4\pi R^3},\quad r<R, \qquad =0,\ r>R \tag{3.18}$$

With this, (3.12) reduces to

$$f_1\cong\frac{3i\gamma k}{q^2}\frac{e^{2iS(0)}}{(q'R^3)}v_f^\dagger v_i\int_{-R}^{R}e^{ix}\frac{G(r)}{[C(r)]^2}rdr. \tag{3.19}$$

This may be expanded by partial integration:

$$f_1=\frac{\gamma k}{q^2}e^{2iS(0)}\frac{3}{(q'^2R^3)}\left\{e^{ix}\left[\frac{rG}{C^3}-\frac{G}{iq'C^4}\right]\right\}_{-R}^{R}v_f^\dagger v_i. \tag{3.20}$$

Let us check this result with the Born approximation, i.e., by neglecting all except the leading terms in γ. In this case $F=G=a$ constant and

$$f_1=(2\gamma k/q^2)F_0(q'R)v_f^\dagger v_i, \tag{3.21}$$

where

$$F_0=\frac{3}{(q'R)^2}\left[\frac{\sin q'R}{q'R}-\cos q'R\right],$$

which is the correct result, with a modified wavenumber.

Going back to the general case, we see that the general result may be expressed by the substitution

$$F_0\to\mathfrak{F}_1, \tag{3.22a}$$

where

$$\mathfrak{F}_1=-[3e^{i\Phi_2}/(q'R)^2]\{(\mu_1+i\mu_2)\cos(q'R+\Phi_1) -(1/q'R)(\nu_1+i\nu_2)\sin(q'R+\Phi_1)\}, \tag{3.22b}$$

and

$$\mu_1=\frac{1}{2}\left\{\frac{G(R)}{[C(R)]^3}+\frac{G(-R)}{[C(-R)]^3}\right\},$$

$$\mu_2=\frac{1}{2}\frac{1}{q'R}\left\{\frac{G(R)}{[C(R)]^4}-\frac{G(-R)}{[C(-R)]^4}\right\},$$

$$\nu_1=\frac{1}{2}\left\{\frac{G(R)}{[C(R)]^4}+\frac{G(-R)}{[C(-R)]^4}\right\},$$

$$\nu_2=\frac{-q'R}{2}\left\{\frac{G(R)}{[C(R)]^3}-\frac{G(-R)}{[C(-R)]^3}\right\}.$$

The orders of magnitude of these coefficients are

$$\mu_1=1+O(\gamma/k'R),\quad \mu_2=O(\gamma/(k'R)^2),$$
$$\nu_1=1+O(\gamma/k'R),\quad \nu_2=O(\gamma).$$

For consistency with previous approximations, μ_2 is replaced by zero in the calculations. The phases Φ_1 and Φ_2, (3.15), are of course evaluated at $r=R$:

$$\Phi_1=-\tfrac{1}{2}aq'(k'^2-\tfrac{1}{6}q'^2)R^3,$$
$$\Phi_2=-\tfrac{1}{2}bK'^2R^2+\tfrac{1}{8}cK'^4R^4. \quad (3.22c)$$

Let us observe some of the interesting features of this result. In contrasting it to the Born approximation, we note first the q'-dependent change in phase Φ_1 of the argument of the sine and cosine. This apparently arises from the variation of V with r and expresses the fact that the average wave number is slightly smaller than its magnitude at the origin. We also see that the real coefficients of the sine and cosine are modified by a relative correction of order $\gamma/k'R$. In electron-nucleus scattering the coefficients are enhanced and in positron-nucleus scattering they are reduced. In the Born approximation, the cosine term dominates for large $q'R$ except near the zeros of the cosine, and the effect of the sine term is merely to shift the positions of the zeros. Now we see that the zeros are filled in somewhat because the sine term has a complex coefficient linear in b and c. This filling in of the zeros may be very important for heavy nuclei. Typical results will be presented and discussed in Sec. IV.

Inelastic Scattering

The matrix element for electron scattering with nuclear excitation, calculated to lowest order in the nuclear transition matrix element, is of the form

$$M_{fi}=e\int\psi_{\mathbf{k}_f}{}^{(-)\dagger}(\mathbf{r})\gamma_\mu A_{fi}{}^\mu(\mathbf{r})\psi_{\mathbf{k}_i}{}^{(+)}(\mathbf{r})d^3r, \quad (3.23)$$

where $A_{fi}{}^\mu$ is the potential associated with the nuclear transition from state i to state f. The initial electron wave function $\psi_{\mathbf{k}_i}{}^{(+)}$ is distorted in the Coulomb potential due to the initial nuclear state, while $\psi_{\mathbf{k}_f}{}^{(-)}$ is distorted in the Coulomb potential from the final nuclear state. Although the eikonal wave function should be useful in the general situation, we shall restrict our attention here to electric multipole transitions in situations where the excitation energy is small relative to the momentum transfer. Then only the transition Coulomb potential is important, and (3.23) reduces to

$$M_{fi}=\int\psi_{\mathbf{k}_f}{}^{(-)\dagger}(\mathbf{r})V_{fi}(\mathbf{r})\psi_{\mathbf{k}_i}{}^{(+)}(\mathbf{r})d^3r. \quad (3.24)$$

Further, V_{fi} may be expanded in a number of multipoles depending on the nuclear states involved.

If the electron wave functions in (3.24) are approximated by eikonal wave functions, an integral of the form (3.1) results immediately. Since the spherically symmetric part of the final Coulomb potential is not greatly different from the incident potential, the same parameters k', a, b, c, may be used for both wave functions in (3.24).

The method of asymptotic expansion we have developed may then be applied immediately to this integral to reduce it to a one-dimensional one, as in the elastic scattering case. If the transition potential is not spherically symmetric, the integral in (3.3) may not be neglected. This situation will be discussed in detail for the case of quadrupole excitation, which may be taken as typical of higher order multipole excitations.

Monopole Excitations

In this case

$$V_{fi}(\mathbf{r})\rightarrow V_{\mathrm{mon}}(r)$$

and (3.6) is an adequate approximation to (3.24), with F given by (3.14). Monopole excitation is a particularly important example, because it permits a numerical check of our approximation with results obtained by a phase shift analysis. Consider two neighboring elastic potentials differing by an amount proportional to the monopole potential

$$V_\lambda=V+\lambda V_{\mathrm{mon}}. \quad (3.25)$$

Then the scattering amplitudes are related by

$$f_\lambda=f+\lambda\frac{k}{2\pi}\int\psi_{\mathbf{k}_f}{}^{(-)\dagger}V_{\mathrm{mon}}\psi_{\mathbf{k}_i}{}^{(+)}(\lambda)d^3r. \quad (3.26)$$

Letting λ tend toward zero, we find

$$f_{\mathrm{mon}}=\frac{k}{2\pi}\int\psi_{\mathbf{k}_f}{}^{(-)\dagger}V_{\mathrm{mon}}\psi_{\mathbf{k}_i}{}^{(+)}d^3r=\frac{\partial f_\lambda}{\partial\lambda}=\frac{f_\lambda-f_{-\lambda}}{2\lambda}. \quad (3.27)$$

Thus the integral associated with the excitation may be estimated by taking the differences between scattering amplitudes associated with slightly different charge distributions.

A particularly simple example to study is the one of radial oscillations of a uniformly charged sphere. This is given by

$$V_{\mathrm{mon}}=\Delta R\frac{\partial V}{\partial R}=\Delta R\frac{3}{2}\frac{\gamma}{R^2}\left(1-\frac{r^2}{R^2}\right) \quad r<R$$
$$=0 \quad r>R. \quad (3.28)$$

In this case

$$f_{\mathrm{mon}}=\Delta R\partial f/\partial R, \quad (3.29)$$

where f is the elastic scattering amplitude calculated for a sphere of radius R, and ΔR represents the amplitude of radial oscillation.[19]

[19] This method can be applied to any potential-scattering problem, e.g., optical-model calculations of monopole excitations caused by protons on nuclei. It can be made flexible enough to accommodate any transition density by including the appropriate functional dependence in the elastic-scattering density and by varying its parameter.

The monopole excitation amplitude may now be calculated approximately by replacing $\psi^{(\pm)}$ by $\phi^{(\pm)}$ in (3.27). This gives

$$f_{\text{mon}}\cong\frac{k}{2\pi}\Delta R\int\phi_{\mathbf{k}_f}{}^{(-)\dagger}\frac{\partial V}{\partial R}\phi_{\mathbf{k}_i}{}^{(+)}d^3r=\Delta R\frac{\partial f_1}{\partial R}\bigg|_{a,b,c,k',S(0)},$$

where the R dependence entering f_1 through the a, b, c, k', and $S(0)$ in $\phi^{(\pm)}$ is held fixed. This apparently differs from (3.29); we shall return to this point later. Approximating to the same order as before, we find

$$\sigma_{\text{mon}}=(k\Delta R)^2(2\gamma k/q^2)^2\cos^2\tfrac{1}{2}\theta|F_{\text{mon}}|^2, \quad (3.30a)$$

where

$$F_{\text{mon}}=\frac{1}{k}\frac{\partial\mathfrak{F}_1}{\partial R}\bigg|_{a,b,c,k',S(0)}$$
$$=-\frac{3e^{i\phi_2}}{kRq'R}\left\{\left[-\mu_1C_1+\frac{3\nu_1}{(q'R)^2}\right]\sin(q'R+\Phi_1)\right.$$
$$-\frac{1}{q'R}[(\nu_1C_1+2\mu_1)+i(\nu_2C_1-\mu_1C_2q'R)]$$
$$\left.\times\cos(q'R+\Phi_1)\right\}, \quad (3.30b)$$

and

$$C_1=1+\frac{1}{q'}\frac{d\Phi_1}{dR}=1-\tfrac{3}{2}a(k'^2-\tfrac{1}{6}q'^2)R^2,$$

$$C_2=\frac{1}{q'}\frac{d\Phi_2}{dR}=-b(K'^2/q')R+\tfrac{1}{2}c(K'^4/q')R^3.$$

Results are presented in Sec. IV, where they are compared with those obtained from the phase-shift calculation.

Now we return to the remark made in the preceding paragraph which may be restated

$$\frac{\partial f_1}{\partial R}\neq\frac{\partial f_1}{\partial R}\bigg|_{a,b,c,k',S(0)}.$$

This indicates that our approximations for elastic scattering and monopole excitation are not quite consistent with each other since they do not satisfy (3.29). We recall that (3.8) was obtained by intuitive arguments rather than by any systematic procedure. On the other hand (3.30) represents the first stage of the expression of $\psi^{(\pm)}$ in the eikonal series. Therefore we believe that (3.30) is a more consistent approximation to monopole excitation than (3.8) is to elastic scattering. We therefore seek a new approximation, f_2, to elastic scattering. This should satisfy

$$\frac{\partial f_2}{\partial R}=\frac{\partial f_1}{\partial R}\bigg|_{a,b,c,k',S(0)}.$$

This is easily solved for f_2; the result amounts to replacing $\mathfrak{F}_1$ by $\mathfrak{F}_2$ where

$$\mathfrak{F}_2=-[3e^{i\Phi_2}/(q'R)^2]\left\{\mu_1C_1(k'/k)\cos(q'R+\Phi_1)\right.$$
$$-\frac{1}{q'R}\left[\nu_1C_1+i\left(\nu_2C_1-\mu_1C_2q'R-2\gamma\mu_1C_1\frac{k'}{k}\right)\right]$$
$$\left.\times\frac{k'}{k}\sin(q'R+\Phi_1)\right\}. \quad (3.31)$$

In arriving at this result, we have made use of the following:

$$\partial\Phi_1/\partial R\cong0, \quad \partial\Phi_2/\partial R\cong0 \quad \text{(to order } \gamma),$$
$$\partial q'R/\partial R=q, \quad \partial S(0)/\partial R=-\gamma/R.$$

Consistent with the previous approximation, terms of the following order have been neglected (but not consistently):

(i) $\gamma/(kR)^4$ in the real part of $\mathfrak{F}_2$;

(ii) $\gamma/(kR)^4$ in the imaginary part of the coefficient of $\cos(q'R+\Phi_1)$. In particular, this permits us to neglect derivatives of k', μ_1, ν_1, ν_2, C_1, C_2 with respect to R.

Quadrupole Excitation

In this case the potential in (3.24) takes the form

$$V_{fi}=Y_{2m}(\hat{r})V_2(r). \quad (3.32a)$$

If nuclear alignment effects are not to be observed, the cross section is to be calculated for each m separately, and the final result is obtained by averaging over m. This leads to great simplifications because the result is independent of the axis used to define m, and it is clearly a great convenience to take this axis along $\hat{q}$. With this choice, only the $m=0$ contribution is of the required order of magnitude, because of the symmetries of the φ integration. Let us make the angular dependence associated with Y_{20} explicit by using the replacement

$$F\rightarrow F\cdot(\tfrac{3}{2}\mu^2-\tfrac{1}{2})(5/4\pi)^{1/2} \quad (3.32b)$$

in (3.3). Then in the partial integrations, the derivatives of F may still be neglected, but those of Y_{2m} may not. One again finds a result of the form (3.6a), but with G replaced by

$$G_Q(r)=\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{1}{1+(\Phi_1'+\Phi_2')/q'r}\right.$$
$$+\frac{3i}{q'r}\frac{1}{[1+(\Phi_1'+\Phi_2')/q'r]^2}-\frac{3}{(q'r)^2}$$
$$\left.\times\frac{1}{[1+(\Phi_1'+\Phi_2')/q'r]^3}\right\}F(r,1,\varphi)d\varphi \quad (3.33a)$$
$$\equiv G_0(r)+\frac{3iG_1(r)}{q'r}-\frac{3G_2(r)}{(q'r)^2}. \quad (3.33b)$$

The quadrupole excitation cross-section is now

$$\sigma_Q=(1/4\pi)|f_Q|^2\cos^2\tfrac{1}{2}\theta, \tag{3.34a}$$

where

$$f_Q=\frac{k}{iq'}e^{2iS(0)}\left(\frac{k'}{k}\right)^2\int_{-\infty}^{\infty}G_Q(r)V_2(r)e^{ix_Q r}dr. \tag{3.34b}$$

A factor $\frac{1}{5}$ has entered (3.34a) because of the averaging over m; f_Q represents the one nonvanishing term in this average.

Next we calculate f_Q asymptotically for the special case of a δ-function quadrupole-transition charge density at radius r_0

$$V_2(r)=\tfrac{1}{5}\rho_2(r_0)\{(r^2/r_0^3)\theta(r_0-r)+(r_0^2/r^3)\theta(r-r_0)\}, \tag{3.35}$$

where θ is the step function,

$$\theta(\lambda)=1 \quad \lambda>0$$
$$=0 \quad \lambda<0.$$

The general result could be recovered by integrating this with respect to r_0. Neglecting small contributions in the same manner as previously, we find

$$f_Q=\rho_2(r_0)(2k/q^2)e^{2iS(0)}\mathfrak{F}_Q, \tag{3.36a}$$

where

$$\mathfrak{F}_Q=\frac{1}{2q'r_0}\left\{\frac{e^{ix}}{iC^2}\left[G_0+\frac{3iG_1}{q'r}-\frac{3G_2}{(q'r)^2}\right]\right\}\Bigg|_{-r_0}^{r_0}$$
$$=\frac{e^{i\Phi_2}}{q'r_0}\left\{\tilde{\mu}_1\sin(q'r_0+\Phi_1)+\frac{1}{q'r_0}(\tilde{\nu}_1+i\tilde{\nu}_2)\cos(q'r_0+\Phi_1)\right\} \tag{3.36b}$$

and

$$\tilde{\mu}_1=\frac{1}{2}\left\{\frac{G_0(r_0)-[3G_2(r_0)/(q'r_0)^2]}{[C(r_0)]^2}+\frac{G(-r_0)-[3G_2(-r_0)/(q'r_0)^2]}{[C(-r_0)]^2}\right\},$$

$$\tilde{\nu}_1=\frac{1}{2}\left\{\frac{3G_1(r_0)}{[C(r_0)]^2}+\frac{3G_1(-r_0)}{[C(-r_0)]^2}\right\},$$

$$\tilde{\nu}_2=-\frac{q'r_0}{2}\left\{\frac{G_0(r_0)-[3G_2(r_0)/(q'r_0)^2]}{[C(r_0)]^2}-\frac{G(-r_0)-[3G_2(-r_0)/(q'r_0)^2]}{[C(r_0)]^2}\right\}.$$

IV. NUMERICAL RESULTS AND DISCUSSION

We now wish to compare the differential cross sections obtained by our methods with the exact results fo partial-wave analyses.[20,21] As has been plain in the development, however, a number of separate approximations have been employed, some inherent in the method, and some for simplicity in the present case. We first summarize this situation, so that in the subsequent comparisons some estimates can be made about possible improvements.

The basic feature of our method is the use of the eikonal approximation to describe the electron wave function in the vicinity of the nucleus. We have restricted our considerations to the lowest-order eikonal, or semiclassical approximation. In addition, we have made analytic expansions about $r=0$ of the quantities involved in the approximation [$S(\mathbf{r})$, $u_0(\mathbf{r})$, etc.], to match the assumed expansion (2.30) of the potential. A partial confirmation of the validity of this expansion is provided by the good agreement shown in Table I between the values of b and c from the eikonal approximation and those from the phase-shift analysis using (2.51) and (2.54).[22] In the perturbation theory for which this is assumed to provide a good approximation to the zero-order wave functions, we have calculated the scattering amplitude to only lowest order (first order) in the scattering potential. The resulting matrix element for the amplitude has been then approximated by the first terms in asymptotic expansions.

We are considering as a special case the uniform charge distribution. This shape contains only one length parameter R, the radius. The number of dimensionless quantities on which the validity of the approximations depend is thus limited to kR (or $k'R$), qR (or $q'R$) and $\gamma=Ze^2/\hbar c$. The eikonal expansion, valid for high energies and smoothly varying potentials, is better for small γ and large kR. From the arguments at the end of Sec. II we expect that the next-order eikonal terms are of order $\gamma/(kR)^2$. In the expansion of the semiclassical wave functions to order γ the a terms in S and u_0 are correct inside the nucleus while the b and c terms are only the first terms in an analytic fit to the phase in the transverse plane through the origin. It is easy to check that inside the nucleus b and c give the phase and its first derivative with respect to r_1 to 3 and 12.5% accuracy, respectively.[23] These are the only quantities

[20] B. C. Clark, R. Herman, and D. G. Ravenhall (unpublished calculations).

[21] J. T. Reynolds and D. S. Onley (unpublished calculations).

[22] The "errors" shown in the phase shift values of b and c reflect the fact that the actual phase shifts fluctuate about the smooth curve given by (2.51) (Ref. 16). (Note that these fluctuations are larger in the case of the uniform charge distribution). In all cases, the phase-shift value of b agrees with the b_{av} to within 2% (for the uniform charge distribution, the difference between b_{av} and b_1+b_2 is negligible). The phase-shift values of c are less well determined, but agree reasonably well with c_1 [Eq. (2.34b)] for the uniform and smoothed uniform cases. The disagreement in the three-parameter case may be attributed to the fact that c_1 is anomalously small so that c_2, which we have not evaluated, might be relatively important.

[23] The treatment of Sec. 3 shows that we need to know the functions S and u_0 and their derivatives at $\pm R\hat{q}$; this gives $r_1=|\hat{k}\times R\hat{q}|=R\cos\frac{1}{2}\theta$. The worst errors therefore occur at the smallest angles. For $\theta>40°$, the error in C_2, for example, is less than 6%. For the uniform charge distribution, S_1' can be calculated exactly and the evaluation of u_0, C_2, etc., can be carried through in the same manner as has been done here. This is being done by L. McDonald, who is also investigating the contributions of other neglected terms.

which enter the present paper; the higher radial derivatives, which would be required for corrections to the asymptotic expansion, are given very poorly by the b and c terms. In general, the number of terms to be kept depends on the rapidity of the variation of $V(r)$, and our somewhat vaguely defined substitution of the smoothed potential $\hat{V}(r)$ for $V(r)$ make this rather hard to assess. The resulting uncertainties produced in the scattering amplitude will presumably depend on the process considered, and the emphasis the transition potential puts on the various regions of $\mathbf{r}$. One expects, for example, that in quadrupole excitation, because of the long range of the interaction potential, the neglected terms in the expansions about the origin would contribute more than in monopole excitation, and hence in elastic scattering. But in general we expect the wave-function approximation to improve as k increases roughly like $(kR)^{-2}$ and at a given kR to be better for small γ.

The adequacy of first-order perturbation theory in calculating the amplitude depends in some way on the form factor associated with the scattering potential. The relative magnitude in our approximation of the second-order contribution depends physically on the importance of two half-angle scatterings compared with the one large-angle scatter. For a uniform charge distribution, whose sharp edge produces abundant high Fourier components (i.e., single large-angle scatters), the neglect of these higher terms will be best justified, whereas for a Gaussian charge distribution, with very few high Fourier components, it would probably be very poor. The applicability of our method to actual nuclear shapes, where the smooth edge of the distribution attenuates the high Fourier components of the uniform shape, needs further examination in this respect. The asymptotic expansions used to evaluate the first-order amplitude introduce errors which depend on the parameter $(qR)^{-1}$, and our final expressions thus omit contributions which are most important in the forward direction. In an expression of the form (3.1) the contribution from different regions of $\mathbf{r}$ depends on the size of q. Hence the errors at large r in our expansion of the eikonal wave function can also introduce q-dependent errors in the amplitude. The exact dependence on γ, kR and qR of the accuracy of our final result is thus somewhat complicated and difficult to predict.

Of the two versions presented for elastic scattering, the approximation derived via monopole excitation (Approximation 2) is certainly more consistent in its treatment of the small terms. The consideration of incoherent amplitudes such as the monopole excitation does not require the somewhat tortuous reasoning involving the separation from $V(r)$ of $\hat{V}(r)$ contained at the beginning of Sec. III. The elastic amplitude derived from the monopole excitation, (3.31), is thus more firmly based, at least to first order in the perturbation expansion employed to obtain (3.1). The derivation is somewhat indirect, however, and in our comparison with exact calculations we include the more direct but less well-founded version (3.22). These are the curves labelled "approximation 2" and "approximation 1," respectively, in our plots of elastic scattering.

We now examine empirically the accuracy of our approximation and its dependence on the three dimensionless parameters $\gamma=Ze^2/\hbar c$, kR, and qR by comparing with results of partial-wave calculations. We recall again that because our comparisons are confined to the case of a uniform charge distribution with but one radial parameter R, these are the only dimensionless numbers involved in the comparison. By plotting the quantity $k^2d\sigma/d\Omega$, explicit dependence on the energy $E=\hbar ck$ is removed, and particular examples are completely specified by the values of γ and kR. Titanium, $_{22}$Ti, with γ taken as 0.16054, represents light nuclei and gold, $_{79}$Au, with $\gamma=0.5675$, heavy nuclei. Since, in the main, terms neglected are of order $\gamma/(kR)^2$, agreement is expected to start at smaller kR values for titanium than for gold. The elastic scattering has thus been compared at $kR=4$ and 8 for tatinium, in Figs. 2 and 3, and for $kR=8$ and 16 for gold, in Figs. 4 and 5. To appreciate the region of physical applicability of our method, we can insert the actual sizes of these nuclei. For titanium, those kR values correspond to scattering at about 200 and 400 MeV, respectively; for gold, $kR=8$ and 16 would be about 250 and 500 MeV, respectively.

We note first, very briefly, the well-known character-

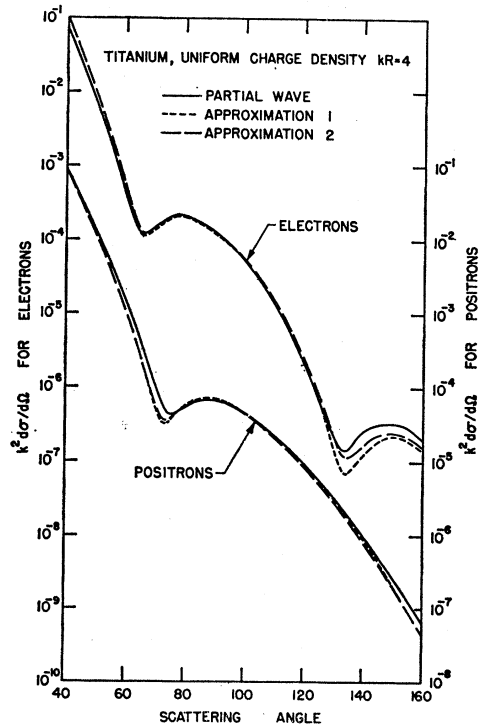

FIG. 2. Electron and positron scattering on titanium, $\gamma=0.16054$, uniform charge density, $kR=4$. Approximation 1 refers to (3.22), and approximation 2 to (3.31).

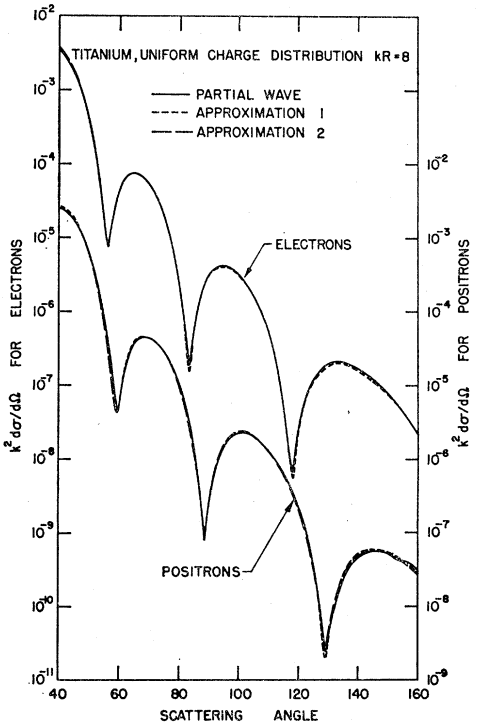

FIG. 3. Electron and positron scattering on titanium, uniform charge density $kR=8$.

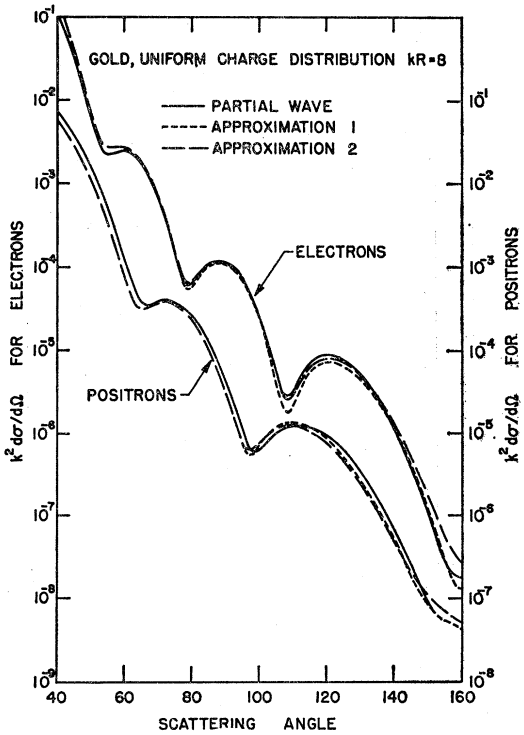

FIG. 4. Electron and positron scattering on gold, $\gamma=0.5765$, uniform charge density $kR=8$.

istics of the exact results which have stimulated approximations such as ours. The pronounced diffraction structure of the Born approximation is reproduced, but is shifted to smaller or larger angles, for electrons or positrons, respectively, by amounts which increase with γ. The zeros are filled in by amounts which increase with γ. Both of these effects are in magnitude the same for electrons and positrons. Looking at the same diffraction dip (as regards its qR value) at different values of kR, which is practically the useful comparison, we see that the filling-in of the zeros decreases as kR increases. (We compare for example, the dip at $\theta\simeq 134°$ in the electron curve of Fig. 2 with the dip at $\theta\simeq 56°$ in Fig. 3.) For a given γ and kR, it decreases as qR increases (i.e.,

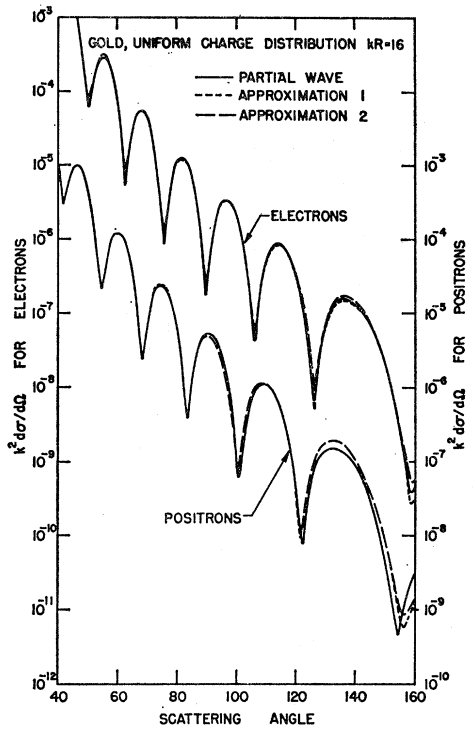

FIG. 5. Electron and positron scattering on gold, uniform charge density, $kR=16$.

as θ increases) as is especially clear from Fig. 5. The effect of the c term in S is illustrated in Fig. 6; we note that it is particularly important at smaller angles.

In over-all behavior our approximate calculations reproduce these features very clearly. The disagreements, however, are also interesting, since they may point the way to possible improvements. The position of the diffraction dips is reproduced very well for electrons in all cases, but for positrons there is a tendency, at the lower kR values for each γ, for the minima to be shifted to slightly smaller angles, as seen in Figs. 2 and 4. This is true for both elastic scattering approximations, which become the same at small qR. This effect is believed to be due to the contribution of δS to Φ_1 and

corrections to ν_1, both of which could alter the position of the zeros of Re f_1. Since δS affects electrons and positrons similarly, and order γ corrections to ν_1 affect them oppositely, one expects an asymmetry in the positioning of the minima relative to the gross shift produced by the change in wave number. The tendency decreases as kR is doubled in each case, however, and is barely perceptible in Fig. 3. Of the filling-in of the Born approximation zeros, the only general feature seems to be that the monopole-derived approximation 2 is better than the earlier approximation 1. The maxima of the diffraction structure, where small correction terms will have less effect, are reproduced most closely, including

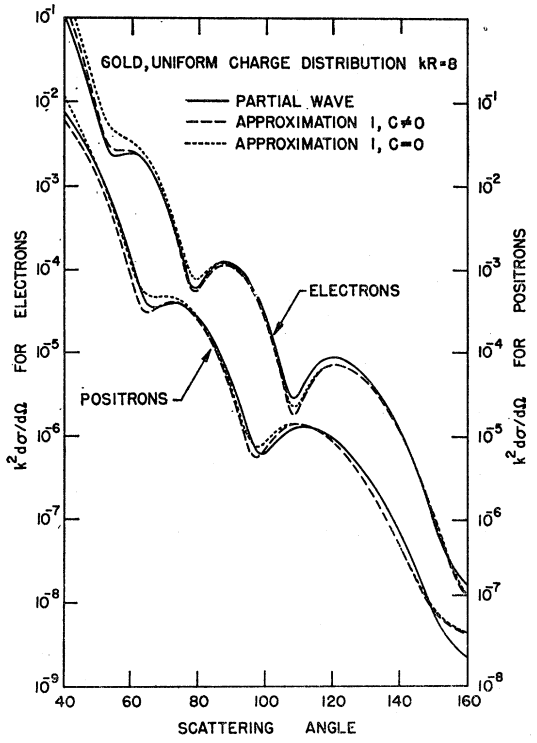

FIG. 6. Electron and positron scattering on gold, uniform charge density $kR=8$. This graph illustrates the effect of the c term in S (2.37).

the middle-angle region in Figs. 2 and 4. There is a tendency for even this feature to be less well reproduced at the very largest angles, however. This feature is not well understood at present, since the asymptotic expansion should be better there; possibly, it is because the cross section has become so small that double scatterings at small distances have become important. Inaccuracies at small angles, or more precisely at small qR, are expected from the asymptotic expansion of the amplitude, of course.

To display more clearly the agreements and differences between the approximate and exact results Figs. 7 and 8 plot on an expanded scale the ratio of the approximate calculation σ_2 to the exact one. We note

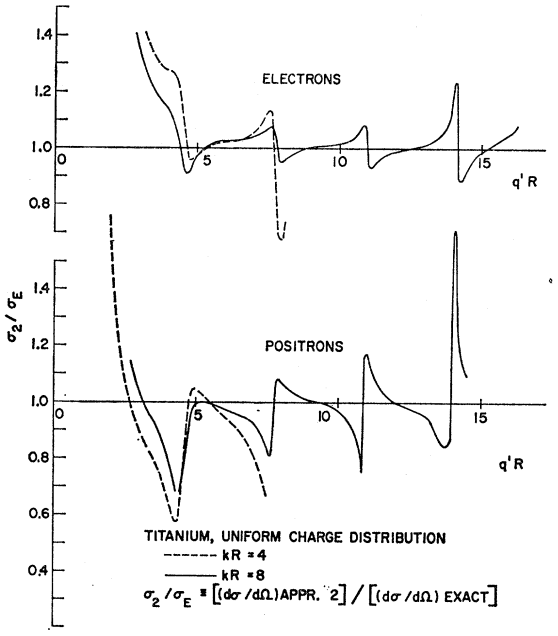

FIG. 7. Ratios of approximate to exact cross sections for electron and positron scattering on titanium.

that the greatest relative errors occur at the minima, where the approximate cross section has not been computed with the same relative accuracy as at the maxima. In the positron cases, especially, the small error in positioning the minima has given rise to a rapid change in the ratio. It is also clear from these figures that there is a general improvement in accuracy with increasing kR.

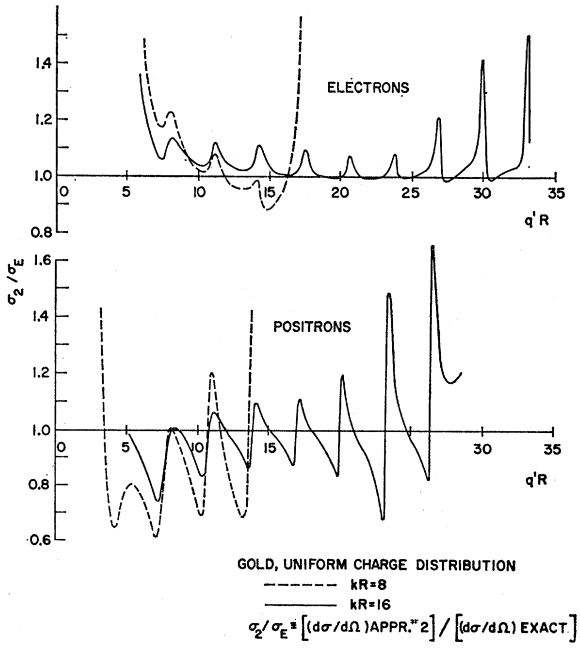

FIG. 8. Ratios of approximate to exact cross sections for electron and positron scattering on gold.

TABLE II. Values of the coefficients in (3.22b) for electron scattering from a uniformly charged gold nucleus with $kR=8$.

θ	μ_1	μ_2	ν_1	ν_2
40°	1.98	0.35	2.67	−8.43
50°	1.69	0.20	2.10	−7.51
60°	1.53	0.12	1.82	−6.94
70°	1.44	0.09	1.66	−6.50
80°	1.38	0.06	1.55	−6.10
90°	1.34	0.05	1.47	−5.74
100°	1.30	0.04	1.41	−5.39
110°	1.27	0.03	1.36	−5.05
120°	1.25	0.02	1.32	−4.74
130°	1.23	0.02	1.29	−4.46
140°	1.21	0.02	1.27	−4.21
150°	1.20	0.01	1.25	−4.01
160°	1.19	0.01	1.24	−3.86

Another way to get some insight into the calculation is to look at the sizes of the coefficients μ_1, μ_2, ν_1, and ν_2, whose leading orders of magnitude are, respectively, 1, $\gamma/(kR)^2$, 1, and γ. These are shown in Tables II and III for the typical cases of electron and positron scattering on gold, $kR=8$. In this case $\gamma\sim 1$, and we see that the deviation of the coefficients from their Born approximation values (1,0,1,0) is considerable and is greatest at the smallest angles. Because these deviations are already so large, it may seem surprising that further corrections are not equally important. This is probably because our calculation is not a systematic one in powers of γ; since it has denominators linear in γ, it really contains many higher powers of γ already. The best determined coefficient is μ_1, and it is the most important one since it determines the magnitude of the cross section at the maxima. Possible large errors in ν_1 would lead to only a small error in the position of the minima. The coefficient ν_2 is important in determining the filling in of the minima and it has not yet been determined to the same relative accuracy as μ_1. The coefficient μ_2 is completely unimportant, and it and similar terms have always been dropped because they are not given consistently in the leading order of the asymptotic expansion. Clearly, to improve the results near the minima better calculations of ν_1 and ν_2, as well as Φ_1, are necessary.[23] The differences between electron and positron scattering also appear in these tables, where we see that the coefficients for electron scattering differ more significantly from the Born approximation than do those for positron scattering. This can be traced to the a term in C, which is negative for electrons, thus enhancing all coefficients and is the reverse for positrons.

TABLE III. Values of the coefficients in (3.22b) for positron scattering from a uniformly charged gold nucleus with $kR=8$.

θ	μ_1	μ_2	ν_1	ν_2
40°	1.01	−0.17	1.07	3.19
50°	0.92	−0.10	0.92	3.01
60°	0.86	−0.07	0.84	2.90
70°	0.83	−0.05	0.80	2.84
80°	0.81	−0.04	0.78	2.80
90°	0.80	−0.03	0.76	2.79
100°	0.79	−0.03	0.75	2.80
110°	0.78	−0.02	0.75	2.83
120°	0.78	−0.02	0.74	2.89
130°	0.77	−0.02	0.73	2.95
140°	0.76	−0.02	0.73	3.03
150°	0.76	−0.02	0.73	3.11
160°	0.75	−0.02	0.72	3.17

For brevity, the monopole excitation comparison is made for electrons and $kR=8$ only; in Fig. 9 the dimensionless quantity $k\Delta R$ in (3.30a) which describes the strength of the transition has been given the value unity. Comparison of Figs. 9 and 3, and Figs. 9 and 4, shows how closely the regular structure of the monopole excitation follows that of the elastic scattering. The feature that the two amplitudes are just $\frac{1}{2}\pi$ out of phase a result which the Born approximation predicts, is a natural consequence of the exact relationship (3.29). We use this relation both for our exact calculation of the monopole excitation from the exact elastic scattering, and for the approximation 2 to the elastic scattering obtained from our approximation to monopole excitation. The comparison of (3.30b) and (3.31) displays the phase difference explicitly. The close connection between elastic and monopole excitation amplitudes which we exploit in both the exact and the approximate calculations explains why the quality of the agreement is the same in the corresponding cases.

The quadrupole-excitation results are shown in Fig. 10, again for $kR=8$ and electrons on gold and

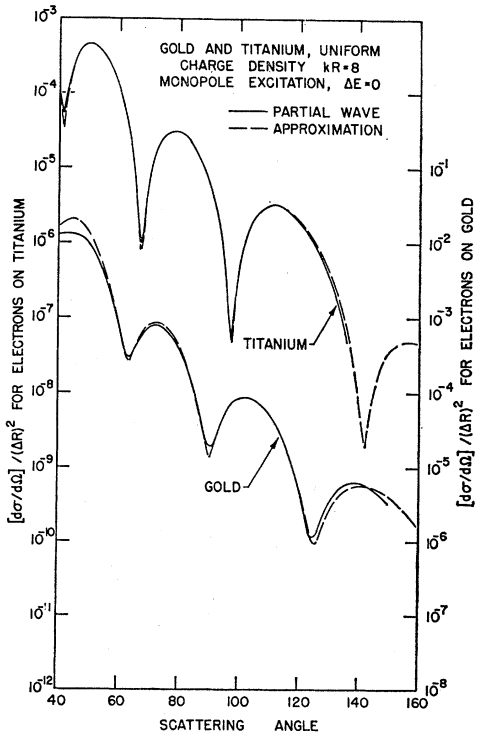

FIG. 9. Monopole excitation cross sections for electrons on gold and titanium, $kR=8$.

titanium.[21] The δ-function excitation is also chosen to be at a radius given by $kr_0=8$. For quadrupole excitation, it is not possible to give a simple exact method of calculation such as that outlined for monopole excitation in Sec. III, using (3.27). Instead, one must expand the scattering eigenfunctions in partial waves and calculate integrals like (3.24) numerically. To the extent that our approximation is sufficiently accurate, it can therefore replace time-consuming calculations. The results shown in Fig. 10 are promising. They indicate that our distorted wave approximation will be valuable in exploratory calculations of excitation cross sections although the exact methods may still be necessary for more refined analyses.

For the special case of the uniform charge distribution our approximations have produced expressions for the scattering amplitudes which display analytically the behavior expected. They turn out to be in close agreement with the partial-wave calculations, without the aid of any parameter adjustments or fudge factors, so that the physical correctness of this approach is clearly demonstrated. Obvious improvements in this case would be to carry further terms both in the Taylor expansion of the eikonal function and the asymptotic expansions of the amplitude. For charge distributions of more physical applicability, such as the Fermi shape, however, the asymptotic expansion (3.20) of the radial integral, which used the discontinuity in $\rho(r)$, must presumably be replaced by an exact evaluation. This is unavoidably less transparent and also lengthier computationally. (For this reason we make no attempt to compare computing times of our present approximation and of the partial-wave calculation.) It would be interesting to see the effect of this change alone. A numerical evaluation of the amplitude (3.1) avoiding completely the asymptotic expansions is also possible. If present disagreement at large angles is due to inaccuracies in the Taylor expansion, however, it would not thereby be improved, unless those expansions were also carried further. These seem to be matters for further exploration. The numerical evaluation of the whole approximation, including the semiclassical wave function, seems profitless. Even for inelastic scattering, the labor involved is equal to that of the partial-wave analysis, and there is still the uncertainty of higher order eikonal corrections.

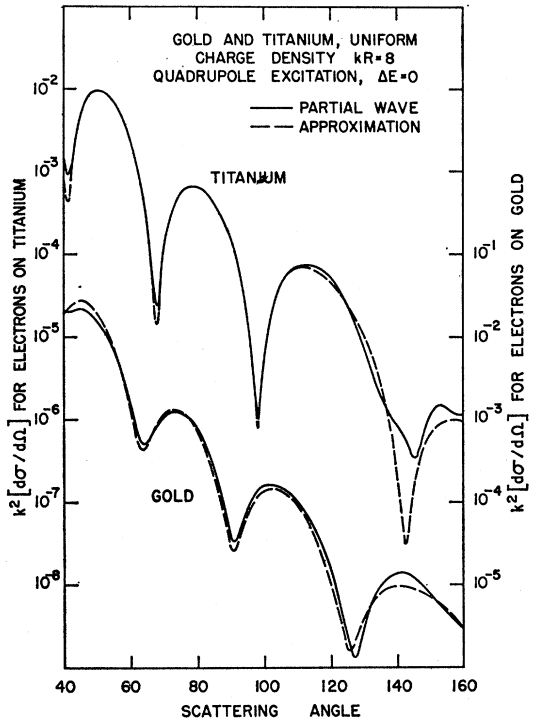

FIG. 10. Quadrupole excitation cross sections for electrons on gold and titanium, $kR=8$.

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